Exercises Measure Theoretic Probability

Chapter 1

1. Prove the following statements.

    (a) The intersection of an arbitrary family of d-systems is again a d-system.

    (b) The intersection of an arbitrary family of σ-algebras is again a σ-algebra.

    (c) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of subsets of Ω with $\mathcal{C}_1 \subset \mathcal{C}_2$, then $d(\mathcal{C}_1) \subset d(\mathcal{C}_2)$.

2. Let $\mathcal{G}$ and $\mathcal{H}$ be two σ-algebras on Ω. Let $\mathcal{C} = \{G \cap H : G \in \mathcal{G},\, H \in \mathcal{H}\}$. Show that $\mathcal{C}$ is a π-system and that $\sigma(\mathcal{C}) = \sigma(\mathcal{G}, \mathcal{H})$.

3. Show that $\mathcal{D}_2$ (Williams, page 194) is a π-system.

4. Let Ω be a countable set. Let $\mathcal{F} = 2^\Omega$ and let $p : \Omega \to [0,1]$ satisfy $\sum_{\omega\in\Omega} p(\omega) = 1$. Put $\mathbb{P}(A) = \sum_{\omega\in A} p(\omega)$ for $A \in \mathcal{F}$. Show that $\mathbb{P}$ is a probability measure.

5. Let Ω be a countable set. Let $\mathcal{A}$ be the collection of $A \subset \Omega$ such that A or its complement has finite cardinality. Show that A is an algebra. What is $d(\mathcal{A})$?

6. Show that a finitely additive map $\mu : \Sigma_0 \to [0, \infty]$ is countably additive if $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$. If μ is countably additive, do we necessarily have $\mu(H_n) \to 0$ for every decreasing sequence of sets $H_n \in \Sigma_0$ with $\bigcap_n H_n = \emptyset$?

Chapter 2

1. Exercise of section 2.9 (page 28).

Chapter 3

1. If h_1 and h_2 are measurable functions, then h_1h_2 is measurable too.

2. Let X be a random variable. Show that $\Pi(X) := \{X^{-1}(-\infty, x] : x \in \mathbb{R}\}$ is a π-system and that it generates $\sigma(X)$.

3. Let $\{Y_\gamma : \gamma \in C\}$ be an arbitrary collection of random variables and $\{X_n : n \in \mathbb{N}\}$ be a countable collection of random variables, all defined on the same probability space.

    (a) Show that $\sigma\{Y_\gamma : \gamma \in C\} = \sigma\{Y_\gamma^{-1}(B) : \gamma \in C, B \in \mathcal{B}\}$.

(b) Let $\mathcal{X}_n = \sigma\{X_1, \ldots, X_n\}$ $(n \in \mathbb{N})$ and $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$. Show that $\mathcal{A}$ is an algebra and that $\sigma(\mathcal{A}) = \sigma\{X_n : n \in \mathbb{N}\}$.

4. Show that the X^+ and X^- are measurable functions and that X^+ is right-continuous and X^- is left-continuous (notation as in section 3.12).

5. Let $\mathcal{F}$ be a σ-algebra on Ω with the property that for all $F \in \mathcal{F}$ it holds that $\mathbb{P}(F) \in \{0, 1\}$. Let $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Show that for some $c \in \mathbb{R}$ one has $\mathbb{P}(X = c) = 1$. (*Hint:* $\mathbb{P}(X \leq x) \in \{0, 1\}$ for all x.)

Chapter 4

1. Williams, exercise E4.1.

2. Williams, exercise E4.6.

3. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be σ-algebras and let $\mathcal{G} = \sigma(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \ldots)$.

   (a) Show that $\Pi = \{G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_k} : k \in \mathbb{N}, i_k \in \mathbb{N}, G_{i_j} \in \mathcal{G}_{i_j}\}$ is a π-system that generates $\mathcal{G}$.

   (b) Assume that $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is an independent sequence. Let M and N be disjoint subsets of $\mathbb{N}$ and put $\mathcal{M} = \sigma(\mathcal{G}_i, i \in M)$ and $\mathcal{N} = \sigma(\mathcal{G}_i, i \in N)$. Show that $\mathcal{M}$ and $\mathcal{N}$ are independent σ-algebras.

