Back to finance:

2. Investment Theory

(Section 19.6)

- Problem formulation.
- An extension of HJB.
- The simplest consumption-investment problem.
- The Merton fund separation results.

Recap of Basic Facts

We consider a market with n assets.

$$
\begin{aligned}
S_t^i &= \text{price of asset No } i, \\
h_t^i &= \text{units of asset No } i \text{ in portfolio} \\
w_t^i &= \text{portfolio weight on asset No } i \quad \text{[previously also } u_t^i\text{]} \\
X_t &= \text{portfolio value} \quad \text{[previously denoted } V_t\text{]} \\
c_t &= \text{consumption rate} \quad \geq 0
\end{aligned}
$$

We have the relations

$$
X_t = \sum_{i=1}^{n} h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=1}^{n} w_t^i = 1.
$$

Basic equation:
Dynamics of self financing portfolio in terms of relative weights

$$
dX_t = X_t \sum_{i=1}^{n} w_t^i \frac{dS_t^i}{S_t^i} - c_t dt
$$

(dual to dividends, see pp. 212, 214, now with a "minus term")

Simplest model

Assume a scalar risky asset and a constant short rate.

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t \\
dB_t &= r B_t dt
\end{aligned}
$$

We want to maximize expected utility of consumption over time

utility function $F(t;)$

$$
\max_{w^0, w^1, c} \quad E\left[\int_0^T F(t, c_t) dt\right]
$$

[also a term $\Phi(T, c_T)$ can be included, or $\Phi(T, X_T)$]

Dynamics

$$
dX_t = X_t \left[w_t^0 r + w_t^1 \alpha\right] dt - c_t dt + w_t^1 \sigma X_t dW_t,
$$

Constraints

$$
\begin{aligned}
c_t &\geq 0, \ \forall t \geq 0, \\
w_t^0 + w_t^1 &= 1, \ \forall t \geq 0.
\end{aligned}
$$

Sensible problem (formulation)?

.... become suspicious ....

Nonsense!

What are the problems?

- We can obtain unlimited utility by simply consuming arbitrary large amounts.
- The wealth will go negative, but there is nothing in the problem formulations which prohibits this.
- We would like to impose a constratin of type $X_t \geq 0$ but this is a **state constraint** and DynP does not allow this. (See p. 324)

Good News:
DynP can be generalized to handle (some) problems of this kind.

The use of stopping times helps!

Generalized problem

Let D be a nice open subset of $[0, T] \times R^n$ and consider the following problem.

note τ !

$$\max_{u \in U} \quad E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi\left(\tau, X_\tau^{\mathbf{u}}\right)\right].$$

Dynamics:

$$\begin{aligned} dX_t &= \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t, \\ X_0 &= x_0, \end{aligned}$$

(as before)

The **stopping time** τ is defined by

$$\tau = \inf\left\{t \geq 0 \left|(t, X_t) \in \partial D\right.\right\} \wedge T. \quad \leq T \quad !$$

a random time!

boundary of D

So, the problem looks as before, but with the difference that the horizon is random!

Generalized HJB

Theorem: Given enough regularity the follwing hold.

as before

boundary condition on ∂D, not just at $t=T$

1. The optimal value function satisfies

$$
\begin{cases}
\dfrac{\partial V}{\partial t}(t,x) + \sup\limits_{u \in U}\left\{F(t,x,u) + \mathcal{A}^u V(t,x)\right\} = 0, & \forall (t,x) \in D \\
V(t,x) = \Phi(t,x), & \forall (t,x) \in \partial D.
\end{cases}
$$

2. We have an obvious verification theorem. replace e.g.

$H(T,x)$ on p.340 by $H(t,x) = \Phi(t,x)$, $\forall (t,x) \in \partial D$ $\boxtimes$

Reformulated problem

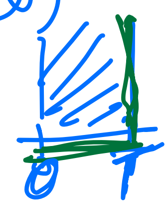
or c_T?

$$\max_{c \geq 0,\ w \in R} \quad E\left[\int_0^{\tau} F(t, c_t)dt + \Phi(X_T)\right]$$

open set in $[0,\infty)\times R$

The "ruin time" τ is defined by

corresponds to $D=[0,T)\times(0,\infty)$



$$\tau = \inf\{t \geq 0 \,|X_t = 0\} \wedge T.$$

$t \leq \tau \Rightarrow X_t \geq 0$!

and $\partial D = \{(t,x)\in D :$
$x = 0$ or $t = T\}$

Notation:

$$\begin{aligned} w^1 &= w, \\ w^0 &= 1 - w \end{aligned}$$

Thus no constraint on w.

