Continuous Time Finance

Dividends,

Forwards, Futures, and Futures Options

Ch 16 & 26

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Contents

1. Dividends

Dividends

Black-Scholes model:

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

New feature:
The underlying stock pays **dividends**.

don't confuse with notation for discount factor on p. 204

$$D_t = \text{The cumulative dividends over the interval } [0, t]$$

(will depend on $S_u, u \leq t$)

Interpretation:
Over the interval $[t, t + dt]$ you obtain the amount dD_t

Two cases

- Discrete dividends (realistic but messy). SKIP!
- Continuous dividends (unrealistic but easy to handle).

Portfolios and Dividends

Consider a market with N assets.

$$
\begin{aligned}
S_t^i &= \text{price at } t\text{, of asset No } i \\
D_t^i &= \text{cumulative dividends for } S^i \text{ over} \\
& \quad \text{the interval } [0, t] \\
h_t^i &= \text{number of units of asset } i \\
V_t &= \text{market value of the portfolio } h \text{ at } t
\end{aligned}
$$

$D_0^i = 0$ (!)

Assumption: We assume that D has continuous trajctories.

Definition: The **value process** V is defined by

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

(as before)

Self financing portfolios

Recall:

$$V_t = \sum_{i=1}^{N} h_t^i S_t^i$$

New **Definition:** The strategy h is **self financing** if

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$

where the **gain** process G^i is defined by

$$dG_t^i = dS_t^i + dD_t^i$$

extra "income", compared to S_t^i

Interpret!

Note: The definitions above rely on the assumption that D is continuous. In the case of a discontinuous D, the definitions are more complicated.

Relative weights

u_t^i = the relative share of the portfolio value, which is invested in asset No i.

$$u_t^i = \frac{h_t^i S_t^i}{V_t}$$ (as before)

$$dV_t = \sum_{i=1}^{N} h_t^i dG_t^i$$ (previous page)

Substitute!

$$dV_t = V_t \sum_{i=1}^{N} u_t^i \frac{dG_t^i}{S_t^i}$$

Continuous Dividend Yield

Definition: The stock S pays a **continuous dividend yield** of q, if D has the form

$$dD_t = qS_t dt$$, here the dividend rate is proportional to S_t

Problem:
How does the dividend affect the price of a European Call? (compared to a non-dividend stock).

Answer:
The price is lower. (why?) you can guess....

Black-Scholes with Cont. Dividend Yield

as before,

$$\left.\begin{array}{rcl} dS_t & = & \alpha S_t dt + \sigma S_t dW_t, \\ dD_t & = & q S_t dt \end{array}\right\} \Rightarrow$$

Gain process:

$$dG_t = (\alpha + q)S_t dt + \sigma S_t dW_t$$

Consider a fixed claim

$$X = \Phi(S_T)$$

and assume that

$$\Pi_t [X] = F(t, S_t)$$

(justified by some Markov property)

Standard Procedure, familiar by now

- Assume that the derivative price is of the form

$$\Pi_t\left[X\right] = F(t, S_t).$$

- Form a portfolio based on underlying S and derivative F, with portfolio dynamics with SF property

$$dV_t = V_t \left\{ u_t^S \cdot \boxed{\frac{dG_t}{S_t}} + u_t^F \cdot \frac{dF}{F} \right\}$$

new

(compare p-212)

- Choose u^S and u^F such that the dW-term is wiped out. This gives us

$$dV_t = V_t \cdot k_t dt$$

- Absence of arbitrage implies

$$k_t = r$$

- This relation will say something about F.

Value dynamics:

$$dV = V \cdot \left\{ u^S \frac{dG}{S} + u^F \frac{dF}{F} \right\},$$

$$dG = S(\alpha + q)dt + \sigma S dW.$$

From Itô we obtain

$$dF = \alpha_F F dt + \sigma_F F dW,$$

where

$$\begin{aligned} \alpha_F &= \frac{1}{F}\left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\}, \\ \sigma_F &= \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}. \end{aligned}$$

Collecting terms gives us

$$\begin{aligned} dV &= V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt \\ &+ V \cdot \left\{ u^S \sigma + u^F \sigma_F \right\} dW, \end{aligned}$$

Define u^S and u^F by the system

$$
\begin{aligned}
u^S \sigma + u^F \sigma_F &= 0, \\
u^S + u^F &= 1.
\end{aligned}
$$

to wipe out the Wiener term

Solution (if $\sigma_F \neq \sigma$)

$$
\begin{aligned}
u^S &= \frac{\sigma_F}{\sigma_F - \sigma}, \\
u^F &= \frac{-\sigma}{\sigma_F - \sigma},
\end{aligned}
$$

