University of Copenhagen
Department of Applied Mathematics and Statistics

Martin Jacobsen
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TWO EXAMPLES OF LOCAL MARTINGALES THAT ARE NOT TRUE MARTINGALES

1. Let (B_1, B_2) be a standard Brownian motion in two dimensions starting from $(0,0)$. Let $a > 0$ and consider the process Y defined by

$$Y_t = \log \sqrt{(B_{1,t} + a)^2 + B_{2,t}^2},$$

which is well defined since the two-dimensional Brownian motion starting from $(a, 0)$ never hits $(0,0)$. Y is $\log(\mathrm{BES}(2))$ starting from $Y_0 = \log a$.

By Itô's formula, Y is a continuous local martingale (log is the scale function for BES(2)), but by computation

$$\mathbb{E}Y_1 = \log a + \int_a^\infty \frac{1}{r} e^{-\frac{1}{2}r^2}\, dr \neq \log a$$

so Y is not a true martingale.

It may be argued that

$$\mathbb{E}e^{\theta|Y_t|} < \infty$$

for all $\theta > 0$, $t > 0$ which shows that no kind of moment conditions on any given Y_t can be used to verify that a given local martingale Y is a true martingale: for that one needs e.g. that $\mathbb{E}\sup_{s:s\leq t} |Y_s| < \infty$ for all t.

2. Consider the SDE

$$dX_t = (a + bX_t)\, dt + \sigma X_t^\gamma \, dB_t, \quad X_0 \equiv x_0 > 0.$$

For suitable choices of the parameters a, b, σ^2, γ (with $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\sigma^2 > 0$, $\gamma \geq \frac{1}{2}$ (necessary for $X > 0$)) the diffusion solving this SDE stays strictly positive at all times and has an invariant distribution: it is a hard grind getting the exact conditions which are as follows,

$\gamma = \frac{1}{2}$	$2a \geq \sigma^2, b < 0$
$\frac{1}{2} < \gamma < 1$	$a > 0, b \leq 0$
$\gamma = 1$	$a > 0, 2b < \sigma^2$
$\gamma > 1$	$a > 0, b \in \mathbb{R}$ or $a = 0, b < 0$.

It is the case $\gamma > 1$ which is of interest for this example. The density of the invariant distribution is proportional to

$$x^{-2\gamma} \exp\left[\tfrac{2a}{\sigma^2(1-2\gamma)} x^{1-2\gamma} + \tfrac{2b}{\sigma^2(2-2\gamma)} x^{2-2\gamma}\right] \quad (x > 0)$$

and has heavy tails but always a finite expectation ξ. So start X according to the invariant distribution and obtain if the local martingale term in the SDE is a true martingale that

$$\mathbb{E}X_t = \mathbb{E}X_0 + \int_0^t (a + b\mathbb{E}X_s)\, ds,$$

i.e. $\xi = -a/b$. But with $X > 0$, for $a > 0, b > 0$ (which is allowed), this is just nonsense! It is quite interesting by the way to simulate X and extract a picture of the local martingale from the simulation!