Dynamics Of simple model on p.350 become

$$dX_t = w_t\left[\alpha - r\right]X_t dt + \left(rX_t - c_t\right)dt + w\sigma X_t dW_t,$$

for optimal $\hat{c}, \hat{w}$: $V_t + F(t, \hat{c}) + \hat{w}\, x(\alpha - r)\frac{\partial V}{\partial x}(\cdot\cdot) + \dots = 0$ (*)

HJB Equation

$$
\begin{aligned}
\frac{\partial V}{\partial t} + \sup_{c \geq 0, w \in R} \left\{ F(t,c) + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2} \right\} &= 0, \\
V(T, x) &= 0, \\
V(t, 0) &= 0.
\end{aligned}
$$

$\Phi = 0 \Rightarrow$

We now specialize (why?) to

$$F(t,c) = e^{-\delta t} c^{\gamma},$$

with $0 < \gamma < 1 \Rightarrow$ $F_c(t,0) = +\infty \Rightarrow$ optimal $c > 0$.

and for simplicity we assume that

$$\Phi = 0,$$

so we have to maximize

$$e^{-\delta t} c^{\gamma} + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

w.r.t. $c \geq 0$ and $w \in R$

Analysis of the HJB Equation

In the embedded static problem we maximize, over c and w, (repeat from p.356)

$$e^{-\delta t}c^{\gamma} + wx(\alpha - r)V_x + (rx - c)V_x + \frac{1}{2}x^2w^2\sigma^2V_{xx},$$

First order conditions:

$$\begin{aligned}
(1) \qquad \gamma c^{\gamma-1} &= e^{\delta t}V_x, && \left(\text{from } \frac{\partial \dots}{\partial c} = 0\right) \\
(2) \qquad w &= \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, && \left(\text{from } \frac{\partial \dots}{\partial w} = 0\right)
\end{aligned}$$

Ansatz:

$$V(t, x) = e^{-\delta t}h(t)x^{\gamma}, \qquad \left(\text{like } F(t, c)\right)$$

Because of the boundary conditions, we must demand that $\Phi = 0$

$$h(T) = 0. \tag{5}$$

alternatively, you can try $V(t,x) = k(t)x^{\gamma}$

Given a V of this form we have (using $\cdot$ to denote the time derivative)

$$
\begin{aligned}
V_t &= e^{-\delta t}\dot{h}x^{\gamma} - \delta e^{-\delta t}hx^{\gamma}, \qquad (h = h(t)) \\
V_x &= \gamma e^{-\delta t}hx^{\gamma-1}, \\
V_{xx} &= \gamma(\gamma-1)e^{-\delta t}hx^{\gamma-2}.
\end{aligned}
$$

giving us

$$
\begin{aligned}
\text{use p.357, (2): } \widehat{w}(t,x) &= \frac{\alpha - r}{\sigma^2(1-\gamma)}, \quad (\text{constant!}) \\
\text{use p.357, (1): } \widehat{c}(t,x) &= xh(t)^{-1/(1-\gamma)}. \quad (\text{linear in } x)
\end{aligned}
$$

Plug all this into HJB! and try to solve

(*) on ~~top~~ of p. 356, or the one on the bottom with $\hat{w}$ and $\hat{c}$ etc.

tedious work, but you can check!

After rearrangements we obtain

$$x^{\gamma}\left\{\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)}\right\}=0,$$

where the constants A and B are given by

tedious computations,

$$\begin{aligned}A &= \frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}+r\gamma-\frac{1}{2}\frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)}-\delta = \frac{\gamma(\alpha-r)^2}{2\sigma^2(1-\gamma)}+r\gamma-\delta\\ B &= 1-\gamma.\end{aligned}$$

If this equation is to hold for all x and all t, then we see that h must solve the ODE

$$\begin{aligned}\dot{h}(t)+Ah(t)+Bh(t)^{-\gamma/(1-\gamma)} &= 0,\\ h(T) &= 0.\end{aligned}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly. See Exercises 19.2, 19.3

We are done.