Value dynamics (dW term wiped out)

$$dV = V \cdot \left\{ u^S(\alpha + q) + u^F \alpha_F \right\} dt.$$

Absence of arbitrage implies (usual argument)

$$u^S(\alpha + q) + u^F \alpha_F = r,$$

We get, using α_F and σ_F of p. 216,

$$\frac{\partial F}{\partial t} + (r - q)S\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.$$

Pricing PDE

Proposition: The pricing function F is given as the solution to the PDE

$$\left\{\begin{array}{rcl} \frac{\partial F}{\partial t} + (r-q)s\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF & = & 0, \\ F(T,s) & = & \Phi(s). \end{array}\right.$$

We can now apply Feynman-Kac to the PDE in order to obtain a risk neutral valuation formula.

under a risk-neutral measure Q:

$$dS_t = (r-q)S_t\,dt + \sigma S_t\,dW_t^Q$$

$$dB_t = rB_t\,dt$$

differences with previous results are due to q.

end of lecture 5a;

Risk Neutral Valuation

The pricing function has the representation

$$F(t,s) = e^{-r(T-t)} E^{Q}_{t,s}\left[\Phi(S_T)\right],$$

where the Q-dynamics of S are given by

$$dS_t = (r-q)S_t dt + \sigma S_t dW^Q_t.$$

Question: Which object is a martingale under the measure Q? Is it $\frac{S_t}{B_t}$ as before?

Martingale Property

Proposition: Under the martingale measure Q the **normalized gain process**

$$G_t^Z = e^{-rt}S_t + \underbrace{\int_0^t e^{-ru}dD_u}_{\text{extra term compared to previous case}}$$

is a Q-martingale.

Proof: Exercise; simply comput $dG^Z_t = * \, dW^Q_t$, no dt term!

Note: The result above holds in great generality.

Interpretation:
In a risk neutral world, today's stock price should be the expected value of all future discounted earnings which arise from holding the stock.

$$S_0 = E^Q\left[\int_0^t e^{-ru}dD_u + e^{-rt}S_t\right], \geq \underbrace{e^{-rt}E^Q S_t}_{\text{the "old" price}} !$$

from Proposition upon noticing $G^Z_0 = S_0$

Pricing formula

Pricing formula for claims of the type

$$\mathcal{Z} = \Phi(S_T)$$

We are standing at time t, with dividend yield q. Today's stock price is s.

- Suppose that you have the pricing function

$$F^0(t, s)$$

  for a non dividend stock.

- Denote the pricing function for the dividend paying stock by

$$F^q(t, s)$$

Proposition: With notation as above we have

$$F^q(t, s) = F^0\left(t, se^{-q(T-t)}\right)$$

This is exercise 16.5

Moral

Use your old formulas, but replace today's stock price s with $se^{-q(T-t)}$.

$\Rightarrow$ explicit expressions in the Black-Scholes case

NB: $se^{-q(T-t)} \leq s$.

If $F^0(t,s)$ is increasing in s (true?)

then $F^q(t,s) \leq F^0(t,s)$

European Call on Dividend-Paying-Stock

$$F^q(t,s) = se^{-q(T-t)}N[d_1] - e^{-r(T-t)}KN[d_2].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)\right\} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

compare to p. 71, and see the role of q;

Martingale Analysis

Basic task: We have a general model for stock price S and cumulative dividends D, under P. How do we find a martingale measure Q, and exactly which objects will be martingales under Q?

needed to define a martingale measure

Main Idea: We attack this situation by reducing it to the well known case of a market without dividends. Then we apply standard techniques.

The Reduction Technique

- Consider the self financing portfolio where you keep 1 unit of the stock and invest all dividends in the bank. Denote the portfolio value by V.

- This portfolio can be viewed as a traded asset **without dividends**. (as they disappear into the bank account)

- Now apply the First Fundamental Theorem to the market (B, V) instead of the original market (B, S).

- Thus there exists a martingale measure Q such that $\frac{\Pi_t}{B_t}$ is a Q martingale for all traded assets (underlying and derivatives) without dividends.

- In particular the process

$$\frac{V_t}{B_t}$$

is a Q martingale.

The V Process

Let h_t denote the number of units in the bank account, where $h_0 = 0$. V is then characterized by

from previous page

$$
\begin{aligned}
V_t &= 1 \cdot S_t + h_t B_t & (1) \\
dV_t &= \underbrace{dS_t + dD_t}_{\text{SF condition, p. 211}} + h_t dB_t & (2)
\end{aligned}
$$

From (1) we obtain

$$dV_t = dS_t + h_t dB_t + B_t dh_t$$

(ordinary product rule, if dh_t makes sense)

Comparing this with (2) gives us

$$B_t dh_t = dD_t$$

Integrating this gives us

$$h_t = \int_0^t \frac{1}{B_s} dD_s$$

We thus have

$$V_t = S_t + B_t \int_0^t \frac{1}{B_s} dD_s \tag{3}$$

and the first fundamental theorem gives us the following result.