Chapter 5

1. Show that the integral is a linear operator on $\mathcal{L}^1(S, \Sigma, \mu)$ by showing first that the result of section 5.5 holds true and then the general case.

2. Prove the second part of Scheffé's lemma (see page 55).

3. Consider a measure space (S, Σ, μ). Let $f \in (m\Sigma)^+$ and define $\nu(E) = \int_S f 1_E \, d\mu$, $E \in \Sigma$. Show that ν is a measure on Σ. Show also that $h \in \mathcal{L}^1(S, \Sigma, \nu)$ iff $fh \in \mathcal{L}^1(S, \Sigma, \mu)$ and that $\int_S h \, d\nu = \int_S fh \, d\mu$. (Use the 'standard machine' of section 5.12).

4. Let $(x_1, x_2, \ldots)$ be a sequence of nonnegative real numbers, let $\ell : \mathbb{N} \to \mathbb{N}$ be a bijection and define the sequence $(y_1, y_2, \ldots)$ by $y_k = x_{\ell(k)}$. Let for each n the n-vector y^n be given by $y^n = (y_1, \ldots, y_n)$. Consider then for each n a sequence of numbers x^n defined by $x_k^n = x_k$ if x_k is a coordinate of y^n. Otherwise put $x_k^n = 0$. Show that $x_k^n \uparrow x_k$ for every k as $n \to \infty$. Show that $\sum_{k=1}^{\infty} y_k = \sum_{k=1}^{\infty} x_k$.

5. In this exercise λ denotes Lebesgue measure on the Borel sets of $[0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Then the Riemann integral $I := \int_0^1 f(x) \, dx$ exists (this is standard Analysis). But also the Lebesgue integral of f exists. (Explain why.). Also explain why (use the definition of the Riemann

integral) there is a decreasing sequence of simple functions U_n with limit U satisfying $U \geq f$ and $\lambda(U^n) \downarrow I$. Prove that $\lambda(f) = I$.

Chapter 6

1. Let X and Y be simple random variables. Show that $\mathbb{E}\, X$ doesn't depend on the chosen representation of X. Show also that $\mathbb{E}\,(X+Y) = \mathbb{E}\, X + \mathbb{E}\, Y$.

2. Show that for $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ it holds that $|\mathbb{E}\, X| \leq \mathbb{E}\, |X|$.

3. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that $\lim_{n\to\infty} n\mathbb{P}(|X| > n) = 0$.

4. Prove the assertions (a)-(c) of section 6.5.

5. Complete the proof of theorem 6.11: show the a.s. uniqueness of Y and show that if $X - Y \perp Z$ for all Z in $\mathcal{K}$, then $||X - Y||_2 = \inf\{||X - Y'||_2 : Y' \in \mathcal{K}\}$.

6. Prove lemma 6.12 with the 'standard machine' of section 5.12. With notation as in lemma 6.12, let $Y = h(X)$. Show also the following equality: $\mathbb{E}\, Y = \int_{\mathbb{R}} y\, \Lambda_Y(dy)$, with Λ_Y the *law* of Y.

Chapter 8

1. Prove part (b) of Fubini's theorem in section 8.2 for $f \in \mathcal{L}^1(S, \Sigma, \mu)$ (you already know it for $f \in \mathrm{m}\Sigma^+$). Explain why $s_1 \mapsto f(s_1, s_2)$ is in $\mathcal{L}^1(S_1, \Sigma_1, \mu_1)$ for all s_2 outside a set N of μ_2-measure zero and that I_2^f is well defined on N^c.

2. If $Z_1, Z_2, \ldots$ is a sequence of nonnegative random variables, then

$$\mathbb{E} \sum_{k=1}^{\infty} Z_k = \sum_{k=1}^{\infty} \mathbb{E}\, Z_k. \tag{1}$$

   Show that this follows from Fubini's theorem (as an alternative to section 6.5). If $\sum_{k=1}^{\infty} \mathbb{E}\, Z_k < \infty$, what is $\mathbb{P}(\sum_{k=1}^{\infty} Z_k = \infty)$. Formulate a result similar to (1) for random variables Z_k that may assume negative values as well.