end of lecture 9a

Merton's Mutal Fund Theorems

Section 19.7

1. The case with no risk free asset

We consider n risky assets with dynamics

$$dS_i = S_i \alpha_i dt + S_i \sigma_i dW, \quad i = 1, \ldots, n \quad , \sigma_i \in \mathbb{R}^{1 \times k}$$

where W is Wiener in R^k. ~~O~~In vector form:

$$dS = D(S)\alpha dt + D(S)\sigma dW.$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \text{ and } \sigma = \begin{bmatrix} - \sigma_1 - \\ \vdots \\ - \sigma_n - \end{bmatrix} \in \mathbb{R}^{n \times k}$$

$D(S)$ is the diagonal matrix

$$D(S) = diag[S_1, \ldots, S_n]. \quad \in \mathbb{R}^n$$

Formal problem

$$\max_{c,w} \quad E\left[\int_0^T F(t, c_t)dt\right]$$

given the dynamics (use the ST condition on p.349)

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW,$$

additional to the dS_t^i equations

and constraints

$$\sum_{i=1}^{n} w_t^i = e'w_t = 1, \quad c_t \geq 0. \qquad w_t = (w_t^1, \ldots, w_t^n)'$$

Assumptions:

- The vector α and the matrix σ are constant and deterministic.
- The volatility matrix σ has full row rank so $\sigma\sigma'$ is positive definite and invertible. $\Rightarrow$ arbitrage free and complete market if $n=k$

Note: S does not turn up in the X-dynamics so V is of the form

$$\underbrace{V(t,x,s)}_{\text{would result from}} = V(t,x)$$

$d\begin{pmatrix} X_t \\ S_t \end{pmatrix} = \ldots dt + \ldots dW_t$

combined state vector

see p. 328 for $\mathcal{A}$

The HJB equation is

$$
\begin{cases}
V_t(t,x) + \sup_{e'w=1,\ c\geq 0} \left\{ F(t,c) + \mathcal{A}^{c,w} V(t,x) \right\} &= 0, \\
V(T,x) &= 0, \\
V(t,0) &= 0.
\end{cases}
$$

boundary conditions as on p. 356

where

$$
\mathcal{A}^{c,w} V = x w' \alpha V_x - c V_x + \frac{1}{2} x^2 w' \Sigma w \, V_{xx},
$$

The matrix Σ is given by

$$
\Sigma = \sigma \sigma'.
$$

The HJB equation ~~is~~ then becomes

$$
\left\{
\begin{array}{rcl}
V_t + \displaystyle\sup_{w'e=1,\ c\geq 0} \left\{ F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} \right\} & = & 0, \\
V(T,x) & = & 0, \\
V(t,0) & = & 0.
\end{array}
\right.
$$

where $\Sigma = \sigma\sigma'$.

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

standard technique for optimization under linear constraints

$$
L = F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\,(1 - w'e)\,.
$$

$$= L(c, w, \lambda)$$

Repeat:

$$L = F(t,c) + (xw'\alpha - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx} + \lambda\,(1 - w'e)\,.$$

The first order condition for c is

$$F_c = V_x.$$

row vector

(recall $\nabla(w'\Sigma w) = 2w'\Sigma$)

The first order condition for w is

$$x\alpha' V_x + x^2 V_{xx} w'\Sigma = \lambda e',$$ (row vector)

so we can solve for w in order to obtain

$$\hat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}} e - \frac{xV_x}{x^2 V_{xx}}\alpha\right].$$ (column vector)

Using the relation $e'w = 1$ this gives λ as

$$\lambda = \frac{x^2 V_{xx} + xV_x e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},$$

$$1 = e'\hat{w} = \lambda \frac{e'\Sigma^{-1}e}{x^2 V_{xx}} - \frac{\cancel{x} V_x e'\Sigma^{-1}\alpha}{x^{\cancel{2}} V_{xx}}$$

in $\hat{w} = \Sigma^{-1}[\cdots]$

Inserting λ gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \frac{V_x}{xV_{xx}}\Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right].$$

We can write this as

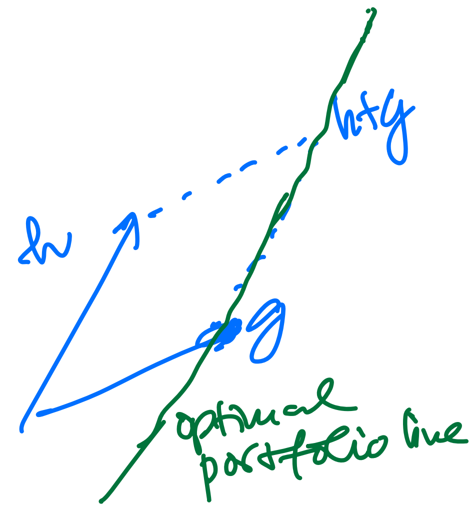
$$\hat{w}(t) = g + Y(t)h,$$

where the fixed vectors g and h are given by

$$\begin{aligned} g &= \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \\ h &= \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \end{aligned}$$

