lecture 4

Continuous Time Finance

The Martingale Approach

II: Pricing and Hedging

(Ch 10-12)

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(lecture 3 applied to finance)

Financial Markets (a recap)

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N\right]$$

Example: (Black-Scholes, $S^0 := B,\ S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N\right]$$

h_t^i = number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = \overbrace{h_t}^{\text{row vector}} \underbrace{S_t}_{\text{column vector}}$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**.

NB! This simple observation is in fact the basis of the following theory.

Itô theory!

discrete time : $\Delta Y_t = h_t \Delta X_t$ predictable;

$E[\Delta Y_t | \mathcal{F}_{t-1}] = E[h_t \Delta X_t | \mathcal{F}_{t-1}]$
$= h_t E[\Delta X_t | \mathcal{F}_{t-1}]$

Arbitrage

The portfolio u is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T \geq 0, \; P - a.s.$ (weaker than on p. 59)
- $P\left(V_T > 0\right) > 0$

Main Question: When is the market free of arbitrage?

First Attempt

not realistic

Proposition: If $S_t^0, \cdots, S_t^N$ are P-martingales, then the market is free of arbitrage.

Proof:
Assume that V is an arbitrage strategy. Since

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i,$$

V is a P-martingale, so (because then expectations remain the same)

$$V_0 = E^P [V_T] > 0.$$

This contradicts $V_0 = 0$.

True, but useless: next page

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

(We would have to assume that $\alpha = r = 0$)

for S and B martingales

We now try to improve on this result.

Choose $\mathbf{S}_0$ as numeraire

Definition:
The **normalized price vector** Z is given by

$$Z_t = \frac{S_t}{S_t^0} = \left[1, Z_t^1, ..., Z_t^N\right]$$

The **normalized value process** V^Z is given by

$$V_t^Z = \sum_0^N h_t^i Z_t^i.$$

Idea:
The arbitrage and self financing concepts should be independent of the accounting unit.

Invariance of numeraire

Proposition: One can show (see the book) that

- S-arbitrage $\Longleftrightarrow$ Z-arbitrage.

- S-self-financing $\Longleftrightarrow$ Z-self-financing. $\left(dV_t = h_t dS_t \Leftrightarrow dV_t^Z = h_t dS_t\right)$

Insight:

- If h self-financing then

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

- Thus, if the **normalized** price process Z is a P-martingale, then V^Z is a martingale.

Second Attempt

the normalized processes

Proposition: If $Z_t^0, \cdots, Z_t^N$ are P-martingales, then the market is free of arbitrage.

True, but still fairly useless.

by argument as on p. 173

Example: (Black-Scholes)

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

$Z_t^1 = \frac{S_t}{B_t}$

$$
\begin{aligned}
dZ_t^1 &= (\alpha - r) Z_t^1 dt + \sigma Z_t^1 dW_t, \\
dZ_t^0 &= 0 dt.
\end{aligned}
$$

We would have to assume "risk-neutrality", i.e. that $\alpha = r$. to have Z^1 is a P-martingale

Arbitrage

Recall that h is an arbitrage if

- h is self financing
- $V_0 = 0.$
- $V_T \geq 0, \; P - a.s.$
- $P(V_T > 0) > 0$

Major insight

This concept is invariant under an **equivalent change of measure!**

$$P \sim Q \Rightarrow \begin{cases} P(A) = 0 \Leftrightarrow Q(A) = 0 \\ P(A) = 1 \Leftrightarrow Q(A) = 1 \\ P(A) > 0 \Leftrightarrow Q(A) > 0 \end{cases}$$

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \ldots, N$$

  are **Q-martingales**.

W~~an~~e now state the main result of arbitrage theory.

First Fundamental Theorem

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

(FTAP 1)

Comments

- It is very easy to prove that existence of EMM imples no arbitrage (see below).

- The other implication is technically very hard.

- For discrete time and finite sample space Ω the hard part follows "easily" from the separation theorem for convex sets.

- For discrete time and more general sample space we need the Hahn-Banach Theorem. (formulated as an infinite dimensional version of the separation theorem

- For continuous time the proof becomes technically very hard, mainly due to topological problems. See the textbook.

end of lecture 4a.

Proof that EMM implies no arbitrage

Assume that there exists an EMM denoted by Q. Assume that $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$. *)

Note: $E^Q(V_T) > 0$!

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale

$$\Downarrow$$

$$V_0 = V_0^Z = E^Q\left[V_T^Z\right] > 0$$

$B_0 = 1$

$$\Downarrow$$

No arbitrage

*) All these statements also true for V_T^Z instead of V_T

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

(generalizes $B_t = e^{rt}$)

Example: The Black-Scholes Model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Look for martingale measure. We set $Z = S/B$.

$$dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t,$$

Girsanov transformation on $[0, T]$:

$$\begin{cases} dL_t &= L_t \varphi_t dW_t, \\ L_0 &= 1. \end{cases}$$

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T$$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process.

