Introduction to stochastic finance in continuous time

Additional exercises
Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E} X_i = 1$ for all $i$. Define $P_n = \prod_{i=1}^n X_i$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ ($n \geq 1$). Show that the $P_n$ form a martingale sequence.

Suppose a continuous model for the stock price is such that $\log(S(T)/S(t))$ has a normal $N((r - \frac{1}{2}\sigma^2)(T - t), \sigma^2(T - t))$ distribution (under $\mathbb{Q}$). Assume that at time $t$ the price $S(t)$ is known to be equal to $s$. Then the price of the usual European call option at time $t$ is known to be
\[
e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(s \frac{S(T)}{S(t)} - K)^+.
\]
Show by computation of an integral that the explicit expression of this price is given by the Black-Scholes formula of Equation (1.27).

Give a direct proof of Proposition 1.5.

Consider the interval $[0, 1]$. A partition $\Pi$ of $[0, 1]$ is a set $\Pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n = 1$. The mesh $\mu(\Pi)$ of $\Pi$ is defined as the biggest ‘gap’: $\mu(\Pi) = \max\{t_j - t_{j-1} : j = 1, \ldots, n\}$.

For $p > 0$ the $p$-th order variation of a function $f : [0, 1] \to \mathbb{R}$ over a partition $\Pi = \{t_0, \ldots, t_n\}$ of $[0, 1]$ with $0 = t_0 < t_1 < \ldots < t_n = 1$ is defined as $V^p(f; \Pi) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p$. Put $V^p(f) = \lim V^p(f; \Pi)$, where the limit is taken over (sequences of) partitions with mesh tending to zero.

(a) Let $p = 1$ and $f \in C^1[0, 1]$, so with bounded (left/right in the endpoints) derivative. Argue (use Riemann sums) that $V^1(f) = \int_0^1 |f'(t)| \, dt$.

(b) Let $p = 2$ and $f \in C^1[0, 1]$. Show first that there exists $C > 0$ such that $|f(t) - f(s)| \leq C|t - s|$ for all $s, t \in [0, 1]$, then $\sum_j (t_j - t_{j-1})^2 \leq \mu(\Pi)$ for $t_j \in \Pi$ and finally that $V^2(f) = 0$.

(c) Let $f(t) = 2$ for $t \leq \frac{1}{2}$ and $f(t) = 0$ for $t > \frac{1}{2}$. Show that $V^p(f) = 2^p$ for any $p > 0$.

Let $X$ have the standard normal distribution and $\phi(\lambda) = \mathbb{E} \exp(i\lambda X)$, for $\lambda \in \mathbb{R}$.

(a) Argue that $\phi'(\lambda) = i\mathbb{E} (X \exp(i\lambda X))$.

(b) Show (use integration by parts) that $\phi'(\lambda) = -\lambda \phi(\lambda)$.

(c) Conclude that $\phi(\lambda) = \exp(-\frac{1}{2} \lambda^2)$.

Let $X$ be a random variable with $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X > 0) > 0$.

(a) Show that there exists $n \in \mathbb{N}$ such that $\mathbb{P}(X > 1/n) > 0$. (Reason by contradiction, assume that $\mathbb{P}(X > 1/n) = 0$ for all $n \in \mathbb{N}$.)

(b) Show that $\mathbb{E} X > 0$. 

Here are some consequences of Example 6.2. Let $E$ be another representation, consider $X$. We see that the change of measure from $P$ under $\phi$ is random variable having the $N(0,1)$ distribution. Let $\phi_a$ denote the density of the $N(a,1)$ distribution. Let $z(x) = \frac{\phi_a(x)}{\phi(x)}$.

Define $Q(F) = E(1_FZ)$ for $F \in \mathcal{F}$, where $Z = z(X)$.

(a) Compute $E Z = 1$ (use an integral).
(b) Show that $Q(X \leq x) = \Phi(x-a)$, where $\Phi$ is the distribution function of $N(0,1)$.
(c) Conclude that $X^Q := X - a$ has the standard normal distribution under $Q$.
(d) Show that $z(x) = \exp(a x - \frac{1}{2} a^2)$ and that $Z$ has the same distribution as $\exp(a W(1) - \frac{1}{2} a^2)$, where $W$ is a Brownian motion. We see that the change of measure from $P$ to $Q$ is ‘neutralized’ by replacing $X$ by $X - a$ in the sense that $X - a$ has, under $Q$, the same distribution as $X$ under $P$.