whereas Y is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}.$$



We had

$$\hat{w}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" Δ, where

$$\Delta = \{w \in R^n \, |e'w = 1\}.$$

If we fix two points on the optimal portfolio line, say $w^a = g + ah$ and $w^b = g + bh$, then any point w on the line can be written as an affine combination of the basis points w^a and w^b. An easy calculation shows that if $w^s = g + sh$ then we can write

$$w^s = \mu w^a + (1-\mu) w^b,$$

where

$$\mu = \frac{s-b}{a-b}.$$

Summary:

Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed s the portfolio w^s stays fixed over time.

2. For fixed a, b with $a \neq b$ the optimal portfolio $\hat{\mathbf{w}}(t)$ is, obtained by allocating all resources between the fixed funds w^a and w^b, i.e.

$$\hat{w}(t) = \mu^a(t)w^a + \mu^b(t)w^b,$$

$$\mu^a(t) = \frac{Y(t)-b}{b-a}, \quad \mu^b(t) = 1 - \mu^a(t)$$

(note $\mu^a(t) + \mu^b(t) = 1$)

The case with a risk free asset

Again we consider the standard model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t),$$

We also assume the risk free asset B with dynamics

$$dB = rBdt.$$

extra term

We denote $B = S_0$ and consider portfolio weights $(w_0, w_1, \ldots, w_n)'$ where $\sum_0^n w_i = 1$. We then eliminate w_0 by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

(method could also be used in previous case)

and use the letter w to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

Note: $w \in R^n$ without constraints. (no "Laplace" needed, as w is <u>not</u> constrained)

HJB

We obtain (again from the SF condition)

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW,$$

where $e = (1, 1, \ldots, 1)'$. (note: $w'e \neq 1$ in general here)

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \{F(t,c) + \mathcal{A}^{c,w}V(t,x)\} &= 0, \\ V(T,x) &= 0, \\ V(t,0) &= 0, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}^c V &= xw'(\alpha - re)V_x(t,x) + (rx - c)V_x(t,x) \\ &+ \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x). \end{aligned}$$

again $\Sigma = \sigma\sigma'$

First order conditions

We maximize

$$F(t,c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2 w' \Sigma w V_{xx}$$

with $c \geq 0$ and $w \in R^n$.

The first order conditions are (parallel to p.364)

$$\begin{aligned} F_c &= V_x, \\ \hat{w} &= -\frac{V_x}{xV_{xx}} \underbrace{\Sigma^{-1}(\alpha - re)}_{w_t \in R^n}, \end{aligned}$$

, only risky weights

with geometrically obvious economic interpretation,

like on p.366

definition on p.371

Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds w^0 and w^f.

2. The fund w^0 consists only of the risk free asset.

3. The fund w^f consists only of the risky assets, and is given by
$$w^f = \Sigma^{-1}(\alpha - re).$$

and relative allocations of wealth are $u^f = -\frac{V_x}{xV_{xx}}$ (everything depending on $t, X(t)$)

$u_0 = 1 - u^f$

More (alternative) theory

Continuous Time Finance

The Martingale Approach to Optimal Investment Theory

Ch 20

Tomas Björk

essential ingredient is completeness of the market

Contents

- $\boxed{\text{Decoupling}}$ the wealth profile from the portfolio choice.
- Lagrange relaxation. (seen before)
- Solving the general wealth problem.
- Example: Log utility.
- Example: The numeraire portfolio.

Problem Formulation

Standard model with internal filtration

$$
\begin{aligned}
dS_t &= D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \\
dB_t &= rB_t dt.
\end{aligned}
$$

Assumptions:

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is **complete**.
- We have a given initial wealth x_0

Problem:

$$
\max_{h \in \mathcal{H}} \quad E^P\left[\Phi(X_T)\right]
$$

(terminal wealth)

where

$$
\mathcal{H} = \{\text{self financing portfolios}\}
$$

given the initial wealth $X_0 = x_0$.

Some observations

- In a complete market, there is a unique martingale measure Q.
- Every claim Z satisfying the budget constraint

$$e^{-rT}E^Q[Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

$h_T' S_T = Z$ and $x_0 = h_0' S_0 = e^{-rT} E^Q h_T' S_T$,

since $E^Q\left[\frac{S_T}{e^{rT}} \middle| \mathcal{F}_t\right] = \frac{S_t}{e^{rt}}$, also for $t=0$!