3. Let the vector of random variables (X, Y) have a joint probability density function f. Let f_X and f_Y be the (marginal) probability density functions of X and Y respectively. Show that X and Y are independent iff $f(x, y) = f_X(x) f_Y(y)$ for all x, y except in a set of Leb$\times$Leb-measure zero.

4. Let f be defined on $\mathbb{R}^2$ such that for all $a \in \mathbb{R}$ the function $y \mapsto f(a, y)$ is Borel and such that for all $b \in \mathbb{R}^2$ the function $x \mapsto f(x, b)$ is continuous.

Show that for all $a, b, c \in \mathbb{R}$ the function $(x, y) \mapsto bx + cf(a, y)$ is Borel-measurable on $\mathbb{R}^2$. Let $a_i^n = i/n$, $i \in \mathbb{Z}, n \in \mathbb{N}$. Define

$$f^n(x,y) = \sum_i 1_{(a_{i-1}^n, a_i^n]}(x)\Big(\frac{a_i^n - x}{a_i^n - a_{i-1}^n} f(a_{i-1}^n, y) + \frac{x - a_{i-1}^n}{a_i^n - a_{i-1}^n} f(a_i^n, y)\Big).$$

Show that the f^n are Borel-measurable on $\mathbb{R}^2$ and conclude that f is Borel-measurable on $\mathbb{R}^2$.

5. Show that for $t > 0$

$$\int_0^\infty \sin x \, e^{-tx} \, \mathrm{d}x = \frac{1}{1+t^2}.$$

Show that $x \mapsto \frac{\sin x}{x}$ is *not* in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}, \mathrm{Leb})$, but that we can use Fubini's theorem to prove that the *Riemann* integral

$$\int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x = \frac{\pi}{2}.$$

6. Let $F, G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right-continuous. Use Fubini's theorem to show the integration by parts formula, valid for all $a < b$,

$$F(b)G(b) - F(a)G(a) = \int_{(a,b]} F(s-) \, dG(s) + \int_{(a,b]} G(s) \, dF(s).$$

Hint: integrate $1_{(a,b]^2}$ and split the square into a lower and an upper triangle.

7. Let F be the distribution function of a nonnegative random variable X and assume that $\mathbb{E}\, X^\alpha < \infty$ for some $\alpha > 0$. Use exercise 6 to show that

$$\mathbb{E}\, X^\alpha = \alpha \int_0^\infty x^{\alpha-1}(1 - F(x)) \, dx.$$

Chapter 9

1. Finish the proof of theorem 9.2: Take arbitrary $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and show that the existence of the conditional expectation of X follows from the existence of the conditional expectations of X^+ and X^-.

2. Prove the conditional version of Fatou's lemma, property (f) on page 88 (Williams).

3. Prove the conditional Dominated Convergence theorem, property (g) on page 88 (Williams).

4. Let (X, Y) have a bivariate normal distribution with $\mathbb{E}\,X = \mu_X$, $\mathbb{E}\,Y = \mu_Y$, $\operatorname{Var} X = \sigma_X^2$, $\operatorname{Var} Y = \sigma_Y^2$ and $\operatorname{Cov}(X, Y) = c$. Let

$$\hat{X} = \mu_x + \frac{c}{\sigma_Y^2}(Y - \mu_Y).$$

Show that $\mathbb{E}\,(X - \hat{X})Y = 0$. Show also (use a special property of the bivariate normal distribution) that $\mathbb{E}\,(X - \hat{X})g(Y) = 0$ if g is a Borel-measurable function such that $\mathbb{E}\,g(Y)^2 < \infty$. Conclude that $\hat{X}$ is a version of $\mathbb{E}\,[X|Y]$.

5. Williams, Exercise E9.1.

6. Williams, Exercise E9.2

7. Let $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Let $\hat{X}$ be a version of the conditional expectation $\mathbb{E}[X|\mathcal{G}]$. Show that

$$\mathbb{E}(X - Y)^2 = \mathbb{E}(X - \hat{X})^2 + \mathbb{E}(Y - \hat{X})^2.$$

Deduce that $\hat{X}$ can be viewed as an orthogonal projection of X onto $(\Omega, \mathcal{G}, \mathbb{P})$.

