Pricing and trading credit default swaps

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Begin at the beginning, and go on till you come to the end. Then, .......

L. Carroll, Alice’s Adventures in Wonderland
A probability space \((\Omega, \mathcal{G}, \mathbb{P})\) is given. All the processes are assumed to be \(\mathcal{G}\)-adapted and càdlàg.

We denote

\[ B_t = \exp(\int_0^t r(s)ds) \]

the savings account, where \(r\) is deterministic.
Self-Financing Trading Strategies and Dividend-paying Assets

Let $S^i$, $i = 1, \ldots, k$ denote the price processes of securities that pay dividends according to a process of finite variation $D^i$, with $D^i_0 = 0$, and $S^j$, $j = k + 1, \ldots, m$ non-dividend-paying assets.

The **wealth process** associated to the strategy $\phi = (\phi^1, \ldots, \phi^m)$ is

$$V_t(\phi) = \sum_{\ell=1}^m \phi^\ell_t S^\ell_t.$$ 

A strategy $\phi$ is said to be **self-financing** if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ where the **gains process** $G(\phi)$ is

$$G_t(\phi) = \sum_{i=1}^k \int_{[0,t]} \phi^i_u dD^i_u + \sum_{\ell=1}^m \int_{[0,t]} \phi^\ell_u dS^\ell_u.$$
We say that $\mathbb{Q}$, equivalent to $\mathbb{P}$, is a \textbf{martingale measure} if

- the discounted price $B_t^{-1}S^i_t$ of any non-dividend paying traded security is a $\mathbb{Q}$-martingale with respect to $\mathcal{G}$
- the ex-dividend price process $S^i_t$ associated with the dividend process $D^i$ satisfies:

$$S^i_t = B_t \mathbb{E}_Q \left( S^i_T B_T^{-1} + \int_{[0,T]} B_u^{-1} dD_u^i \bigg| \mathcal{G}_t \right).$$

The processes $S^i_t B_t^{-1} + \int_{[0,t]} B_u^{-1} dD_u^i$ are $\mathbb{Q}$-martingales.

For any self-financing trading strategy $\phi$, the discounted wealth process $B_t^{-1}V_t(\phi)$ is a $\mathbb{Q}$-martingale.
Defaultable Market

The probability space is endowed with a reference filtration $\mathcal{F}$.

The \textbf{default time} $\tau$ is a \textbf{non-negative random variable}.

$H_t = 1_{\{\tau \leq t\}}$ is the \textbf{default process}, with natural filtration $\mathcal{H}$. Note that $\mathcal{H}_t = \sigma(t \wedge \tau)$ and that $\tau$ is a $\mathcal{H}$-stopping time.

We set $\mathcal{G} = \mathcal{F} \vee \mathcal{H}$.
Some examples

- $\tau$ is a stopping time in a Brownian filtration
- $\lambda$ is a given non-negative $F$-adapted process and

$$\tau = \inf\{t : \int_0^t \lambda_u du \geq U\}$$

where $U$ is a non-negative r.v. independent of $F$. 


Defaultable claim

A defaultable claim maturing at $T$ is a quadruple $(X, A, Z, \tau)$, where

- $X$ is an $\mathcal{F}_T$-measurable random variable,
- $A$ is an $\mathcal{F}$-adapted continuous process of finite variation
- $Z$ is an $\mathcal{F}$-predictable process.

The payoff $X$ is done at time $T$ if $\tau > T$

The payoff $Z_\tau$ is done at default time $\tau$ if $\tau \leq T$

The process $A$ corresponds to a cumulative continuous payment till default time.
The **dividend process** $D$ of a defaultable claim $(0, A, Z, \tau)$ equals, on $t \leq T$,

\[
D_t = A_{\tau \wedge t} + 1_{\tau \leq t} Z_{\tau} \\
= \int_{[0,t]} (1 - H_u) dA_u + \int_{[0,t]} Z_u dH_u
\]
A credit default swap with a constant rate $\kappa$ and recovery at default is a defaultable claim $(0, A, Z, \tau)$, where

- $Z_t \equiv \delta(t)$
- $A_t = -\kappa t$ for every $t \in [0, T]$.