- We can thus write our problem as

$$\max_Z \quad E^P[\Phi(Z)]$$

subject to the constraint

$$e^{-rT}E^Q[Z] = x_0.$$

- We can forget the wealth dynamics! (for the time being, see step 2 below)

Basic Ideas

Our problem was

$$\max_{Z} \quad E^P[\Phi(Z)]$$

subject to

$$e^{-rT} E^Q[Z] = x_0.$$

Idea I:

We can **decouple** the optimal portfolio problem into:

1. Finding the optimal wealth profile $\hat{Z}$.
2. Given $\hat{Z}$, find the replicating portfolio.

Idea II:

- Rewrite the constraint under the measure P.
- Use Lagrangian techniques to relax the constraint.

end of lecture 9b

Lagrange formulation

Recall

Problem:

$$\max_{Z} \quad E^P[\Phi(Z)]$$

$Z \in \mathcal{F}_T$

subject to

$$e^{-rT}E^P[L_T Z] = x_0.$$

(constraint in terms of measure P)

Here L is the likelihood process, i.e.

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t, \quad 0 \le t \le T$$

Recall $E^Q Z = E^P[L_T Z]$

The Lagrangian of the problem is

$$\mathcal{L} = E^P[\Phi(Z)] + \lambda\left\{x_0 - e^{-rT}E^P[L_T Z]\right\}$$

i.e.

$$\mathcal{L} = E^P\left[\Phi(Z) - \lambda e^{-rT}L_T Z\right] + \lambda x_0$$

note: both expectations under P!

The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable λ, to find the optimal Z by maximizing

$$\mathcal{L} = E^P\left[\Phi(Z) - \lambda e^{-rT} L_T Z\right] + \lambda x_0$$

over unconstrained Z, i.e. to maximize

$$\int_\Omega \left\{\Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega)\right\} dP(\omega)$$

This is a trivial problem! (if you look at it the right way)

We can simply maximize $Z(\omega)$ for each ω separately.

$$\max_z \quad \left\{\Phi(z) - \lambda e^{-rT} L_T z\right\} \qquad (L_T = L_T(\omega))$$

under the integral

The optimal wealth profile

Our problem: (repeat from previous slide)

$$\max_{z} \quad \left\{\Phi(z) - \lambda e^{-rT} L_T z\right\}$$

First order condition

$$\Phi'(z) = \lambda e^{-rT} L_T$$

The optimal Z is thus given by

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right)$$

$\hat{z}$ depends on λ!

where

$$G(y) = \left[\Phi'\right]^{-1}(y).$$

[if Φ' has inverse with y in its domain]

The dual variable λ is determined by the constraint

implicitly

$$e^{-rT} E^P\left[L_T \hat{Z}\right] = x_0.$$

You have to solve λ from this equation ($\hat{z} = \hat{z}(\lambda)$) and hope that a (unique) solution exists

(hope $\to$ prove)

Example – log utility

Assume that

$$\Phi(x) = \ln(x) \quad , \; \Phi'(x) = \frac{1}{x} \Rightarrow$$

Then inverse of Φ' is

$$g(y) = \frac{1}{y} \quad , \; \text{for all } y > 0$$

Thus

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right) = \frac{1}{\lambda} e^{rT} L_T^{-1}$$

Finally λ is determined by

$$e^{-rT} E^P \left[L_T \hat{Z} \right] = x_0.$$

i.e.

$$e^{-rT} E^P \left[L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0.$$

so $\lambda = x_0^{-1}$ and

$$\hat{Z} = x_0 e^{rT} L_T^{-1}$$

(to be interpreted as optimal wealth at time T)

The optimal wealth process

- We have computed the optimal **terminal** wealth profile

$$\widehat{Z} = \widehat{X}_T = x_0 e^{rT} L_T^{-1} \qquad (1)$$

- What does the optimal wealth **process** $\widehat{X}_t$ look like?

We have (why?) (discounted traded assets are Q-martingales)

$$\widehat{X}_t = e^{-r(T-t)} E^Q \left[\widehat{X}_T \middle| \mathcal{F}_t \right] \qquad (2)$$

so we obtain from (1) and (2):

$$\widehat{X}_t = x_0 e^{rt} E^Q \left[L_T^{-1} \middle| \mathcal{F}_t \right]$$

abstract theory, $L_T^{-1} = dP/dQ$ on $\mathcal{F}_T$

But L^{-1} is a Q-martingale (why?) so we obtain

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

The Optimal Portfolio

- We have computed the optimal wealth process: $\hat{X}_t$
- How do we compute the optimal portfolio?