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma},$$ then $dZ_t = Z_t \sigma dW_t^Q$

Q-dynamics of S: (side remark)

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage.

Pricing

We consider a market $B_t, S_t^1, \ldots, S_t^N$.

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

Def: $\Pi_t(X)$ is an arbitrage free price process if the extended market is arbitrage free

Solution

extra asset

The $\boxed{\text{extended}}$ market

$$B_t, S_t^1, \ldots, S_t^N, \Pi_t[X]$$

must be arbitrage free, so there must exist a martingale measure Q for $(S_t, \Pi_t[X])$. In particular

$$\frac{\Pi_t[X]}{B_t}$$

must be a Q-martingale, i.e.

$$\frac{\Pi_t[X]}{B_t} = E^Q\left[\left.\frac{\Pi_T[X]}{B_T}\right| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T[X] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right],$$

if $dB_t = r_t B_t dt$.

NB: if $r_t \equiv r$, then

$$\Pi_t(X) = e^{-r(T-t)} E^Q\left[X \middle| \mathcal{F}_t\right]$$

Example: The Black-Scholes Model

Q-dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

NB: S is a Markov process under Q!

Simple claim:

$$X = \Phi(S_T),$$

$$\Pi_t [X] = e^{-r(T-t)} E^Q [\Phi(S_T) | \mathcal{F}_t]$$

Kolmogorov $\Rightarrow$? (Markov property)

$$\Pi_t [X] = F(t, S_t)$$

where $F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T, s) = \Phi(s). \end{cases}$$

[Feynman - Kac]

Problem

Recall the valuation formula
(p.188)

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

What if there are several different martingale measures Q?

This is connected with the **completeness** of the market.

Hedging

Def: A portfolio h is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a arbitrage free natural way of pricing X is

$$\Pi_t [X] = V_t^h$$

(see p. 101 for a justification)

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

Fix T-claim X.

If h is a hedge for X then

- $V_T^Z = \frac{X}{B_T}$

- h is self financing, i.e.

$$dV_t^Z = \sum_1^K h_t^i dZ_t^i$$

martingales!

Thus V^Z is a Q-**martingale**. $\Rightarrow$

$$V_t^Z = E^Q\left[\left.\frac{X}{B_T}\right| \mathcal{F}_t\right]$$

Lemma:
Fix T-claim X. Define martingale M by

$$M_t = E^Q \left[\frac{X}{B_T} \right| \left. \mathcal{F}_t \right]$$

Suppose that there exist predictable processes $h^1, \cdots, h^N$ such that

$$M_t = x + \sum_{i=1}^{N} \int_0^t h_s^i dZ_s^i,$$

Then X can be replicated.

Proof

We guess that

$$M_t = V_t^Z = h_t^B \cdot 1 + \sum_{i=1}^{N} h_t^i Z_t^i$$

Define: h^B by

$$h_t^B = M_t - \sum_{i=1}^{N} h_t^i Z_t^i.$$

We have $M_t = V_t^Z$, and we get , by assumption,

$$dV_t^Z = dM_t = \sum_{i=1}^{N} h_t^i dZt^i,$$

so the portfolio is self financing. Furthermore:

$$V_T^Z = M_T \underset{\text{def of } M_T}{=} E^Q\left[\left.\frac{X}{B_T}\right| \mathcal{F}_T\right] = \frac{X}{B_T}.$$

→ hedge!

end of lecture 4b.

Second Fundamental Theorem

[FTAP 2]

The second most important result in arbitrage theory is the following.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

?

Proof: It is obvious (why?) that if the market is complete, then Q must be unique. The other implication is very hard to prove. It basically relies on duality arguments from functional analysis.

For all $A \in \mathcal{F}_T$, $1_A B_T$ can be hedged and hence has a unique price: for any Q:

$$\Pi_t(1_A B_T) = \frac{E^Q[1_A B_T \mid \mathcal{F}_t]}{B_t}.$$

For $t=0$: unique $\Pi_0(1_A B_T) = Q(A) \Rightarrow Q$ unique

Black-Scholes Model

Q-dynamics (recall $Z_t = \frac{S_t}{B_t}$)

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\ dZ_t &= Z_t \sigma dW_t^Q \end{aligned}$$

$$M_t = E^Q \left[e^{-rT} X \middle| \mathcal{F}_t \right],$$

Representation theorem for Wiener processes
$\Downarrow$
there exists g such that

$$M_t = M(0) + \int_0^t g_s dW_s^Q.$$

Thus

$$M_t = M_0 + \int_0^t h_s^1 dZ_s,$$

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result: from lemma on p. 194, 195
X can be replicated using the portfolio defined by

$$
\begin{aligned}
h_t^1 &= g_t/\sigma Z_t, \\
h_t^B &= M_t - h_t^1 Z_t.
\end{aligned}
$$

Moral: The Black Scholes model is complete.