The function (or process) $\delta : [0, T] \rightarrow IR$ represents the default protection, and the constant $\kappa \in IR$ represents the CDS rate (also termed the spread, premium or annuity of a CDS).
We assume here that \textbf{F is the trivial filtration}. Let

\[ G(t) = \mathbb{Q}(\tau > t) = \int_t^\infty f(u)du \]

be the \(\mathbb{Q}\)-survival probability. In that case, for any function \(h\),

\[
\mathbb{E}(h(\tau)|\mathcal{H}_t) \mathbb{1}_{\{t<\tau\}} = \mathbb{1}_{\{t<\tau\}} \frac{1}{\mathbb{P}(t < \tau)} \mathbb{E}(h(\tau) \mathbb{1}_{\{t<\tau\}})
\]

\[
= \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \mathbb{E} \left( \int_t^\infty h(u)f(u)du \right)
\]
We assume that $r = 0$.

The ex-dividend price of a CDS maturing at $T$ with rate $\kappa$ is

$$S_t(\kappa) = \mathbb{E}_Q \left( \mathbb{1}_{\{t<\tau\leq T\}} \delta(\tau) \mid \mathcal{H}_t \right) - \mathbb{E}_Q \left( \mathbb{1}_{\{t<\tau\}} \kappa((\tau \wedge T) - t) \mid \mathcal{H}_t \right)$$

$$= \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left( \int_t^T \delta(u) f(u) \, du - \kappa \int_t^T G(u) \, du \right).$$

For a CDS initiated at time 0, the value $\kappa$ is determined so that $S_0(\kappa) = 0$, hence

$$\int_0^T \delta(u) f(u) \, du = \kappa \int_0^T G(u) \, du$$

Note that the price $S_t$ can take negative values.
The process

\[ M_t = H_t - \int_0^t (1 - H_u) \gamma(u) \, du = H_t - \int_0^{t \wedge \tau} \gamma(u) \, du, \]

where \( \gamma(u) = \frac{f(u)}{G(u)} \) is a (\( \mathbb{Q}, \mathbb{H} \))-martingale.

The process

\[ L_t = \mathbb{1}_{\{t < \tau\}} \frac{1}{G(t)} \]

is a (\( \mathbb{Q}, \mathbb{H} \))-martingale which satisfies \( dL_t = -L_t \, dM_t \).
Using

\[ S_t(\kappa) = L_t \left( \int_t^T \delta(u) f(u) du - \kappa \int_t^T G(u) du \right) \]

and IP formula, one proves that the dynamics of the ex-dividend price \( S_t(\kappa) \) are

\[ dS_t(\kappa) = -S_t- (\kappa) dM_t + (1 - H_t)(\kappa - \delta(t) \gamma(t)) dt . \]
The dividend process associated with the CDS is
\[
dD_t = -\kappa (1 - H_t) dt + \delta(t) dH_t
\]
hence,
\[
d(S_t(\kappa) + D_t) = -S_{t-}(\kappa) dM_t + (1 - H_t) (\kappa - \delta(t) \gamma(t)) dt
\]
\[
-\kappa (1 - H_t) dt + \delta(t) dH_t
\]
\[
= (\delta(t) - S_{t-}(\kappa)) dM_t
\]
The function \( \tilde{S}_t(\kappa) \) such that \( \mathbb{1}_{\{t < \tau\}} \tilde{S}_t(\kappa) = \mathbb{1}_{\{t < \tau\}} S_t(\kappa) \) is the \textbf{predefault-price}, it satisfies
\[
d\tilde{S}_t(\kappa) = \left( \tilde{S}_t(\kappa) \gamma(t) + (\kappa - \delta(t) \gamma(t)) \right) dt,
\]
We assume that \( \tilde{S}_t(\kappa) \neq \delta(t) \) for every \( t \in [0, T] \).
Hedging with CDS

Our aim is to find a replicating strategy for the defaultable claim $(X, 0, Z, \tau)$, where $X$ is a constant and $Z_t = z(t)$.