Proposition: For a market with dividends, the martingale measure Q is characterized by the fact that the **normalized gain process**

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a Q martingale. (as on p. 221)

Quiz: Could you have guessed the formula (3) for V?

Continuous Dividend Yield

Model under P

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dD_t &= q S_t dt \end{aligned}$$

We recall (p. 221)

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

Easy calculation gives us

$$dG_t^Z = Z_t (\alpha - r + q) dt + Z_t \sigma dW_t,$$

where $Z = S/B$.

Girsanov transformation $dQ = LdP$, where

$$dL_t = L_t \varphi_t dW_t \quad \text{(for some } \varphi_t\text{)}$$

We have

$$dW_t = \varphi_t dt + dW_t^Q \quad \text{(see p. 166)}$$

Insert this into dG^Z

The Q dynamics for G^Z are

$$dG_t^Z = Z_t \left(\alpha - r + q + \sigma\varphi_t\right) dt + Z_t \sigma dW_t^Q$$

Martingale condition

$$\alpha - r + q + \sigma\varphi_t = 0$$

$\alpha + \sigma\varphi_t = r - q$ ✓

Q-dynamics of S

$$dS_t = S_t \left(\alpha + \sigma\varphi\right) dt + S_t \sigma dW_t^Q$$

Using the martingale condition this gives us the Q-dynamics of S as

$$dS_t = S_t \left(r - q\right) dt + S_t \sigma dW_t^Q$$

(note again the role of the dividend rate q)

Risk Neutral Valuation

Theorem: For a T-claim X, the price process $\Pi_t[X]$ is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q[X \mid \mathcal{F}_t],$$

where the Q-dynamics of S are given by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t^Q.$$

end of lecture 5b

2. Forward and Futures Contracts

Forward Contracts

A **forward contract** on the T-claim X, **contracted at** t, is defined by the following payment scheme.

- The holder of the forward contract receives, at time T, the stochastic amount X from the underwriter.
- The holder of the contract pays, at time T, the **forward price** $f(t; T, X)$ to the underwriter.
- The forward price $f(t; T, X)$ is determined at time t. (will be $\mathcal{F}_t$- measurable)
- The forward price $f(t; T, X)$ is determined in such a way that the price of the forward contract equals zero, at the time t when the contract is made.

(compensating cash flows, swap of products that net to zero)

General Risk Neutral Formula

Suppose we have a bank account B with dynamics

$$dB_t = r_t B_t dt, \quad B_0 = 1$$

with a (possibly stochastic) short rate r_t. Then

$$B_t = e^{\int_0^t r_s ds}$$

and we have the following risk neutral valuation for a T-claim X

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t\right] \qquad (*)$$

Setting $X = 1$ we have the price, at time t, of a zero coupon bond maturing at T as

$$p(t,T) = E^Q\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t\right] = B_t E^Q\left[\frac{1}{B(T)} \middle| \mathcal{F}_t\right]$$

[see also book, Section 29.1]

$\frac{p(t,T)}{B(t)}$ is Q-martingale

generalizes $e^{-r(T-t)}$!

Forward Price Formula

Theorem: The forward price of the claim X is given by

$$f(t,T) = \frac{1}{p(t,T)} E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]$$

where $p(t,T)$ denotes the price at time t of a zero coupon bond maturing at time T.

In particular, if the short rate r is deterministic we have

$$f(t,T) = E^Q \left[X \middle| \mathcal{F}_t \right]$$

note the normalization factor,
not B_t !

Proof

definition of forward contract

The net cash flow at maturity is $X - f(t, T)$. If the value of this at time t equals zero we obtain

$$\Pi_t [X] = \Pi_t [f(t, T)]$$

We have (from p. 234)

$$\Pi_t [X] = E^Q \left[e^{-\int_t^T r_s ds} \cdot X \middle| \mathcal{F}_t \right]$$

and, since $f(t, T)$ is known at t, we obviously (why?) have

$$(\Pi_t[X] =) \; \Pi_t [f(t, T)] = p(t, T) f(t, T).$$

This proves the main result. If r is deterministic and constant then $p(t, T) = e^{-r(T-t)}$ which gives us the second formula.

$$\Pi_t [f(t,T)] = f(t,T) \underbrace{E^Q \left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t \right]}_{p(t,T)}.$$

Futures Contracts

A **futures contract** on the T-claim X, is a financial asset with the following properties.

(i) At every point of time t with $0 \leq t \leq T$, there exists in the market a quoted object $F(t; T, X)$, known as the **futures price** for X at t, for delivery at T.

(ii) At the time T of delivery, the holder of the contract pays $F(T; T, X)$ and receives the claim X.

(iii) During an arbitrary time interval $(s, t]$ the holder of the contract receives the amount $F(t; T, X) - F(s; T, X)$.

