# Proof of the Verification theorem

Look at Slide 340 for the formulation of the theorem, or at Theorem 19.6 in Björk's book (3rd Edition).

**Proof** We will show that the solution $H$ of the HJB equation is in fact the optimal value function $V$, and, parallel to that, that the optimal control $\hat{\mathbf{u}}$ is given by $\hat{\mathbf{u}}(t,x) = g(t,x)$.

Consider an arbitrary control law $\mathbf{u}$ and recall the dynamics of the process $X^{\mathbf{u}}$, in abbreviated notation,

$$\mathrm{d}X^{\mathbf{u}} = \mu^{\mathbf{u}}\ \mathrm{d}t + \sigma^{\mathbf{u}}\ \mathrm{d}W.$$

Apply the Itô formula to $H(t, X^{\mathbf{u}}_t)$ to get, again in abbreviated notation and with $H_t$ and $H_x$ the first order partial derivatives,

$$H(T, X^{\mathbf{u}}_T) = H(t,x) + \int_t^T (H_t + \mathcal{A}^{\mathbf{u}} H)\ \mathrm{d}s + \int_t^T H_x \sigma^{\mathbf{u}}\ \mathrm{d}W. \tag{1}$$

Since $H$ solves the HJB equation, one has the equality

$$H_t(t,x) + \sup_u \left( F(t,x,u) + \mathcal{A}^u H(t,x) \right) = 0.$$

Hence, for any $u$ one gets

$$H_t(t,x) + F(t,x,u) + \mathcal{A}^u H(t,x) \leq 0,$$

which also implies (note the alternative notation, see the slides)

$$H_t(t, X^{\mathbf{u}}_t) + F^{\mathbf{u}}(t, X^{\mathbf{u}}_t) + \mathcal{A}^{\mathbf{u}} H(t, X^{\mathbf{u}}_t) \leq 0.$$

Use this inequality in (1) to get, now written in full detail

$$H(T, X^{\mathbf{u}}_T) \leq H(t,x) - \int_t^T F^{\mathbf{u}}(s, X^{\mathbf{u}}_s)\ \mathrm{d}s + \int_t^T H_x(s, X^{\mathbf{u}}_s) \sigma^{\mathbf{u}}(s, X^{\mathbf{u}}_s)\ \mathrm{d}W.$$

Recall the boundary condition for $H$, $H(T,x) = \Phi(x)$, to rewrite this inequality as

$$\Phi(X^{\mathbf{u}}_T) \leq H(t,x) - \int_t^T F^{\mathbf{u}}(s, X^{\mathbf{u}}_s)\ \mathrm{d}s + \int_t^T H_x(s, X^{\mathbf{u}}_s) \sigma^{\mathbf{u}}(s, X^{\mathbf{u}}_s)\ \mathrm{d}W.$$

In the next step we take expectations in the above inequality. Under sufficient technical conditions, the expectation of the Itô integral will vanish, and we end up with

$$\mathbb{E}\Phi(X^{\mathbf{u}}_T) \leq H(t,x) - \mathbb{E} \int_t^T F^{\mathbf{u}}(s, X^{\mathbf{u}}_s)\ \mathrm{d}s,$$

which we rewrite as

$$H(t,x) \geq \mathbb{E}\Phi(X_T^{\mathbf{u}}) + \mathbb{E}\int_t^T F^{\mathbf{u}}(s, X_s^{\mathbf{u}})\ \mathrm{d}s.$$

The right hand side of the latter inequality we recognize, see slide 329, as the value function $\mathcal{J}(t,x,\mathbf{u})$. So, we obtain

$$H(t,x) \geq \mathcal{J}(t,x,\mathbf{u}).$$

This inequality is valid for any control law $\mathbf{u}$. Hence, one also has

$$H(t,x) \geq \sup_{\mathbf{u}} \mathcal{J}(t,x,\mathbf{u}).$$

But, here the right hand side is by definition the optimal value function $V$ at $(t,x)$, see slide 329 again. So we conclude

$$H(t,x) \geq V(t,x).$$

Recall our aim, showing that $H(t,x) = V(t,x)$. We accomplish this by showing the companion inequality $H(t,x) \leq V(t,x)$. This will be done as follows.

Choose the function $g$ as given by the statement of the Verification theorem, i.e., this function attains the supremum in $H_t(t,x) + \sup_u (F(t,x,u) + \mathcal{A}^u H(t,x))$, so

$$H_t(t,x) + F(t,x,g(t,x)) + \mathcal{A}^g H(t,x) = 0.$$

This relation we substitute in (1) to get (with the $\mathbf{u}$ now replaced with $\mathbf{g}$)

$$H(T, X_T^{\mathbf{g}}) = H(t,x) - \int_t^T F^{\mathbf{g}}(s, X_s^{\mathbf{g}})\ \mathrm{d}s + \int_t^T H_x(s, X_s^{\mathbf{g}})\sigma^{\mathbf{g}}(s, X_s^{\mathbf{g}})\ \mathrm{d}W.$$

Taking expections, one now gets by similar reasoning as above,

$$\mathbb{E}\Phi(X_T^{\mathbf{g}}) = H(t,x) - \mathbb{E}\int_t^T F^{\mathbf{g}}(s, X_s^{\mathbf{g}})\ \mathrm{d}s,$$

rearranged to

$$H(t,x) = \mathbb{E}\int_t^T F^{\mathbf{g}}(s, X_s^{\mathbf{g}})\ \mathrm{d}s + \mathbb{E}\Phi(X_T^{\mathbf{g}}),$$

where we recognize the right hand side as $\mathcal{J}(t,x,\mathbf{g})$. By the optimality of the function $V$, one trivially has $\mathcal{J}(t,x,\mathbf{g}) \leq V(t,x)$, and it follows that

$$H(t,x) \leq V(t,x),$$

as desired. We conclude that $H(t,x) = V(t,x)$.

Moreover, we now also obtain the equality $V(t,x) = \mathcal{J}(t,x,\mathbf{g})$, which shows the optimality of the control law $\mathbf{g}$. $\square$