Let $\hat{g}$ and $\phi^1$ be two functions defined as

$$\hat{g}(t) = \frac{1}{G(t)} \left( \int_0^t z(s) dG(s) + XG(T) \right)$$

$$\phi^1(t) = \frac{h(t) - \hat{g}(t)}{\delta(t) - \tilde{S}_t(\kappa)}$$

Let $\phi^0_t = V_t(\phi) - \phi^1(t)S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_Q(Y|\mathcal{H}_t)$ and

$$Y = 1_{\{T \geq \tau\}}z(\tau) + 1_{\{T < \tau\}}X$$

Then the self-financing strategy $\phi = (\phi^0, \phi^1)$ based on the savings account and the CDS is a replicating strategy.
Proof: The terminal value of the wealth is

\[ V_T = z(\tau)\mathbb{1}_{\tau<T} + X\mathbb{1}_{T<\tau} \]

On the one hand

\[
E(V_T|\mathcal{H}_t) = V_t = z(\tau)\mathbb{1}_{\tau\leq t} + \mathbb{1}_{\tau<t} \frac{1}{G(t)} \left( XG(T) + \int_0^t z(s)dG(s) \right)
\]

\[
= \int_0^t z(s)dH_s + (1 - H_t) \frac{1}{G(t)} \left( XG(T) + \int_0^t z(s)dG(s) \right)
\]

hence \( dV_t = (z(t) - \hat{g}(t))dM_t \) with \( \hat{g}(t) = \frac{1}{G(t)}(\int_0^t z(s)dG(s) + XG(T)) \).

On the other hand, \( dV_t = \phi^1_t dS_t(\kappa) = \phi^1_t(\delta(t) - S_{t-}(\kappa))dM_t \).
First to default

We assume again that $F$ is the trivial filtration.

We now assume that two CDS’s with default times $\tau_1$ and $\tau_2$ are given. Let $G$ be the survival probability of the pair $(\tau_1, \tau_2)$

$$G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s).$$

We assume that the pair $(\tau_1, \tau_2)$ admits a density $f$. Some easy computation lead to $\mathbb{P}(t < \tau_1 | \tau_2) = h(t, \tau_2)$ where:

$$h(t, s) = \frac{\partial_2 G(t, s)}{\partial_2 G(0, s)}$$
Martingales

- **Filtration** $H^i = \sigma(\tau_i \wedge t)$ The processes

\[ M^i_t = H^i_t - \int_0^{t \wedge \tau_i} \frac{f_i(s)}{1 - F_i(s)} ds \]

where

\[ F_i(s) = \mathbb{P}(\tau_i \leq s) = \int_0^s f_i(u) du \]

are $H^i$-martingales. In terms of $G$:

\[ 1 - F_1(t) = G(t, 0), \quad f_1(t) = -\partial_1 G(t, 0) \]
Filtration $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ Let $F^{(1)}$ be the $\mathbb{H}^2$-submartingale

$$F_t^{(1)} = \mathbb{P}(\tau_1 \leq t | \mathcal{H}^2_t)$$

with decomposition $F_t^{(1)} = Z_t^{(1)} + \int_0^t a_s^{(1)} ds$ where $Z^{(1)}$ is an $\mathbb{H}^2$-martingale.

The process

$$M_t^{(1)} = H_t^1 - \int_0^{t \wedge \tau_1} \frac{a_s^{(1)}}{1 - F_s^{(1)}} ds$$

is a $\mathbb{H}$-martingale
In a closed form, the process

\[ M_t^{(1)} = H_t^1 - \int_0^{t \wedge \tau_1} \frac{a_s^{(1)}}{1 - F_s^{(1)}} ds \]

is a $\mathbf{H}$-martingale, where

- $a_t^{(1)} = -H_t^2 \partial_1 h^{(1)}(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)}$