Chapter 10

1. Let X be an adapted process and T a stopping time that is finite. Show that X_T is $\mathcal{F}$-measurable. Show also that for arbitrary stopping times T (so the value infinity is also allowed) the stopped process X^T is adapted.

2. For every n we have a measurable function f_n on $\mathbb{R}^n$. Let $Z_1, Z_2, \ldots$ be independent random variables and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Show that (you may assume sufficient integrability) that $X_n = f_n(Z_1, \ldots, Z_n)$ defines a martingale under the condition that $\mathbb{E}f_n(z_1, \ldots, z_{n-1}, Z_n) = f_{n-1}(z_1, \ldots, z_{n-1})$ for every n.

3. If S and T are stopping times, then also $S + T$, $S \vee T$ and $S \wedge T$ are stopping times. Show this.

4. Show that an adapted process X is a martingale iff $\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n$ for all $n, m \geq 0$.

5. (a) If X is a martingale is and f a convex function such that $\mathbb{E}|f(X_n)| < \infty$, then Y defined by $Y_n = f(X_n)$ is a submartingale. Show this.
   (b) Show that Y is a submartingale, if X is a submartingale and f is a convex increasing function.

6. Prove Corollaries (c) and (d) on page 101.

7. Let $X_1, X_2, \ldots$ be an *iid* sequence of Bernoulli random variables. Put $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$. Let M be a martingale adapted to the generated filtration. Show that the *Martingale Representation Property* holds: there exists a constant m and a predictable process Y such that $M_n = m + (Y \bullet X)_n$, $n \geq 1$.

Chapter 11

1. Let X be an adapted process and $a < b$ real numbers. Let $S_1 = \inf\{n : X_n < a\}$, $T_1 = \inf\{n > S_1 : X_n > b\}$, etc. Show that the S_k and T_k are stopping times. Show also that the process C of section 11.1 is previsible (synonymous for predictable).

2. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0, 1)$, $\mathcal{F}$ the Borel sets of $[0, 1)$ and $\mathbb{P}$ the Lebesgue measure. Let $I_k^n = [k2^{-n}, (k+1)2^{-n})$ for $k = 0, \ldots, 2^n - 1$ and $\mathcal{F}_n$ be the σ-algebra by the I_k^n for $k = 0, \ldots, 2^n - 1$. Define $X_n = 1_{I_0^n} 2^n$. Show that X_n is a martingale is and that the conditions of theorem 11.5 are satisfied. What is X_∞ in this case? Do we have $X_n \xrightarrow{\mathcal{L}^1} X_\infty$? (This has something to do with 11.6).

3. Let X be a submartingale with $\sup_{n \geq 0} \mathbb{E}|X_n| < \infty$. Show that there exists a random variable X_∞ such that $X_n \to X_\infty$ a.s.

4. Show that for a supermartingale X the condition $\sup\{\mathbb{E}\,|X_n| : n \in \mathbb{N}\} < \infty$ is equivalent to the condition $\sup\{\mathbb{E}\,X_n^- : n \in \mathbb{N}\} < \infty$.

Chapter 12

1. Exercise 12.1.

2. Exercise 12.2

3. Let (H_n) be a predictable sequence of random variables with $\mathbb{E}\,H_n^2 < \infty$ for all n. Let (ε_n) be a sequence with $\mathbb{E}\,\varepsilon_n^2 = 1$, $\mathbb{E}\,\varepsilon_n = 0$ and ε_n independent of $\mathcal{F}_{n-1}$ for all n. Let $M_n = \sum_{k \leq n} H_k \varepsilon_k$, $n \geq 0$. Compute the conditional variance process A of (M_n). Take $p > 1/2$ and consider $N_n = \sum_{k \leq n} \frac{1}{(1+A_k)^p} H_k \varepsilon_k$. Show that there exists a random variable N_∞ such that $N_n \to N_\infty$ a.s. Show (use Kronecker's lemma) that $\frac{M_n}{(1+A_n)^p}$ has an a.s. finite limit.

