Continuous Time Finance

Incomplete Markets

Ch 15

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Derivatives on Non Financial Underlying

Recall: The Black-Scholes theory assumes that the market for the underlying asset has (among other things) the following properties.

- The underlying is a liquidly traded asset.
- Shortselling allowed.
- Portfolios can be carried forward in time.

There exists a large market for derivatives, where the underlying does not satisfy these assumptions.

Examples: (see next page)

- Weather derivatives.
- Derivatives on electric energy.
- CAT-bonds.

Typical Contracts

Weather derivatives:

"Heating degree days". Payoff at maturity T is given by

$$\mathcal{Z} = \max\{X_T - 30, 0\}$$

where X_T is the (mean) temperature at some place.

Electricity option:

The right (but not the obligation) to buy, at time T, at a predetermined price K, a constant flow of energy over a predetermined time interval.

CAT bond:

A bond for which the payment of coupons and nominal value is contingent on some (well specified) natural disaster to take place.

Problems

Weather derivatives:
The temperature is not the price of a traded asset.

Electricity derivatives:
Electric energy cannot easily be stored.

CAT-bonds:
Natural disasters are not traded assets.

We will treat all these problems within a **factor model**.

Typical Factor Model Setup

Given:

- An underlying factor process X, which is **not** the price process of a traded asset, with dynamics under the objective probability measure P as

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t.$$

- A risk free asset with dynamics

$$dB_t = rB_t dt,$$

Problem:

Find arbitrage free price $\Pi_t[\mathcal{Z}]$ of a derivative of the form

$$\mathcal{Z} = \Phi(X_T)$$

Note similarity AND difference with earlier set up.

Concrete Examples

Assume that X_t is the temperature at time t at the village of Peniche (Portugal).

Heating degree days:

$$\Phi(X_T) = 100 \cdot \max\{X_T - 30, 0\}$$

Holiday Insurance:

$$\Phi(X_T) = \begin{cases} 1000, & \text{if } X_T < 20 \\ 0, & \text{if } X_T \geq 20 \end{cases}$$

Question

Is the price $\Pi_t[\Phi]$ uniquely determined by the P-dynamics of X, and the requirement of an arbitrage free derivatives market?

[again a question that should make you suspicious]

NO!!

WHY?

find the difference(s)
between current and
~~pr~~ previous set ups

Stock Price Model $\overset{\text{vs}}{\sim}$ Factor Model

Black-Scholes:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

Factor Model: (similar equation)

$$
\begin{aligned}
dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \\
dB_t &= r B_t dt.
\end{aligned}
$$

(r can be time dependent too!)

What is the difference?

Answer

- X is **not the price of a traded asset!**
- We can not form a portfolio based on X.

1. **Rule of thumb:**

$$\begin{aligned} N &= 0, \quad \text{(no risky asset)} \\ R &= 1, \quad \text{(one source of randomness, } W\text{)} \end{aligned}$$

   We have $N < R$. The exogenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price X-derivatives.

- There is **not** a unique price for a **particular** derivative. typical for incomplete markets

- In order to avoid arbitrage, **different** derivatives have to satisfy **internal consistency** relations.

- If we take **one** "benchmark" derivative as given, then all other derivatives can be priced **in terms of** the market price of the benchmark. "replaces the stock"

We consider two given claims $\Phi(X_T)$ and $\Gamma(X_T)$. We assume they are traded with prices

$$
\begin{aligned}
\Pi_t[\Phi] &= f(t, X_t) \\
\Pi_t[\Gamma] &= g(t, X_t)
\end{aligned}
$$

same underlying factor

Program:

self financing

- Form portfolio based on Φ and Γ. Use Itô on f and g to get portfolio dynamics.

$$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}$$

u^f, u^g relative weights

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot kdt,$$

("synthetic bank" with k as the short rate)

- Absence of arbitrage implies

$$k = r$$

- Read off the relation $k = r$!

end of lecture 7a

Carrying out the program:

Recall $\Pi_t[\Phi] = f(t, X_t)$, assumption

From Itô:

$$df = f\mu_f dt + f\sigma_f dW,$$

where

$$\begin{cases} \mu_f &= \frac{f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}}{f}, \\ \sigma_f &= \frac{\sigma f_x}{f}. \end{cases}$$

(see equation for dX_t)

Portfolio dynamics [use $\Pi_t[\Gamma] = g(t, X_t)$ too]

$$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}.$$

$dg = g\mu_g dt + g\sigma_g dW$

Reshuffling terms gives us

$$dV = V \cdot \left\{u^f \mu_f + u^g \mu_g\right\} dt + V \cdot \left\{u^f \sigma_f + u^g \sigma_g\right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^f + u^g &= 1, \\ u^f \sigma_f + u^g \sigma_g &= 0. \end{cases}$$

kill dW contribution

$$\begin{aligned} u^f &= -\frac{\sigma_g}{\sigma_f - \sigma_g}, \\ u^g &= \frac{\sigma_f}{\sigma_f - \sigma_g}, \end{aligned}$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^f \mu_f + u^g \mu_g \right\} dt. \quad + 0\, dW$$

i.e.

$$dV = V \cdot \left\{ \frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} = r$$

which can be written as

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Repeat:

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Note!
The quotient does **not** depend upon the particular choice of contract. (same for $f, g, \dots$)

Consider this as an internal consistency relation (see p. 300)

This relation should also hold when X is the price of a traded asset. True? Think of this!

Result

Assume that the market for X-derivatives is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

holds for all t and for every choice of contract f.

NB: The same λ for all choices of f.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{“Market Price of Risk” (cf. CAPM).} \\ &= \text{Sharpe Ratio} \end{aligned}$$

different terminology

Slogan:
“On an arbitrage free market all X-derivatives have the same market price of risk.”

The relation

$$\frac{\mu_f - r}{\sigma_f} = \lambda$$

is actually a PDE! : Look at the equations for μ_f and σ_f on p.302

Pricing Equation

note that all these functions depend on t, x:
$\mu = \mu(t,x)$, $f_t = f_t(t,x)$ etc.

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf &= 0 \\ f(T,x) &= \Phi(x), \end{cases}$$

note also that μ (drift of X under P) is present in this equation, not in the Black-Scholes PDE. Why?

P-**dynamics:**

$$dX = \mu(t,X)dt + \sigma(t,X)dW.$$

Can we solve the PDE?

No!!

Why??

Answer

Recall the PDE

$$
\begin{cases}
f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\
f(T,x) & = & \Phi(x),
\end{cases}
$$

- In order to solve the PDE **we need to know λ.**
- λ is not given exogenously.
- λ is not determined endogenously.

looks hopeless, way out?

Question:

Who determines λ?

Answer:

THE MARKET!

Interpreting λ

$\left(\Pi_t[\Phi] = f(t, X_t) \right)$

Recall that the f dynamics are

$$df = f\mu_f dt + f\sigma_f dW_t$$

and λ is defined as

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

- λ **measures the aggregate risk aversion** in the market.
- If λ is big then the market is highly risk averse.
- If λ is zero then the market is **risk neutral**.
- If you make an assumption about λ, then you implicitly make an assumption about the aggregate risk aversion of the market. and v.v. you may learn about λ from market (agents) behaviour, perhaps by analyzing a particular derivative.

Moral

- Since the market is incomplete the requirement of an arbitrage free market will **not** lead to unique prices for X-derivatives.

- Prices on derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free derivative market. **All** pricing functions satisfies the **same** PDE. but with different boundary conditions
  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

end of lecture 7b

Risk Neutral Valuation

We recall the PDE

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf &= 0 \\ f(T,x) &= \Phi(x), \end{cases}$$

Using Feynman-Kac we obtain a risk neutral valuation formula.