- $h^{(1)}(t, s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$

- $F_t^{(1)} = \mathbb{P}(\tau_1 \leq t | \mathcal{H}_t^2)$
Indeed, some easy computation enables us to write

\[
F_t^{(1)} = H_t^2 \mathbb{P}(\tau_1 \leq t | \tau_2) + (1 - H_t^2) \frac{\mathbb{P}(\tau_1 \leq t < \tau_2)}{\mathbb{P}(\tau_2 > t)}
\]

\[
= H_t^2 (1 - h^{(1)}(t, \tau_2)) + (1 - H_t^2) \frac{G(0, t) - G(t, t)}{G(0, t)}
\]

where

\[
h^{(1)}(t, v) = \frac{\partial^2 G(t, v)}{\partial v \partial G(0, v)}.
\]
\[
M_t^{(1)} = H_t - \int_0^{t \wedge \tau_1 \wedge \tau_2} \gamma_1(s) \, ds - \int_0^{t \wedge \tau_1 \wedge \tau_2} \gamma_1^2(s, \tau_2) \, ds
\]

with
\[
\gamma_1(s) = -\frac{\partial_1 G(s, s)}{G(s, s)}
\]
\[
\gamma_1^2(t, s) = -\frac{f(t, s)}{\partial_2 G(t, s)}
\]

Note that \(\gamma_1\) is the intensity of \(\tau_1\) before \(\tau_2\) and \(\gamma_1^2(t, \tau_2)\) is the intensity of \(\tau_1\) after \(\tau_2\).
The process

\[ M_t^2 = H_t^2 - \int_0^{t \wedge \tau_2} \frac{a_s^{(2)}}{1 - F_s^{(2)}} ds \]

where

- \( a_t^{(2)} = -H_t^1 \partial_2 h^{(2)}(\tau_1, 1) - (1 - H_t^1) \frac{\partial_2 G(t, t)}{G(t, 0)} \)

- \( h^{(2)}(t, s) = \frac{\partial_1 G(t, s)}{\partial_1 G(t, 0)} \).

- \( F_t^{(2)} = \mathbb{P}(\tau_2 \leq t | \mathcal{H}_t^1) \)

is a \( \mathbb{H} \)-martingale.
It is rather easy to find the dynamics of $S^1$. One starts from the fact that, on the set $\{\tau_1 > t, \tau_2 > t\}$

$$S^1_t = \frac{1}{G(t, t)} \left( - \int_t^T \delta(u) G(du, t) - \kappa \int_t^T du \ G(u, t) \right)$$

$$= V^1(t)$$

and, on the set $\{\tau_1 > t > \tau_2\}$

$$S^1_t = \frac{1}{\partial_2 G(t, \tau_2)} \left( - \int_t^T du \ \delta(u) f(u, \tau_2) - \kappa \int_t^T du \ \partial_2 G(u, \tau_2) \right)$$

$$= V^2(t, \tau_2)$$
Hence

\[ S^1_t = (1 - H^1_t)(1 - H^2_t) V^1(t) + (1 - H^1_t) H^2_t V^2(t, \tau_2) \]

and

\[
\begin{align*}
\text{d}S^1_t &= (1 - H^1_t)(1 - H^2_t) \text{d}V^1(t) + (1 - H^1_t) H^2_t \text{d}V^2(t, \tau_2) \\
&\quad - S^1_t \text{d}H^1_t - (1 - H^1_t) \{ V^1(t) - V^2(t, \tau_2) \} \text{d}H^2_t
\end{align*}
\]

where

\[
\begin{align*}
\text{d}V^1(t) &= \left( (\gamma_1(t) + \gamma_2(t)) V^1(t) + \kappa_1 - \delta_1(t)\gamma_1(t) - S^1_t|2(\kappa_1)\gamma_2(t) \right) \text{d}t \\
\text{d}V^2(t, \tau_2) &= \left( \gamma^1|2(t, \tau_2) V^2(t, \tau_2) - \gamma^1|2(t, \tau_2)\delta_1(t) + \kappa_1 \right) \text{d}t
\end{align*}
\]
and the function $S_{t|2}^1(\kappa_1)$ equals

$$S_{t|2}^1(\kappa_1) = \frac{\int_t^T \delta_1(u)f(u,t)du}{\int_t^\infty f(u,t)du} - \kappa_1 \frac{\int_t^T du \int_u^\infty dz f(z,t)}{\int_t^\infty f(u,t)du}.$$ 