Show that (the value of) the Itô integral of (6.3) doesn’t depend on the specific representation of the simple process $a$. Hint: If $a(t) = \sum_{i=1}^m b_i \mathbf{1}_{(s_i-1,s_i)}(t)$ is another representation, consider $a(t) = \sum_{i=1}^m \sum_{j=1}^n b_i \mathbf{1}_{(s_i-1,s_i]}(t_j-1,t_j]\cap (t_{i-1},t_i]) (t)$.

Here are some consequences of Example 6.2.

(a) Show Equality (6.4).
(b) What is the limit of its right hand side when the $t_i$ come from a sequence of partitions of $[0,T]$ with mesh tending to zero? What is in the same situation the limit of $W^n(t)$?
(c) What is the ‘reasonable’ value of $\int_0^T W(s) \, dW(s)$?

Let $\{M(t) = \sum_{i=1}^m b_i \mathbf{1}_{(s_i-1,s_i]}(t)\}$ be a martingale with $E M(t)^2 < \infty$ for all $t$. Let $A(\cdot)$ be another adapted process with $E A(t) < \infty$ for all $t$.

(a) Show that $E [\{M(t) - M(s)\}^2 | \mathcal{F}_s] = E [M(t)^2 - M(s)^2 | \mathcal{F}_s]$ for all $t > s$.
(b) Suppose $E [\{M(t) - M(s)\}^2 - (A(t) - A(s)) | \mathcal{F}_s] = 0$ for all $t > s$. Show that $\{M(t)^2 - A(t), t \in [0,T]\}$ is a martingale.

This exercise concerns Theorem 6.4.

(a) Show that $I_T(a)$ is a linear functional of $a \in \mathcal{S}$.
(b) Show that $I_T(a)$ is a linear functional of $a \in \mathcal{P}$.
(c) Consider the stochastic integrals $I_t(a)$ with $a \in \mathcal{P}$. Denote by $\langle I(a) \rangle_t$ the quadratic variation process of $I(a)$ (i.e. $\langle I(a) \rangle_t$ is the quadratic variation of $I(a)$ over the interval $[0,t]$). Show that $I(a)_t^2 - \langle I(a) \rangle_t, t \in [0,T]$ is a martingale.

Let $M(t) = \int_0^t W(s) \, dW(s)$.

(a) Use Theorem 6.4 to show that $M(t)$ and $M(t)^2 - \int_0^t W(s)^2 \, ds$ ($t \in [0,T]$) are martingales.
(b) Let \( A(t), t \in [0,T] \), be an adapted positive process with piecewise continuous paths and \( \mathbb{E} A(t) < \infty \) for all \( t \). Show/argue that \( \mathbb{E} \left[ \int_0^t A(u) \, du \mid \mathcal{F}_s \right] = \int_0^s A(u) \, du + \int_s^t \mathbb{E} [A(u) \mid \mathcal{F}_s] \, du \), by using the definition of conditional expectation. (You may use that for positive \( A(u) \) it holds that \( \mathbb{E} \int_0^T A(u) \, du = \int_0^T \mathbb{E} A(u) \, du \).)

(c) Show by a direct computation, not involving stochastic integration theory, that \( M(t) = \int_0^t W(s)^2 \, ds \) \((t \in [0,T])\) is a martingale. (Here it is often useful to write \( W(t) = (W(t) - W(s)) + W(s) \) and that \( W(t) - W(s) \) is independent of \( \mathcal{F}_s \). Recall that \( \mathbb{E} X^4 = 3\sigma^4 \) if \( X \) has the \( \mathcal{N}(0,\sigma^2) \) distribution.)

(a) Verify that (6.15) is true.
(b) There is another way to arrive at (6.15) for a more general situation. Let
\( X \) be a semimartingale and \( \langle X \rangle \) its quadratic variation process. Consider
the product \( Y(t) = \exp(X(t)) \exp(-\frac{1}{2} \langle X \rangle_t) \). Use the Itô product formula
to arrive at (6.15).

15 Prove Proposition 7.2.

16 Here you prove the converse of Theorem 7.6. Suppose (the context of the theorem applies) that a claim \( F(S(T), Z(T)) \) can be hedged by a self-financing portfolio with value process \( V \) if the type \( V(t) = v(t, S(t), Z(t)) \), where \( v \) is sufficiently differentiable. Show that \( v \) satisfies Equation (7.7).