Chapter 13

1. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be uniformly integrable collections of random variables on a common probability space. Show that $\bigcup_{k=1}^n \mathcal{C}_k$ is uniformly integrable. (In particular is a finite collection in $\mathcal{L}^1$ uniformly integrable).

2. Williams, exercise E13.1.
3. Williams, exercise E13.2.
4. Let $\mathcal{C}$ be a uniformly integrable collection of random variables.
   (a) Consider $\bar{\mathcal{C}}$, the closure of $\mathcal{C}$ in $\mathcal{L}^1$. Use E13.1 to show that also $\bar{\mathcal{C}}$ is uniformly integrable.
   (b) Let $\mathcal{D}$ be the convex hull of $\mathcal{C}$, the smallest convex set that contains $\mathcal{C}$. Then both $\mathcal{D}$ and its closure in $\mathcal{L}^1$ are uniformly integrable
5. In this exercise you prove (fill in the details) the following characterization: a collection $\mathcal{C}$ is uniformly integrable iff there exists a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{\mathbb{E}G(|X|) : X \in \mathcal{C}\} < \infty$.
   The necessity you prove as follows. Let $\varepsilon > 0$ choose $a = M/\varepsilon$ and c such that $\frac{G(t)}{t} \geq a$ for all $t > c$. To prove uniform integrability of $\mathcal{C}$ you use that $|X| \leq \frac{G(|X|)}{a}$ on the set $\{|X| \geq c\}$.
   It is less easy to prove sufficiency. Proceed as follows. Suppose that we have a sequence (g_n) with $g_0 = 0$ and $\lim_{n\to\infty} g_n = \infty$. Define $g(t) = \sum_{n=0}^{\infty} 1_{[n,n+1)}(t) g_n$ and $G(t) = \int_0^t g(s)ds$. Check that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$.
   With $a_n(X) = \mathbb{P}(|X| > n)$, it holds that $\mathbb{E}\, G(|X|) \leq \sum_{n=1}^{\infty} g_n a_n(|X|)$. Furthermore, for every $k \in \mathbb{N}$ we have $\int_{|X|\geq k} |X|\, d\mathbb{P} \geq \sum_{m=k}^{\infty} a_m(X)$. Pick for every n a constant $c_n \in \mathbb{N}$ such that $\int_{|X|\geq c_n} |X|\, d\mathbb{P} \leq 2^{-n}$. Then $\sum_{m=c_n}^{\infty} a_m(X) \leq 2^{-n}$ and hence $\sum_{n=1}^{\infty}\sum_{m=c_n}^{\infty} a_m(X) \leq 1$. Choose then the sequence (g_n) as the 'inverse' of (c_n): $g_n = \#\{k : c_k \leq n\}$.
6. Prove that a collection $\mathcal{C}$ is uniformly integrable iff there exists an *increasing and convex* function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to\infty} \frac{G(t)}{t} = \infty$ and $M := \sup\{\mathbb{E}\, G(|X|) : X \in \mathcal{C}\} < \infty$. (You may use the result of exercise 5.) Let $\mathcal{D}$ be the closure of the convex hull of a uniformly integrable collection $\mathcal{C}$ in $\mathcal{L}^1$. With the function G as above we have $\sup\{\mathbb{E}G(|X|) : X \in \mathcal{D}\} = M$, whence also $\mathcal{D}$ is uniformly integrable.
7. Let $p \geq 1$ and let $X, X_1, X_2, \ldots$ be random variables. Then X_n converges to X in $\mathcal{L}^p$ iff the following two conditions are satisfied.
   (a) $X_n \to X$ in probability,
   (b) The collection $\{|X_n|^p : n \in \mathbb{N}\}$ is uniformly integrable.
8. Exercise E13.3.

Chapter 14

1. Let $Y \in \mathcal{L}^1$, $(\mathcal{F}_n)$ and define for all $n \in \mathbb{N}$ the random variable $X_n = \mathbb{E}\,[Y|\mathcal{F}_n]$. We know that there is X_∞ such that $X_n \to X_\infty$ a.s. Show that for $Y \in \mathcal{L}^2$, we have $X_n \xrightarrow{\mathcal{L}^2} X_\infty$. Find a condition such that $X_\infty = Y$. Give also an example in which $P(X_\infty = Y) = 0$.