Note that $V^2(\tau_2, \tau_2) = S_{\tau_2|2}^1(\kappa_1)$
Let \( S^1_t(\kappa_1) \mathbb{I}_{\{t < \tau(1)\}} = \tilde{S}^1_t(\kappa_1) \mathbb{I}_{\{t < \tau(1)\}} \) where \( \tau(1) = \tau_1 \wedge \tau_2 \).

The dynamics of the pre-default price \( \tilde{S}^1_t(\kappa_1) \) are

\[
d\tilde{S}^1_t(\kappa_1) = (\gamma_1(t) + \gamma_2(t))\tilde{S}^1_t(\kappa_1) \, dt + (\kappa_1 - \delta_1(t)\gamma_1(t) - S^1_{t|2}(\kappa_1)\gamma_2(t)) \, dt,
\]
The pre-default price of a FtD claim \((X, 0, Z, \tau_{(1)})\), where \(Z = (Z_1, Z_2)\) and \(X = c(T)\), equals

\[
\tilde{\pi}(t) = \int_t^T du Z_1(u) \int_t^\infty dv f(u, v) + \int_t^T dv Z_2(v) \int_t^\infty du f(u, v) \frac{G(t, t)}{G(t, t)} + c(T) \frac{G(T, T)}{G(t, t)}.
\]

Moreover,

\[
d\tilde{\pi}(t) = (\gamma_1(t) + \gamma_2(t))\tilde{\pi}(t) dt - \sum_{i=1}^n Z_i(t) \gamma_i(t) dt,
\]

\[
= \sum_{i=1}^n \gamma_i(t)(\tilde{\pi}(t) - Z_i(t)) dt.
\]
Assume that the linear system
\[
\phi_1^1 (\tilde{S}_t^1(\kappa_1) - \delta_1(t)) + \phi_2^2 (\tilde{S}_t^2(\kappa_2) - S_{t|1}(\kappa_2)) = Z_1(t) - \tilde{\pi}(t),
\]
\[
\phi_2^2 (\tilde{S}_t^2(\kappa_2) - \delta_2(t)) + \phi_1^1 (\tilde{S}_t^1(\kappa_1) - S_{t|2}(\kappa_1)) = Z_2(t) - \tilde{\pi}(t),
\]
admits a unique solution \( \phi_t = (\phi_1^1, \phi_2^2) \) and let
\[
\phi_0^t = V_t(\phi) - \phi_1^1 S_{t|1}^1(\kappa_1) - \phi_2^2 S_{t|2}^2(\kappa_2)
\]
where
\[
dV_t(\phi) = \sum_{i=1}^{2} \phi^i_t (dS^i_t(\kappa_i) - \kappa_i dt), \quad V_0(\phi) = E_Q(Y)
\]
Then the self-financing strategy \( \phi \) replicates the first-to-default claim \((X, 0, Z, \tau(1))\).
Stochastic intensity

We now assume that some \textbf{reference filtration} $\mathbf{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ is given. We set $\mathbf{G} = \mathbf{F} \lor \mathbf{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \lor \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbf{G}$ is referred to as to the \textbf{full filtration}.

We define the process

$$F_t = \mathbb{Q}\{\tau \leq t \mid \mathcal{F}_t\},$$

and the \textbf{survival process} $\mathcal{G}_t = 1 - F_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}$. 

