1 Functions of bounded variation and Stieltjes integrals

In this section we define functions of bounded variation and review some basic properties. Stieltjes integrals will be discussed subsequently. We consider functions defined on an interval $[a, b]$. Next to these we consider partitions Π of $[a, b]$, finite subsets $\{t_0, \ldots, t_n\}$ of $[a, b]$ with the convention $t_0 \leq \cdots \leq t_n$, and $\mu(\Pi)$ denotes the mesh of Π. Extended partitions, denoted Π^*, are partitions Π, together with additional points τ_i, with $t_{i-1} \leq \tau_i \leq t_i$. By definition $\mu(\Pi^*) = \mu(\Pi)$. Along with a function α, a partition Π, we define

$$V(\alpha; \Pi) := \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|,$$

the variation of α over the partition Π.

Definition 1.1 A function α is said to be of bounded variation if $V(\alpha) := \sup_\Pi V(\alpha; \Pi) < \infty$, the supremum taken over all partitions Π. The variation function $v_\alpha : [a, b] \to \mathbb{R}$ is defined by $v_\alpha(t) = V(\alpha \mathbf{1}_{[a,t]})$.

A refinement Π' of a partition Π satisfies by definition the inclusion $\Pi \subset \Pi'$. In such a case, one has $\mu(\Pi') \leq \mu(\Pi)$ and $V(\alpha; \Pi') \geq V(\alpha; \Pi)$. It follows from the definition of $V(\alpha)$, that there exists a sequence (Π_n) of partitions (which can be taken as successive refinements) such that $V(\alpha; \Pi_n) \to V(\alpha)$.

Example 1.2 Let α be continuously differentiable and assume that $\int_a^b |\alpha'(t)| \, \mathrm{d}t$ is finite. Then $V(\alpha) = \int_a^b |\alpha'(t)| \, \mathrm{d}t$. This follows, since $V(\alpha; \Pi)$ can be written as a Riemann sum

$$\sum_{i=1}^{n} |\alpha'(\tau_i)| \, (t_i - t_{i-1}),$$

where the τ_i satisfy $t_{i-1} \leq \tau_i \leq t_i$ and $\alpha'(\tau_i) = \frac{\alpha(t_i) - \alpha(t_{i-1})}{t_i - t_{i-1}}$.

Note that v_α is an increasing function with $v_\alpha(a) = a$ and $v_\alpha(b) = V(\alpha)$. Any monotone function α is of bounded variation and in this case $V(\alpha) = |\alpha(b) - \alpha(a)|$ and $v_\alpha(t) = |\alpha(t) - \alpha(a)|$. Also the difference of two increasing functions is of bounded variation. This fact has a converse.

Proposition 1.3 *Let α be of bounded variation. Then there exists increasing functions v_α^+ and v_α^- such that $v_\alpha^+(a) = v_\alpha^-(a) = a$, $\alpha(t) - \alpha(a) = v_\alpha^+(t) - v_\alpha^-(t)$. Moreover, one can choose them such that $v_\alpha^+ + v_\alpha^- = v_\alpha$.*

Proof Define

$$v_\alpha^+(t) = \frac{1}{2}(v_\alpha^+(t) + \alpha(t) - \alpha(a))$$

$$v_\alpha^-(t) = \frac{1}{2}(v_\alpha^+(t) - \alpha(t) + \alpha(a)).$$

We only have to check that these functions are increasing, since the other statements are obvious. Let $t' > t$. Then $v_\alpha^+(t') - v_\alpha^+(t) = \frac{1}{2}(v_\alpha^+(t') - v_\alpha^+(t) + \alpha(t') - \alpha(t))$. The difference $v_\alpha^+(t') - v_\alpha^+(t)$ is the variation of α over the interval $[t, t']$, which is greater than or equal to $|\alpha(t') - \alpha(t)|$. Hence $v_\alpha^+(t') - v_\alpha^+(t) \geq a$, and the same holds for $v_\alpha^-(t') - v_\alpha^-(t)$. □

The decomposition in this proposition enjoys a minimality property. If w^+ and w_- are increasing functions, $w^+(a) = w^-(a) = a$ and $\alpha(t) - \alpha(a) = w^+(t) - w^-(t)$, then for all $t' > t$ one has $w^+(t') - w^+(t) \geq v_\alpha^+(t') - v_\alpha^+(t)$ and $w^-(t') - w^-(t) \geq v_\alpha^-(t') - v_\alpha^-(t)$. This property is basically the same as its counterpart for the Jordan decomposition of signed measures.