2. Let $X = (X_n)_{n \leq 0}$ a (backward) supermartingale.

   (a) Show equivalence of the next two properties:
   (i) $\sup_n \mathbb{E}|X_n| < \infty$ and (ii) $\lim_{n \to -\infty} \mathbb{E}X_n < \infty$.
   (Use that $x \mapsto x^+$ is convex and increasing.)

   (b) Under the condition $\sup_n \mathbb{E}|X_n| =: A < \infty$ the supermartingale X is uniformly integrable. To show this, you may proceed as follows (*but other solutions are equally welcome).* Let $\varepsilon > 0$ and choose $K \in \mathbb{Z}$ such that for all $n < K$ one has $0 \leq \mathbb{E}X_n - \mathbb{E}X_K < \varepsilon$. It is then sufficient to show that $(X_n)_{n \leq K}$ is uniformly integrable. Let $c > 0$ be arbitrary and $F_n = \{|X_n| > c\}$. Using the supermartingale inequality you show that

   $$\int_{F_n} |X_n| \, d\mathbb{P} \leq \int_{F_n} |X_K| \, d\mathbb{P} + \varepsilon.$$

   Because $\mathbb{P}(F_n) \leq \frac{A}{c}$ you conclude the proof.

3. Show that R, defined on page 142 of Williams, is equal to $q\mathbb{E}\,||X^{p-1}Y||$. Show also that the hypothesis of lemma 14.10 is true for $X \wedge n$ if it is true for X and complete the proof of this lemma.

4. Exercise E14.1.

5. Exercise E14.2. Find the error in the statement of what you have to prove in (b).

6. Suppose that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} \ll \mathbb{P}$ with $d\mathbb{Q}/d\mathbb{P} = M_\infty$. Denote by $\mathbb{P}_n$ and $\mathbb{Q}_n$ the restrictions of $\mathbb{P}$ and $\mathbb{Q}$ to $\mathcal{F}_n$ $(n \geq 1)$. Show that $\mathbb{Q}_n \ll \mathbb{P}_n$ and that

   $$\frac{d\mathbb{Q}_n}{d\mathbb{P}_n} = M_n,$$

   where $M_n = \mathbb{E}_P[M_\infty|\mathcal{F}_n]$.

7. Let M be a nonnegative martingale with $\mathbb{E}\,M_n = 1$ for all n. Define $\mathbb{Q}_n(F) = \mathbb{E}\,1_F M_n$ for $F \in \mathcal{F}_n$ $(n \geq 1)$. Show that for all n and k one has $\mathbb{Q}_{n+k}(F) = \mathbb{Q}_n(F)$ for $F \in \mathcal{F}_n$. Assume that M is uniformly integrable. Show that there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ that is absolutely continuous w.r.t. $\mathbb{P}$ and that is such that for all n the restriction of $\mathbb{Q}$ to $\mathcal{F}_n$ coincides with $\mathbb{Q}_n$. Characterize $d\mathbb{Q}/d\mathbb{P}$.

8. Consider the set up of section 14.17 (Williams). Assume that

   $$\prod_{k=1}^{n} \mathbb{E}_{\mathbb{P}} \sqrt{\frac{g_k(X_k)}{f_k(X_k)}} \to 0.$$

   Suppose one observes $X_1, \ldots, X_n$. Consider the testing problem H_0: the densities of the X_k are the f_k against H_1: the densities of the X_k are the

g_k and the test that rejects H_0 if $M_n > c_n$, where $\mathbb{P}(M_n > c_n) = \alpha \in (0,1)$ (likelihood ratio test). Show that this test is *consistent*: $\mathbb{Q}(M_n \leq c_n) \to 0$. (Side remark: the content of the Neyman-Pearson lemma is that this test is most powerful among all test with significance level less than or equal to α.)

9. Finish the proof of theorem 14.11: Show that $||Z_n||_p$ is increasing in n and that $||Z_\infty||_p = \sup\{||Z_n||_p : n \geq 1\}$.