The process $G$

$$G_t = \mathbb{Q}\{\tau > t \mid \mathcal{F}_t\}$$

is a supermartingale and admits a decomposition as

$$G_t = Z_t - A_t$$

where $Z$ is an $\mathcal{F}$-martingale and $A$ an $\mathcal{F}$ predictable increasing process. We assume that $G$ is a continuous process with $G_0 = 1$ and $G_t > 0$. 
From the remark that, if \((Y_t, t \geq 0)\) is a \(G\)-adapted process, there exists an \(F\) adapted process \((y_t, t \geq 0)\) such that

\[
Y_t \mathbb{1}_{t < \tau} = y_t \mathbb{1}_{t < \tau}
\]

we obtain the key formulae:

- For any integrable \(G_T\) measurable r.v. \(Y\)

\[
\mathbb{E}(\mathbb{1}_{\{T < \tau\}} Y \mid G_t) = \mathbb{1}_{\{t < \tau\}} \frac{1}{G_t} \mathbb{E}(G_T Y \mid F_t).
\]

- Let \(y\) be an \(F\)-predictable process. Then,

\[
\mathbb{E}(y_\tau \mathbb{1}_{\tau < T} \mid G_t) = y_\tau \mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau > t\}} \frac{1}{G_t} \mathbb{E}\left(\int_t^T y_u dF_u \mid F_t\right).
\]
The ex-dividend price of a credit default swap, with a rate process \( \kappa \) and a protection payment \( \delta_\tau \) at default, equals, for every \( t \in [s, T] \),

\[
S_t(\kappa) = \mathbb{E}_Q \left( \mathbb{1}_{\{t < \tau \leq T\}} \delta_\tau \bigg| G_t \right) - \mathbb{E}_Q \left( \mathbb{1}_{\{t < \tau \}} \int_t^{\tau \wedge T} \kappa_s ds \bigg| G_t \right),
\]

\[
= \mathbb{1}_{\{t < \tau \}} \frac{1}{G_t} \mathbb{E}_Q \left( -\int_t^T \delta_u dG_u + \int_t^\infty dG_u \int_t^{u \wedge T} \kappa_v dv \bigg| \mathcal{F}_t \right).
\]
We now assume that \((\text{H})\) hypothesis holds between \(F\) and \(G\), that is \(F\)-martingales are \(G\)-martingales. It is known that if the \(F\) market is complete and arbitrage free, and if using \(G\)-adapted strategies in the \(F\)-market does not induce arbitrage opportunities, then this hypothesis holds. It is well known that \((\text{H})\) hypothesis is equivalent to

\[
P(\tau \leq t | F_t) = P(\tau \leq t | F_\infty)
\]

hence the process \(F\) is increasing (\(G\) is decreasing). We assume that \(F\) is a Brownian filtration and that \(F\) is absolutely continuous wrt Lebesgue measure.
The process

\[ M_t = H_t - \int_0^{t \wedge \tau} \gamma_u \, du, \]

with \( \gamma_t \, dt = \frac{dF_t}{G_t} \) is a \( \mathbf{G} \)-martingale. The dynamics of the ex-dividend price \( S_t(\kappa) \) are

\[ dS_t(\kappa) = -S_t(\kappa) \, dM_t + (1-H_t)B_tG_t^{-1} \, dm_t + (1-H_t)(r_tS_t(\kappa)+\kappa-\delta_t\gamma_t) \, dt, \]

where \( m \) is the \((\mathbb{Q}, \mathbf{F})\)-martingale given by

\[
m_t = \mathbb{E}_Q \left( \int_0^T B_u^{-1} \delta_u G_u \gamma_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \ \bigg| \mathcal{F}_t \right).
\]
Hedging defaultable claims

