Introduction to stochastic finance in continuous time

Additional exercises
1 Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}X_i = 1$ for all $i$. Define $P_n = \prod_{i=1}^n X_i$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ ($n \geq 1$). Show that the $P_n$ form a martingale sequence.

2 Suppose a continuous model for the stock price is such that $\log(S(T)/S(t))$ has a normal $N((r - \frac{1}{2}\sigma^2)(T - t), \sigma^2(T - t))$ distribution (under $\mathbb{Q}$). Assume that at time $t$ the price $S(t)$ is known to be equal to $s$. Then the price of the usual European call option at time $t$ is known to be

$$e^{-r(T-t)}\mathbb{E}_Q\left(s\frac{S(T)}{S(t)} - K\right)^+. $$

Show by computation of an integral that the explicit expression of this price is given by the Black-Scholes formula of Equation (1.27).

3 Give a direct proof of Proposition 1.5.

4 Suppose one doesn’t use the risk-neutral probabilities $q_u(N)$ and $q_d(N)$ in Theorem 1.4, but instead $p_u(N) = \frac{1}{2} + (\mu - \frac{1}{2}\sigma^2)\frac{\Delta}{2\sigma}$ and the corresponding $p_d(N)$ for some $\mu \in \mathbb{R}$ and sufficiently small positive $\Delta = \frac{T}{n}$. What would then be the limit laws of $\log S_N(t)$ and $\log S_N(t) - \log S_N(s)$ (for $t > s$)?

5 Consider the interval $[0, 1]$. A partition $\Pi$ of $[0, 1]$ is a set $\Pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < t_1 < \ldots < t_n = 1$. The mesh $\mu(\Pi)$ of $\Pi$ is defined as the biggest ‘gap’: $\mu(\Pi) = \max\{t_j - t_{j-1} : j = 1, \ldots, n\}$.

For $p > 0$ the $p$-th order variation of a function $f : [0, 1] \to \mathbb{R}$ over a partition $\Pi = \{t_0, \ldots, t_n\}$ of $[0, 1]$ with $0 = t_0 < t_1 < \ldots < t_n = 1$ is defined as

$$V^p(f; \Pi) = \sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p. $$

Put $V^p(f) = \lim V^p(f; \Pi)$, where the limit is taken over (sequences of) partitions with mesh tending to zero.

(a) Let $p = 1$ and $f \in C^1[0, 1]$, so with bounded (left/right in the endpoints) derivative. Argue (use Riemann sums) that $V^1(f) = \int_0^1 |f'(t)|\,dt$.

(b) Let $p = 2$ and $f \in C^1[0, 1]$. Show first that there exists $C > 0$ such that $|f(t) - f(s)| \leq C|t - s|$ for all $s, t \in [0, 1]$, then $\sum_{j}(t_j - t_{j-1})^2 \leq \mu(\Pi)$ for $t_j \in \Pi$ and finally that $V^2(f) = 0$.

6 Let $X$ have the standard normal distribution and $\phi(\lambda) = \mathbb{E}\exp(i\lambda X)$, for $\lambda \in \mathbb{R}$.

(a) Argue that $\phi'(\lambda) = i\mathbb{E}(X\exp(i\lambda X))$.

(b) Show (use integration by parts) that $\phi'(\lambda) = -\lambda\phi(\lambda)$.

(c) Conclude that $\phi(\lambda) = \exp(-\frac{1}{2}\lambda^2)$.

7 Let $X$ be a random variable with $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X > 0) > 0$.

(a) Show that there exists $n \in \mathbb{N}$ such that $\mathbb{P}(X > 1/n) > 0$. (Reason by contradiction, assume that $\mathbb{P}(X > 1/n) = 0$ for all $n \in \mathbb{N}$.)

(b) Show that $\mathbb{E}X > 0$.

(c) Suppose $X$ is such that $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}X = 0$. Show that it follows that $\mathbb{P}(X > 0) = 0$, equivalently $\mathbb{P}(X = 0) = 1$. 

Consider a probability space \((\Omega, \mathcal{F}, P)\) and that \(X\) is random variable having the \(N(0, 1)\) distribution. Let \(\phi\) denote the density of the \(N(0, 1)\) distribution and let \(\phi_a\) denote the density of the \(N(a, 1)\) distribution. Let \(z(x) = \frac{\phi_a(x)}{\phi(x)}\).

Define \(Q(F) = \mathbb{E} (1_F Z)\) for \(F \in \mathcal{F}\), where \(Z = z(X)\).

(a) Compute \(\mathbb{E} Z = 1\) (use an integral).
(b) Show that \(Q(X \leq x) = \Phi(x - a)\), where \(\Phi\) is the distribution function of \(N(0, 1)\).
(c) Conclude that \(X^Q := X - a\) has the standard normal distribution under \(Q\).
(d) Show that \(z(x) = \exp(ax - \frac{1}{2}a^2)\) and that \(Z\) has the same distribution as \(\exp(aW(1) - \frac{1}{2}a^2)\), where \(W\) is a Brownian motion.

We see that the change of measure from \(P\) to \(Q\) is ‘neutralized’ by replacing \(X\) by \(X - a\) in the sense that \(X - a\) has, under \(Q\), the same distribution as \(X\) under \(P\).

9 Show that (the value of) the Itô integral of (6.3) doesn’t depend on the specific representation of the simple process \(a\). Hint: If \(a(t) = \sum_{i=1}^m b_i 1_{(s_{i-1}, s_i]}(t)\) is another representation, consider \(a(t) = \sum_{i=1}^m \sum_{j=1}^n b_i 1_{(s_{i-1}, s_i]} \cap (t_j-1, t_j](t)\).

10 Here are some consequences of Example 6.2.
(a) Show Equality (6.4).
(b) What is the limit of its right hand side when the \(t_i\) come from a sequence of partitions of \([0, T]\) with mesh tending to zero? What is in the same situation the limit of \(W^n(t)\)?
(c) What is the ‘reasonable’ value of \(\int_0^T W(s) \, dW(s)\)?