Stochastic Finance in Continuous Time

Additional exercises based on notes by José Cruz

Exercises

1 Solve the Black-Scholes PDE for the pricing function F for a claim Φ and show that the solution coincides with the Feynman-Kac formula. Do this at the hand of the following steps.

(a) Define a new function u by $u(t,x) = F(t, e^{x+(r-\frac{1}{2}\sigma^2)t})$ and show that u satisfies the *backward heat equation*

$$u_t + \frac{1}{2}\sigma^2 u_{xx} = 0.$$

Express the boundary condition on $u(T,x)$ in terms of Φ.

(b) Let $p_{t,x}(\cdot)$ be the density of a normal $N(x, T-t)$ distribution for $0 \leq t < T$. Show that p also satisfies the backward heat equation.

(c) Put $u(t,x) = \int_{-\infty}^{\infty} u(T,y)p_{t,x}(y)\,\mathrm{d}y$. Show that this u solves the backward heat equation.

(d) Write $u(t,x)$ as the expectation of a certain function (in terms of Φ) of a random variable with an appropriate normal distribution.

(e) Recognize the Feynman-Kac formula by recalling that in the Black-Scholes model $\log S(T) - \log S(t)$ has a normal $N((r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$ distribution under the measure $\mathbb{Q}$.

2 Let W be a Wiener process under a probability measure $\mathbb{P}$. Consider a market consisting of two assets with prices $S^1(t)$ and $S^2(t)$. Consider also a bank account $B(t)$ with constant short rate r. Consider then the following model.

$$\begin{aligned}
\mathrm{d}S^1(t) &= \alpha_1 S^1(t)\,\mathrm{d}t + \sigma_1 S^1(t)\,\mathrm{d}W(t)\\
\mathrm{d}S^2(t) &= \alpha_2 S^2(t)\,\mathrm{d}t + \sigma_2 S^2(t)\,\mathrm{d}W(t)\\
\mathrm{d}B(t) &= rB(t)\,\mathrm{d}t.
\end{aligned}$$

(a) Discuss informally absence of arbitrage and completeness in the market in this model using the Meta theorem.

(b) Use the martingale approach to provide precise conditions for absence of arbitrage and completeness in the market.

3 Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and a measure $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$, for some $T > 0$. Let $L_t, 0 \leq t \leq T$ denote the likelihood process. Also let X be an adapted process. Prove (for $0 \le t \le T$) that X is a $\mathbb{Q}$-martingale if and only if LX is a $\mathbb{P}$-martingale. [This is Exercise C.9.]

4 Let D be a stochastic discount factor for a financial market. Prove that the process $\Pi_t D_t, (t \ge 0)$ is a $\mathbb{P}$-martingale for every price process $\Pi_t, (t \ge 0)$.

5 Consider the SDE

$$\mathrm{d}X_t = \alpha f(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t,\ X_0 = x_0.$$

We assume that f, σ are known functions and α is an unknown parameter. Also the SDE possesses a unique solution for every fixed choice of α. Construct a dynamical statistical model and compute the maximum likelihood estimator for α based upon observations of X_t. [This is also Exercise 11.2.]

6 Consider a financial market with arbitrary number of assets and a bank account with constant short rate r. Consider the following price process S_t with dynamics under the measure $\mathbb{P}$,

$$\mathrm{d}S_t = \alpha_t S_t\,\mathrm{d}t + \sigma_t S_t\,\mathrm{d}W_t.$$

We also consider the T-claim Y of the form $Y = S_T X$ where X is some random variable such that $X \in \mathcal{F}_T$. As usual we consider the standard risk-neutral formula $\Pi_t[Y] = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[S_T X]$.

(a) Prove that $E^{\mathbb{Q}} e^{-rT}\frac{S_T}{S_0} = 1$.

(b) Define the random variable L_T by $L_T = e^{-rT}\frac{S_T}{S_0}$. Since L_T and $E^{\mathbb{Q}}[L_T] = 1$, we can use L_T as a Radon-Nikodym derivative and define a new measure by $\mathrm{d}\mathbb{Q}^S = L_T \mathrm{d}\mathbb{Q}$ on $\mathcal{F}_T$. Write down a pricing formula $\Pi_0[Y]$ using the new measure $\mathbb{Q}^S$.

(c) Define the process L by L_t equal to $\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}$ on $\mathcal{F}_t$. Derive a formula for L_t and prove that L has $\mathbb{Q}$-dynamics given by $\mathrm{d}L_t = \sigma_t L_t\,\mathrm{d}W_t^{\mathbb{Q}}$.

(d) Write down a pricing formula $\Pi_t[Y]$ using the new measure $\mathbb{Q}^S$.