Risk Neutral Valuation

$$f(t,x) = e^{-r(T-t)} E^Q_{t,x}\left[\Phi(X_T)\right]$$

Q-**dynamics:** note: these are not constants, see p.293

$$dX_t = \{\mu - \lambda\sigma\}\, dt + \sigma dW_t^Q$$

- Price = expected value of future payments

- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Think of Girsanov. If L_t ($\frac{dQ}{dP}$ on $\mathcal{F}_t$) satisfies $dL_t = L_t \varphi_t dW_t$, then

$$\begin{cases} dW_t = dW_t^Q + \varphi_t\, dt \\ dX_t = \mu\, dt + \sigma\, dW_t \end{cases} \Rightarrow dX_t = (\mu + \sigma\varphi)dt + dW_t^Q$$

so $\varphi = -\lambda$, as on p. 266

(see also pp. 318, 319)

Interpretation of the risk adjusted probabilities, i.e. Q

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.
- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.
- This does **not** mean that we live in, or think that we live in, a risk neutral world.
- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.

Diversification argument about λ

product dependent

- If the risk factor is **idiosyncratic** and **diversifiable**, then one can argue that the factor should not be priced by the market. Compare with APT, arbitrage pricing theory, e.g. CAPM
- Mathematically this means that $\lambda = 0$, i.e. $P = Q$. i.e. **the risk neutral distribution coincides with the objective distribution**.

see p. 314

- We thus have the **"actuarial pricing formula"**

$$f(t, x) = e^{-r(T-t)} E_{t,x}^{P} \left[\Phi(X_T) \right]$$

where we use the objective probabiliy measure P.

Modeling Issues

Temperature:

A standard model is given by (Ornstein-Uhlenbeck process)

$$dX_t = \{m(t) - bX_t\}\, dt + \sigma dW_t,$$

where m is the mean temperature capturing seasonal variations. This often works reasonably well. Here X_t has a normal distribution if X_0 is normal (independent of W)

Electricity:

A (naive) model for the spot electricity price is

$$dS_t = S_t\, \{m(t) - a \ln S_t\}\, dt + \sigma S_t dW_t$$

This implies lognormal prices (why?). Electricty prices are however very far from lognormal, because of "spikes" in the prices. Complicated. log S_t is an OU process

CAT bonds:

Here we have to use the theory of point processes and the theory of extremal statistics to model natural disasters. Complicated.

natural tool!
simple case: Poisson process

Martingale Analysis

(like in p.314)

Model: Under P we have

$$
\begin{aligned}
dX_t &= \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\
dB_t &= r B_t dt,
\end{aligned}
$$

We look for martingale measures. Since B is the only traded asset we need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q martingale.

Result: In this model, **every** $Q \sim P$ is a martingale measure.

Girsanov

$$dL_t = L_t \varphi_t dW_t$$

P-dynamics

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t,$$

$$dL_t = L_t \varphi_t dW_t$$

$dQ = L_t dP$ on $\mathcal{F}_t$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q$$

Martingale pricing: for appropriate T-claim Z

$$F(t,x) = e^{-r(T-t)} E^Q\left[Z \middle| \mathcal{F}_t\right]$$

Q-dynamics of X:

$$dX_t = \left\{\mu\left(t, X_t\right) + \sigma\left(t, X_t\right)\varphi_t\right\} dt + \sigma\left(t, X_t\right) dW_t^Q,$$

Result: We have $\lambda_t = -\varphi_t$, i.e,. the Girsanov kernel φ equals minus the market price of risk.

Several Risk Factors

We recall the dynamics of the f-derivative

$$df = f\mu_f dt + f\sigma_f dW_t$$

also
$df = f(r + \lambda\sigma_f)dt$
$+ \sigma_f dW$

and the Market Price of Risk

$$\frac{\mu_f - r}{\sigma_f} = \lambda, \quad \text{i.e.} \quad \mu_f - r = \lambda\sigma_f.$$

In a multifactor model of the type

$$dX_t = \mu(t, X_t)\, dt + \sum_{i=1}^{n} \sigma_i(t, X_t)\, dW_t^i,$$

it follows from Girsanov that for every risk factor W^i there will exist a market price of risk $\lambda_i = -\varphi_i$ such that

$$\mu_f - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Compare with CAPM. (If you know what that is)

end of lecture 7c