Assume for simplicity that we have a standard Black-Scholes model (complete!)

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \\
dB_t &= r B_t dt
\end{aligned}
$$

Recall that

$$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

L_t^{-1} satisfies (Girsanov theory) an equation like $dL_t^{-1} = L_t^{-1} \psi_t dW_t^Q$ for some ψ_t

$= L_t^{-1} (\dots) dt + L_t^{-1} \psi_t dW_t$

Basic Program

$Z_t = z_0 e^{rt} L_t^{-1}$, you "know" L_t

1. Use Ito and the formula for $\widehat{X}_t$ to compute $d\widehat{X}_t$ like

$$d\widehat{X}_t = \widehat{X}_t(\quad)dt + \widehat{X}_t\beta_t dW_t$$

(find β_t later)

where we do not care about ( ).

2. Recall that (for some $\hat{u}_t$, portfolio weight)

involves dW_t

$$d\widehat{X}_t = \widehat{X}_t\left\{(1-\hat{u}_t)\frac{dB_t}{B_t} + \hat{u}_t\frac{dS_t}{S_t}\right\}$$

which we write as

$$d\widehat{X}_t = \widehat{X}_t\{\quad\}dt + \widehat{X}_t\hat{u}_t\sigma dW_t$$

3. We can identify $\hat{u}$ as

$$\hat{u}_t = \frac{\beta_t}{\sigma}$$

ψ_t on p. 382 in thm
$\psi_t = -\varphi$

We recall

(1) $$\widehat{X}_t = x_0 e^{rt} L_t^{-1}.$$

We also recall that

$\left(L_t \text{ is } \frac{dQ}{dP} \text{ on } \mathcal{F}_t\right)$ $$dL_t = L_t \varphi dW_t,$$

where

$$\varphi = \frac{r - \mu}{\sigma}$$

(many lectures ago)

From this we have (Itô for L_t^{-1})

(2) $$dL_t^{-1} = \varphi^2 L_t^{-1} dt - L_t^{-1} \varphi dW_t = -\varphi L_t^{-1} dW_t$$

and we obtain from (1) and (2):

$$d\widehat{X}_t = \widehat{X}_t \{ \quad \} dt - \widehat{X}_t \varphi dW_t \rightarrow \beta_t = -\varphi$$

Result: The optimal portfolio is given by $\hat{u}_t = \frac{\beta_t}{\sigma}$

$$\Rightarrow \quad \hat{u}_t = \frac{\mu - r}{\sigma^2}$$

Note that $\hat{u}$ is a "myopic" portfolio in the sense that it does not depend on the time horizon T.

A Digression: The Numeraire Portfolio

Standard approach:

- Choose a fixed numeraire (portfolio) N.
- Find the corresponding martingale measure, i.e. find Q^N s.t.

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are Q^N -martingales.

some Q

Alternative approach:

- Choose a fixed measure $Q \sim P$.
- Find numeraire N such that $Q = Q^N$:

$\frac{X_t}{N_t}$ is Q^N-martingale if X_t is value of traded asset

Special case:

- Set $Q = P$
- Find numeraire N such that $Q^N = P$ i.e. such that

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are $\overset{P}{\underset{\shortparallel}{}}Q^N$-martingales under the **objective** measure P.
- This N is called the **numeraire portfolio**.

Log utility and the numeraire portfolio

Definition:
The **growth optimal portfolio** (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date T.

wealth process is $\hat{X}_t = x_0 e^{rt} \hat{L}_t$ (p. 381)

Theorem:
Assume that X is GOP. Then X is the numeraire portfolio.

Proof:
We have to show that the process

$$Y_t = \frac{S_t}{X_t}$$

is a P martingale. (and also $\frac{B_t}{X_t} = x_0^{-1} L_t$)

We have

$$\frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t$$

likelihood ratio $\frac{dQ}{dP}$ on $\mathcal{F}_t$

which is a P martingale, since $x_0^{-1} e^{-rt} S_t$ is a Q martingale. use "Bayes" (Additional exercise 3 = exercise C.9 in the book)

end of lecture 9c.

and this is also the
end of the course

Thank you for your attention

and I hope it will
be useful for you!