Chapter 17

1. Let $\mu, \mu_1, \mu_2, \ldots$ be probability measures on $\mathbb{R}$. Show that $\mu_n \xrightarrow{w} \mu$ iff for all bounded Lipschitz continuous functions one has $\int f\, d\mu_n \to \int f\, d\mu$. (*Hint: for one implication the proof of lemma 17.2 is instructive.*)

2. Show the 'if part' of lemma 17.2 without referring to the Skorohod representation. *First you take for given $\varepsilon > 0$ a $K > 0$ such that $F(K) - F(-K) > 1 - \varepsilon$. Approximate a continuous f on the interval $(-K, K]$ with a piecewise constant function and you compute the integrals of this approximating function and use the convergence of the $F_n(x)$ at continuity points x of F etc.*

3. If the random variables $X, X_1, X_2, \ldots$ are defined on the same probability space and if $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{w} X$. Prove this.

4. Suppose that $X_n \xrightarrow{w} X$ and that the collection $\{X_n, n \geq 1\}$ is uniformly integrable (you make a minor change in the definition of this notion if the X_n are defined on different probability spaces). Use the Skorohod representation to show that $X_n \xrightarrow{w} X$ implies $\mathbb{E}X_n \to \mathbb{E}X$.

5. Show the following variation on Fatou's lemma: if $X_n \xrightarrow{w} X$, then $\mathbb{E}|X| \leq \liminf_{n\to\infty} \mathbb{E}|X_n|$.

6. Show that the weak limit of a sequence of probability measures is unique.

7. Look at the proof of the Helly-Bray lemma. You show that F is right-continuous (*use that for every $\varepsilon > 0$ and $x \in \mathbb{R}$ there is a $c \in C$ such that $c > x$ and $F(x) > H(c) - \varepsilon$, take $y \in (x, c)$*) and that $F_{n_k}(x)$ (the n_k were obtained by the Cantor-type diagonalization procedure) converges to $F(x)$ at all continuity points x of F (*take $c_1 < x < c_2$, $c_i \in C$ and use that the $F_{n_k}(c_i)$ converge*).

8. Consider the $N(\mu_n, \sigma_n^2)$ distributions, where the μ_n are real numbers and the σ_n^2 nonnegative. Show that this family is tight iff the sequences (μ_n) and (σ_n^2) are bounded. Under what condition do we have that the $N(\mu_n, \sigma_n^2)$ distributions converge to a (weak) limit? What is this limit?

Central limit theorem

1. For each n we have a sequence $\xi_{n1}, \dots, \xi_{nk_n}$ of independent random variables with $\mathbb{E}\xi_{nj} = 0$ and $\sum_{j=1}^{k_n} \operatorname{Var} \xi_{nj} = 1$. If $\sum_{j=1}^{k_n} \mathbb{E}|\xi_{nj}|^{2+\delta} \to 0$ as $n \to \infty$ for some $\delta > 0$, then $\sum_{j=1}^{k_n} \xi_{nj} \xrightarrow{w} N(0,1)$. Show that this follows from the Lindeberg Central Limit Theorem.

2. The classical central limit theorem says that $\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} (X_j - \mu) \xrightarrow{w} N(0,1)$, if the X_j are *iid* with $\mathbb{E}X_j = \mu$ and $0 < \operatorname{Var} X_j = \sigma^2 < \infty$. Show that this follows from the Lindeberg Central Limit Theorem.

3. Show that $X_n \xrightarrow{w} X$ iff $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for all bounded uniformly continuous functions f.

4. Let X and Y be independent, assume that Y has a $N(0,1)$ distribution. Let $\sigma > 0$. Let ϕ be the characteristic function of X: $\phi(u) = \mathbb{E}\exp(\mathrm{i}uX)$.

   (a) Show that $Z = X + \sigma Y$ has density $p(z) = \frac{1}{\sigma\sqrt{2\pi}} \mathbb{E}\exp(-\frac{1}{2\sigma^2}(z - X)^2)$.

   (b) Show that $p(z) = \frac{1}{2\pi\sigma} \int \phi(-y/\sigma) \exp(\mathrm{i}yz/\sigma - \frac{1}{2}y^2)\, dy$.