Here we didn't need (as on p. 102) that X is of the form $X = \Phi(S_T)$, but see next page(s).

Special Case: Simple Claims

Assume X is of the form $X = \Phi(S_T)$

$$M_t = E^Q\left[e^{-rT}\Phi(S_T)\middle|\mathcal{F}_t\right],$$

normalized price !

Kolmogorov backward equation $\Rightarrow M_t = f(t, S_t)$ (S is Q-Markov)

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} & = & 0, \\ f(T, s) & = & e^{-rT}\Phi(s). \end{cases}$$

Itô $\Rightarrow$ $dM_t = f_t dt + f_s dS_t = [\text{use PDE}] =$

$$dM_t = \sigma S_t \frac{\partial f}{\partial s} dW_t^Q,$$

compare to pp 103-106, $F(t,s) = e^{rt} f(t,s) = B_t f(t,s)$

so

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s},$$

Replicating portfolio h:

$$h_t^B = f - S_t \frac{\partial f}{\partial s},$$

$$h_t^1 = \frac{g_t}{\sigma Z_t} = \frac{S_t \frac{\partial f}{\partial s}}{Z_t} = h_t^1 = B_t \frac{\partial f}{\partial s}.$$

Interpretation: $f(t, S_t) = V_t^Z$, normalized price process of X

Define $F(t, s)$ by

nominal, unnormalized price

$$F(t, s) = e^{rt} f(t, s),$$

so $F(t, S_t) = V_t$. Then, from previous page,

$$\begin{cases} h_t^B = \frac{F(t,S_t) - S_t \frac{\partial F}{\partial s}(t,S_t)}{B_t}, \\ h_t^1 = \frac{\partial F}{\partial s}(t, S_t) \end{cases}$$

where F solves the **Black-Scholes equation**

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T, s) = \Phi(s). \end{cases}$$

Use PDE on p.199 and

$\frac{\partial F}{\partial t} = rB_t f + B_t \frac{\partial f}{\partial t}$, $\frac{\partial F}{\partial s} = B_t \frac{\partial f}{\partial s}$ and PDE for f

Main Results

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q

- The market is complete $\Leftrightarrow$ Q is unique.

- Every X must be priced by the formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$, complete or not

  for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.

- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t[X] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

because $\Pi_t(X)$ = the value of a portfolio!

Completeness vs No Arbitrage
Rule of Thumb

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$
\begin{aligned}
R &= \text{number of random sources} \\
N &= \text{number of risky assets}
\end{aligned}
$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim. (as on p.108)

Rule of thumb

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

For B-S we have $N = R = 1$. Thus the Black-Scholes model is arbitrage free and complete.

Stochastic Discount Factors / pricing formula under P

Given a model under P. For every EMM Q we define the corresponding **Stochastic Discount Factor**, or **SDF**, by

$$D_t = e^{-\int_0^t r_s ds} L_t, \quad = L_t / B_t$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

There is thus a one-to-one correspondence between EMMs and SDFs.

The risk neutral valuation formula for a T-claim X can now be expressed under P instead of under Q.

Proposition: With notation as above we have

$$\Pi_t[X] = \frac{1}{D_t} E^P \left[D_T X \middle| \mathcal{F}_t \right]$$

Start from $\frac{\Pi_t(X)}{B_t} = E^Q\left[\frac{X}{B_T}\middle|\mathcal{F}_t\right]$

Proof: Bayes' formula:

$$= \frac{E^P\left[X L_T / B_T \middle| \mathcal{F}_t\right]}{L_t}$$

$$= E^P\left[X D_T \middle| \mathcal{F}_t\right] / L_t$$

etc.

Martingale Property of $S \cdot D$

Proposition: If S is an arbitrary price process, then the process

$$S_t D_t$$

is a P-martingale.

Proof: Bayes' formula:

Same trick: we know $E^Q\left[\frac{S_T}{B_T} \Big| \mathcal{F}_t\right] = \frac{S_t}{B_t}$

$$\frac{E^P\left[\frac{S_T}{B_T} L_T \Big| \mathcal{F}_t\right]}{L_t} = \frac{S_t}{B_t}$$

D_T

[end of lecture 4c]