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_{\tau} + \mathbb{1}_{\{T < \tau\}} X.$$ 

using two CDS with maturities $T_i$, rates $\kappa_i$ and protection payment $\delta^i$. We assume $r = 0$. Let $\zeta^i_t$ defined as

$$m^i_t = \mathbb{E}_Q \left( \int_0^T \delta^i_u G_u \gamma_u \, du - \kappa_i \int_0^T G_u \, du \, \bigg| \mathcal{F}_t \right), \quad dm^i_t = \zeta^i_t dW_t$$

and

$$m^Z_t = \mathbb{E}_Q \left( - \int_0^\infty Z_u \, dG_u + G_T X \big| \mathcal{F}_t \right), \quad dm^Z_t = \zeta^Z_t dW_t$$
Assume that there exist $\mathbf{F}$-predictable processes $\phi^1, \phi^2$ such that

$$ \sum_{i=1}^{2} \phi^i_t (\delta^i_t - \tilde{S}^i_t(\kappa_i)) = Z_t - \hat{g}_t, \quad \sum_{i=1}^{2} \phi^i_t \zeta^i_t = \zeta_t, $$

where $\hat{g}$ is given by

$$ \hat{g}_t = \frac{1}{G_t} \mathbb{E}_Q \left( - \int_t^T Z_u dG_u + G_T X \bigg| \mathcal{F}_t \right). $$

Let $\phi^0_t = V_t(\phi) - \sum_{i=1}^{2} \phi^i_t S^i_t(\kappa_i)$, where the process $V(\phi)$ is given by

$$ dV_t(\phi) = \sum_{i=1}^{2} \phi^i_t (dS^i_t(\kappa_i) + dD^i_t) $$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$. 
Pricing First to default claims

We now assume that some reference filtration $\mathbf{F}$ is given. Let the default times $\tau_i, i = 1, 2$ be such that (H) hypothesis holds between $\mathbf{F}$ and $\mathbf{G} = \mathbf{F} \vee H^1 \vee H^2$ hence between $\mathbf{F}$ and $\mathbf{G}^1 = \mathbf{F} \vee H^1$ (resp. $\mathbf{G}^2 = \mathbf{F} \vee H^2$). We denote by

$$G(t, s; u) = P(\tau_1 > t, \tau_2 > s|\mathcal{F}_u)$$

Under H hypothesis, $G(t, t; t)$, $G(0, t; t)$ and $G(t, 0; t)$ are increasing processes, supposed to be continuous. Furthermore, for $t < u, s < u$

$$\mathbb{P}(\tau_1 \leq t, \tau_2 \leq s|\mathcal{F}_u) = \mathbb{P}(\tau_1 \leq t, \tau_2 \leq s|\mathcal{F}_\infty)$$
Then, one can generalize the previous results, established in the case of trivial filtration. In the case $r = 0$, the dynamics of the pre-default price $\tilde{S}_t^1(\kappa_1)$ are

$$d\tilde{S}_t^1(\kappa_1) = \left( (\gamma_1(t) + \gamma_2(t))\tilde{S}_t^1(\kappa_1) + \kappa_1 - \delta_1(t)\gamma_1(t) - S_{t|2}^1(\kappa_1)\gamma_2(t) \right) dt + G_t^{-1} dm_t,$$

with

$$\gamma_1(t) = -\frac{\partial_1 G(t, t; t)}{G(t, t; t)}$$
Assume that the recovery $Z$, paid at first default time, is a $\mathbf{F}$-predictable process. The first default time $\tau_{(1)}$ satisfies

$$\mathbb{P}(\tau_{(1)} > t | \mathcal{F}_t) = G(t, t; t) = G(t, t; \infty) = G^{(1)}(t)$$

and

$$\mathbb{E}(Z(\tau_{(1)}) \mathbb{1}_{t < \tau_{(1)} < T} | \mathcal{G}_t) = \mathbb{1}_{\tau_{(1)} > t} \mathbb{E}(\int_t^T Z_u dG^{(1)}(u) | \mathcal{F}_t)$$
In the case where $\tau_1$ and $\tau_2$ are conditionally independent with respect to $\mathcal{F}_t$, then $G^{(1)}(u) = G^1(u)G^2(u)$ with $G^i(t) = \mathbb{P}(\tau_i > t|\mathcal{F}_t)$, hence

$$dG^{(1)}(u) = G^1(u)dG^2(u) + G^2(u)dG^2(u)$$
Begin at the beginning, and go on till you come to the end. *Then, stop.*

L. Carroll, *Alice’s Adventures in Wonderland*
Thank you for your attention