5. Let $X, X_1, X_2, \dots$ be a sequence of random variables and Y a $N(0,1)$-distributed random variable independent of that sequence. Let ϕ_n be the characteristic function of X_n and ϕ that of X. Let p_n be the density of $X_n + \sigma Y$ and p the density of $X + \sigma Y$.

   (a) If $\phi_n \to \phi$ pointwise, then $p_n \to p$ pointwise. Invoke the previous exercise and the dominated convergence theorem to show this.

   (b) Let $f \in C_b(\mathbb{R})$ be bounded by B. Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \leq 2B \int (p(z) - p_n(z))^+ \, dz$.

   (c) Show that $|\mathbb{E}f(X_n + \sigma Y) - \mathbb{E}f(X + \sigma Y)| \to 0$ if $\phi_n \to \phi$ pointwise.

   (d) Prove the following theorem: $X_n \xrightarrow{w} X$ iff $\phi_n \to \phi$ pointwise.

6. Let $X_1, X_2, \dots, X_n$ be an *iid* sequence having a distribution function F, a continuous density (w.r.t. Lebesgue measure) f. Let m be such that $F(m) = \frac{1}{2}$. Assume that $f(m) > 0$ and that n is odd, $n = 2k - 1$, say ($k = \frac{1}{2}(n+1)$).

   (a) Show that m is the unique solution of the equation $F(x) = \frac{1}{2}$. We call m the *median* of the distribution of X_1.

   (b) Let $X_{(1)} = \min\{X_1, \dots, X_n\}$, $X_{(2)} = \min\{X_1, \dots, X_n\} \setminus \{X_{(1)}\}$, etc. The resulting $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ is called the ordered sample. The *sample median* M_n of $X_1, \dots, X_n$ is by definition $X_{(k)}$. Show that with $U_{nj} = 1_{\{X_j \leq m + n^{-1/2}x\}}$ we have

   $$\mathbb{P}(n^{1/2}(M_n - m) \leq x) = \mathbb{P}(\sum_j U_{nj} \geq k).$$

(c) Let $p_n = \mathbb{P}U_{nj}$, $b_n = (np_n(1-p_n))^{1/2}$, $\xi_{nj} = (U_{nj} - p_n)/b_n$, $Z_n = \sum_{j=1}^n \xi_{nj}$, $t_n = (k - np_n)/b_n$. Rewrite the probabilities in part 6b as $\mathbb{P}(Z_n \geq t_n)$ and show that $t_n \to t := -2xf(m)$.

(d) Show that $\mathbb{P}(Z_n \geq t) \to 1 - \Phi(t)$, where Φ is the standard normal distribution.

(e) Show that $\mathbb{P}(Z_n \geq t_n) \to \Phi(2f(m)x)$ and conclude that the *Central Limit Theorem for the sample median* holds:

$$2f(m)n^{1/2}(M_n - m) \xrightarrow{w} N(0,1).$$

Brownian motion

1. Consider the sequence of 'tents' (X^n), where $X_t^n = nt$ for $t \in [0, \frac{1}{2n}]$, $X_t^n = 1 - nt$ for $t \in [\frac{1}{2n}, \frac{1}{n}]$, and zero elsewhere (there is no randomness here). Show that all finite dimensional distributions of the X^n converge, but X^n does not converge in distribution.

2. Show that ρ as in (1.1) defines a metric.

3. Suppose that the ξ_i of section 4 of the lecture notes are *iid* normally distributed random variables. Use Doob's inequality to obtain $\mathbb{P}(\max_{j \leq n} |S_j| > \gamma) \leq 3\gamma^{-4}n^2$.

4. Show that a finite dimensional projection on $C[0,\infty)$ (with the metric ρ) is continuous.

5. Consider $C[0,\infty)$ with the Borel σ-algebra $\mathcal{B}$ induced by ρ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $X : (\Omega, \mathcal{F}) \to (C[0,\infty), \mathcal{B})$ is measurable, then all maps $\omega \mapsto X_t(\omega)$ are random variables. Show this, as well as its converse. For the latter you need separability that allows you to say that the Borel σ-algebra $\mathbb{B}$ is a product σ-algebra (see also Williams, page 82).

6. Prove proposition 2.2 of the lecture notes.