7 Derive the relevant PDE when X is a non traded asset,

$$F_t + (\mu - \lambda\sigma)F_x + \frac{1}{2}\sigma^2 F_{xx} - rF = 0.$$

8 Consider the Black-Scholes model and compute the price of an asset-or-nothing option where the payoff is given by $X = S(T)\mathbf{1}_{\{S(T)\leq K\}}$ by using the asset price process S as a numéraire.

9 Let the stock prices $S^1(t), S^2(t)$ and the bank account $B(t)$ have the following dynamics

$$\begin{aligned} \mathrm{d}S^1(t) &= \alpha_1 S^1(t)\,\mathrm{d}t + \sigma_1 S^1(t)\,\mathrm{d}W^1(t) \\ \mathrm{d}S^2(t) &= \alpha_2 S^2(t)\,\mathrm{d}t + \sigma_2 S^2(t)\,\mathrm{d}W^2(t) \\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t, \end{aligned}$$

where the Wiener processes W^1 and W^2 are assumed to be independent. Compute the price of the T-claim X defined by $X = S^1(T)\mathbf{1}_{\{S^2(T)\leq K\}}$. You may use S^1 a numéraire.

10 Let the stock prices $S^1(t), S^2(t)$ and the bank account $B(t)$ have the following dynamics

$$\begin{aligned} \mathrm{d}S^1(t) &= \alpha_1 S^1(t)\,\mathrm{d}t + \sigma_1 S^1(t)\,\mathrm{d}W^1(t) \\ \mathrm{d}S^2(t) &= \alpha_2 S^2(t)\,\mathrm{d}t + \sigma_2 S^2(t)\,\mathrm{d}W^2(t) \\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t, \end{aligned}$$

where the Wiener processes W^1 and W^2 are assumed to be independent. Compute the price of an exchange option defined by $X = \max\{S^2(T) - S^1(T), 0\}$.

11 Let the stock prices $S^1(t), S^2(t)$ and the bank account $B(t)$ have the following dynamics

$$\begin{aligned} \mathrm{d}S^1(t) &= \alpha_1 S^1(t)\,\mathrm{d}t + \sigma_1 S^1(t)\,\mathrm{d}W^1(t) \\ \mathrm{d}S^2(t) &= \alpha_2 S^2(t)\,\mathrm{d}t + \sigma_2 S^2(t)\,\mathrm{d}W^2(t) \\ \mathrm{d}B(t) &= rB(t)\,\mathrm{d}t, \end{aligned}$$

where the Wiener processes W^1 and W^2 are assumed to be independent. Compute the price of a maximum option defined by $X = \max\{S^2(T), S^1(T)\}$.

12 Derive the hedging portfolio for a claim of the form $\Phi(S_T)$ under the Black-Scholes model in the case of dividends.

13 Prove for the Black-Scholes model, in the case of dividends, that if f gives the arbitrage free price of a derivative then $f(t, S_t)$ is a $\mathbb{Q}$-martingale. [Use the Black-Scholes PDE for this situation.]

14 Deduce the price $F(t, T, \mathcal{Y})$ of a futures contract based on some claim $\mathcal{Y}$, $F(t, T, \mathcal{Y}) = E^{\mathbb{Q}}_{t,X_t}[\mathcal{X}]$ where X is the underlying factor process and $\mathcal{Y}$ is an appropriate function of X_T. [This is basically the content of Proposition 29.6.]

15 Compute the price of a futures option with maturity date T, strike price K, where the Futures contract is a future on S with delivery date T_1, $T_1 > T$. [This has everything to do with Proposition 7.13.]

16 Assume the following exchange rate model,

$$\begin{aligned} \mathrm{d}X(t) &= \alpha_X X(t)\,\mathrm{d}t + \sigma_X X(t)\,\mathrm{d}W(t), \\ \mathrm{d}B_d(t) &= r_d B_d(t)\,\mathrm{d}t, \\ \mathrm{d}B_f(t) &= r_f B_f(t)\,\mathrm{d}t. \end{aligned}$$

All markets are assumed to be frictionless.

(a) Derive an expression for L_t, the Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}^f}{\mathrm{d}\mathbb{Q}^d}$ on $\mathcal{F}_t$.

(b) Find the Girsanov transformation between the domestic martingale measure $\mathbb{Q}^d$ and the foreign martingale measure $\mathbb{Q}^f$, i.e., how $W^{\mathbb{Q}^d}$ and $W^{\mathbb{Q}^f}$ are related.

(c) Finally give the expression for the dynamics of the exchange rate X under the martingale measure $\mathbb{Q}^d$.