(iv) The spot price, at any time t prior to delivery, for buying or selling the futures contract, is by definition equal to zero.

Futures Price Formula (Section 29.2)

in particular (iii)

From the definition it is clear that a futures contract is a **price-dividend pair** (S, D) with

$$S \equiv 0, \quad dD_t = dF(t, T)$$

(~~not~~ of the form $\xi S_t dt$)

From general theory, the normalized gains process (p. 228)

$$G_t^Z = \frac{S_t}{B_t} + \int_0^t \frac{1}{B_s} dD_s$$

is a Q-martingale.

Since $S \equiv 0$ and $dD_t = dF(t, T)$ this implies that

$$\frac{1}{B_t} dF(t, T)$$

"general theory"

is a martingale increment, which implies (why?) that $dF(t, T)$ is a martingale increment. Thus F is a Q-martingale (in t) and we have

$$F(t, T) = E^Q \left[F(T, T) \middle| \mathcal{F}_t \right] = E^Q \left[X \middle| \mathcal{F}_t \right]$$

leads to.

see p. 237 (ii)

Theorem: The futures price process is given by

$$F(t, T) = E^Q [X | \mathcal{F}_t].$$

(no discounting!)

Corollary. If the short rate is deterministic, then the futures and forward prices coincide.

see p. 235

3. Futures Options

Futures Options

We denote the futures price process, at time t with delivery time at T by

$$F(t, T).$$

When T is fixed we sometimes suppress it and write F_t, i.e. $F_t = F(t, T)$

not a partial derivative (in spite of same notation)

Definition:
A European futures call option, with strike price K and exercise date T, on a futures contract with delivery date T_1 will, if exercised at T, pay to the holder:

$T < T_1$

- The amount $F(T, T_1) - K$ in **cash**.

$X = (F(T, T_1) - K)^+$

- A long postition in the underlying futures contract.

NB! The long position above can immediately be closed at no cost, so focus on $(F(T, T_1) - K)^+$

Institutional fact:
The exercise date T of the futures option is typcally very close to the date of delivery of the underlying T_1 futures contract.

Why do Futures Options exist?

- On many markets (such as commodity markets) the futures market is much more liquid than the underlying market.
- Futures options are typically settled in **cash**. This relieves you from handling the underlying (tons of copper, hundreds of pigs, etc.). tons of potatoes
- The market place for futures and futures options is often the same. This facilitates hedging etc.

Pricing Futures Options – Black-76

We consider a futures contract with delivery date T_1 and use the notation $F_t = F(t, T_1)$. We assume the following dynamics for F.

$$dF_t = \mu F_t dt + \sigma F_t dW_t$$

Now suppose we want to price a derivative with exercise date T with the T_1-futures price F as underlying, i.e. a claim of the form

$T_1 > T$

$$\Phi(F_T)$$

This turns out to be quite easy.

From risk neutral valuation we know that the price process $\Pi_t[\Phi]$ is of the form (by analogy)

$$\Pi_t[\Phi] = f(t, F_t)$$

where f is given by

$$f(t, F) = e^{-r(T-t)} E^Q_{t,F}[\Phi(F_T)]$$

→ conditional expectation under Q, with $F_t = F$

so it only remains to find the Q-dynamics for F.

We now recall from p.238

Proposition: The futures price process F_t is a Q-martingale.

! Thus the Q-dynamics of F are given by

$$dF_t = \sigma F_t dW^Q_t$$

(no dt term, and note that volatility remains the same for $Q \sim P$)

We thus have

$$f(t,F) = e^{-r(T-t)} E^Q_{t,F}\left[\Phi(F_T)\right]$$

with Q-dynamics

$$dF_t = \sigma F_t dW^Q_t$$

Now recall from p. 230 the formula for a stock with continuous dividend yield q.

$$f(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right]$$

with Q-dynamics

$$dS_t = (r-q)S_t + \sigma S_t dW^Q_t$$

Note: If we set $q = r$ the formulas are **identical**!

Pricing Formulas

Let $f^0(t, s)$ be the pricing function for the contract $\Phi(S_T)$ for the case when S is a stock without dividends. Let $f(t, F)$ be the pricing formula for the claim $\Phi(F_T)$.

Proposition: With notation as above we have

(book Prop 7.13)

$$f(t, F) = f^0(t, Fe^{-r(T-t)})$$

Moral: Reset today's futures price F to $Fe^{-r(T-t)}$ and use your formulas for stock options.

Compare to p. 222 for dividends

[take there r=0, and then q=r and s=F

Black-76 Formula

The price of a futures option with exercise date T and exercise price K is given by

$$c = e^{-r(T-t)} \left\{ FN\left[d_1\right] - KN\left[d_2\right] \right\}.$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t) \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

(derive from p.46 with $r = q$).
and compare to p.222

end of lecture 5c