Exercise 12.2 Let μ and ν be probability measures on $(\mathbb{R}, \mathcal{B}))$ having corresponding characteristic functions ϕ and ψ.

1. Show that $\int_{\mathbb{R}} \exp(-\mathrm{i}uy)\phi(y)\,\nu(\mathrm{d}y) = \int_{\mathbb{R}} \psi(x-u)\,\mu(\mathrm{d}x)$.

2. Assume that ν is the $N(0, \frac{1}{\sigma^2})$ distribution, then $\psi(u) = \exp(-\frac{1}{2}u^2/\sigma^2)$. Let f_{σ^2} be the density of the $N(0, \sigma^2)$ distribution. Show that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \exp(-\mathrm{i}uy)\phi(y)\exp(-\frac{1}{2}\sigma^2 y^2)\,\mathrm{d}y = \int_{\mathbb{R}} f_{\sigma^2}(u-x)\,\mu(\mathrm{d}x),$$

   and show that the right hand side gives the *density* of $X + Y$, where X has distribution μ, Y has the $N(0, \sigma^2)$ distribution and X and Y are independent (see also Exercise 5.6).

3. Write $Y = \sigma Z$, with Z having the standard normal distribution (X and Z independent). It follows that ϕ and σ determine the distribution of $X + \sigma Z$ for all $\sigma > 0$. Show that ϕ uniquely determines μ. (This gives an alternative proof of the assertion of Corollary 12.4).