The following definition generalizes the concept of Riemann integral.

Definition 1.4 Let $f, \alpha : [a, b] \to \mathbb{R}$ and Π^* be an extended partition of $[a, b]$. Write

$$S(f, \alpha; \Pi^*) = \sum_{i=1}^{n} f(\tau_i)\,(\alpha(t_i) - \alpha(t_{i-1})).$$

We say that $S(f, \alpha) = \lim_{\mu(\Pi^*) \to 0} S(f, a; \Pi^*)$, if for all $\varepsilon > a$, there exists $\delta > a$ such that $\mu(\Pi^*) < \delta$ implies $|S(f, \alpha) - S(f, \alpha; \Pi^*)| < \varepsilon$. If this happens, we say that f is integrable w.r.t. α and we commonly write $\int f \, \mathrm{d}\alpha$ for $S(f, \alpha)$, and call it the Stieltjes integral of f w.r.t. α.

Proposition 1.5 *Let $f, \alpha : [a, b] \to \mathbb{R}$, f continuous and α of bounded variation. Then f is integrable w.r.t. α. Moreover, the triangle inequality $|\int f \, \mathrm{d}\alpha| \leq \int |f| \, \mathrm{d}v_\alpha$ holds.*

Proof (sketch) To show integrability of f w.r.t. α, the idea is to compare $S(f, \alpha; \Pi_b^*)$ and $S(f, \alpha; \Pi_2^*)$ for two extended partitions Π_b^* and Π_2^*. By constructing another extended partition Π^* that is a *refinement* of Π_b^* and Π_2^* in the sense that all t_i and t_i from Π_b^* and Π_2^* belong to Π^*, one can show from

$$|S(f, \alpha; \Pi_b^*) - S(f, \alpha; \Pi_2^*)| \leq |S(f, \alpha; \Pi_b^*) - S(f, \alpha; \Pi^*)| + |S(f, \alpha; \Pi^*) - S(f, \alpha; \Pi_2^*)|$$

that the $S(f, \alpha; \Pi^*)$ form a Cauchy sequence if one chooses Π^* from a sequence (Π_n^*) with $\mu(\Pi_n^*) \to 0$. The limit can be shown to exist independent of the chosen sequence. The triangle inequality for the integrals is almost trivial. □

Proposition 1.6 *Let $f, \alpha : [a, b] \to \mathbb{R}$, be continuous and of bounded variation. Then the following integration by parts formula holds.*

$$\int f \, \mathrm{d}\alpha + \int \alpha \, \mathrm{d}f = f(b)\alpha(b) - f(a)\alpha(a).$$

Proof Choose points $t_a \leq \cdots \leq t_n$ in $[a,b]$ and $\tau_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, to which we add $\tau_0 = a$ and $\tau_{n+1} = b$. By Abel's summation formula, we have

$$\sum_{i=1}^{n} f(\tau_i)\,(\alpha(t_i) - \alpha(t_{i-1})) =$$

$$f(b)\alpha(b) - f(a)\alpha(a) - \sum_{i=1}^{n+1} \alpha(t_{i-1})\,(f(\tau_i) - f(\tau_{i-1})).$$

The result follows by application of Proposition 1.5. □

This proposition can be used to *define* $\int \alpha \,\mathrm{d}f$ for functions α of bounded variation and continuous functions f, simply by putting

$$\int \alpha \,\mathrm{d}f := f(b)\alpha(b) - f(a)\alpha(a) - \int f \,\mathrm{d}\alpha.$$

It follows from Proposition 1.3 that α admits finite left and right limits at all t in $[a,b]$. By α_+ we denote the function given by $\alpha_+(t) = \lim_{u \downarrow t} \alpha(u)$ for $t \in [a,b)$ and $\alpha_+(b) = \alpha(b)$. Note that α_+ is right-continuous. We close this section with a result relating Stieltjes and Lebesgue integrals. The proposition below gives an example of such a connection, but can be substantially generalized.

Proposition 1.7 *Let $f, \alpha : [a,b] \to \mathbb{R}$, f continuous and α of bounded variation. Let $\mu = \mu_\alpha$ be the signed measure on $([a,b], \mathcal{B}([a,b]))$ that is uniquely defined by $\mu((a,b]) = \alpha_+(b) - \alpha_+(a)$ for all $a \leq a < b \leq b$. Then the Lebesgue integral $\int f \,\mathrm{d}\mu$ and the Stieltjes integral $\int f \,\mathrm{d}\alpha$ are equal, $\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\alpha$.*

Proof Exercise. □