On the Tail Probability for Discounted Sums of Heavy-tailed Losses

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Main references

This talk is mainly based on


Outline

● The general problem;

● Classes of heavy-tailed distributions;

● The main results;

● Examples.
The general problem

Consider the randomly weighted sum \( \sum_{i=1}^{n} \theta_i X_i \), with

- \( \{X_n, n = 1, 2, \ldots\} \) a sequence of i.i.d. r.v.’s;
- \( \{\theta_n, n = 1, 2, \ldots\} \) a sequence of non-negative dependent r.v.’s;
- the sequences \( \{X_n, n = 1, 2, \ldots\} \) and \( \{\theta_n, n = 1, 2, \ldots\} \) being independent.

We want to investigate its tail probability and functionals (risk measures) thereof.
The general problem: interpretation

- $X_n$: represents the net loss or payoff of an insurance or financial product (or portfolio, line of business, conglomerate,...) in (development) year $n$.
  
  - Is assumed to be independent across time.
  
  - In insurance, typically heavy-tailed.

- $\theta_n$: represents the stochastic discount factor for year $n$.
  
  - Case 1: $\theta_n = Y_1 \cdots Y_n$, with $\{Y_n, n = 1, 2, \ldots\}$ a sequence of non-negative i.i.d. r.v.’s.
  
  - Case 2: no assumption on the dependence structure. Includes e.g., GARCH models.
The general problem: possible solutions

- Monte Carlo simulation;

- Easy-computable bounds or approximations à la Roger & Shi (1995);

- Asymptotics.
Classes of heavy-tailed distributions [1]

- Class $S$:

$$\lim_{x \to +\infty} \frac{F^n(x)}{F(x)} = n,$$

for any (or equivalently, for some) $n \geq 2$.

- Class $L$:

$$\lim_{x \to +\infty} \frac{F(x+y)}{F(x)} = 1,$$

for any real number $y$ (or equivalently, for $y = 1$).
Classes of heavy-tailed distributions [2]

- Class $\mathcal{D}$:
  \[
  \limsup_{x \to +\infty} \frac{F(xy)}{F(x)} < +\infty,
  \]
  for any $0 < y < 1$ (or equivalently for some $0 < y < 1$).

- $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$; see e.g., Embrechts, Klüppelberg & Mikosch (1997).
Classes of heavy-tailed distributions [3]

- Class $\mathcal{R}_{-\alpha}$:
  \[
  \lim_{x \to +\infty} \frac{F(xy)}{F(x)} = y^{-\alpha},
  \]
  for any $y > 0$.

- Class $\mathcal{R}_{-\infty}$:
  \[
  \lim_{x \to +\infty} \frac{F(xy)}{F(x)} = \begin{cases}
    0, & y > 1; \\
    +\infty, & 0 < y < 1.
  \end{cases}
  \]
Asymptotic results [1]

Let

- \{Y_n, n = 1, 2, \ldots\} i.i.d. supported on \((0, +\infty)\);

- \(Z_n := Y_1Y_2 \cdots Y_n\);

- \(0 < a_n < +\infty, n = 1, 2, \ldots\)

If \(F_Y \in S \cap \mathcal{R}_{-\infty}\), then it holds for each \(n = 1, 2, \ldots\) that

\[
\mathbb{P} \left( \sum_{i=1}^{n} a_iZ_i > x \right) \sim \sum_{i=1}^{n} \mathbb{P}(a_iZ_i > x).
\]
Asymptotic results [2]

Let

- \{X_n, n = 1, 2, \ldots\} i.i.d. supported on \((-\infty, +\infty)\).

If \(F_X \in D \cap L\) and \(F_Y \in R_{-\infty}\), then it holds for each \(n = 1, 2, \ldots\) that

\[
P \left( \sum_{i=1}^{n} (a_i + X_i)Z_i > x \right) \sim \sum_{i=1}^{n} P((a_i + X)Z_i > x)
\]

and that

\[
P \left( \sum_{i=1}^{n} (a_iX_i)Z_i > x \right) \sim \sum_{i=1}^{n} P((a_iX)Z_i > x).
\]
Asymptotic results [3]

If $X$ and $Y$ follow a lognormal law with $\sigma_Y < \sigma_X$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P} \left( \sum_{i=1}^{n} (a_i + X_i)Z_i > x \right) \sim \sum_{i=1}^{n} \mathbb{P} ((a_i + X)Z_i > x)$$

and that

$$\mathbb{P} \left( \sum_{i=1}^{n} (a_iX_i)Z_i > x \right) \sim \sum_{i=1}^{n} \mathbb{P} ((a_iX)Z_i > x).$$
Asymptotic results [4]

Let

- \( \{\theta_n, n = 1, 2, \ldots\} \) non-negative and dependent.

If \( F_X \in \mathcal{R}_{-\alpha} \) for some \( \alpha > 0 \) and there exists some \( \delta > 0 \) such that \( \mathbb{E}[\theta_i^{\alpha+\delta}] < +\infty \) for each \( 1 \leq i \leq n \), then it holds for each \( n = 1, 2, \ldots \) that

\[
\mathbb{P} \left( \sum_{i=1}^{n} \theta_i X_i > x \right) \sim \sum_{i=1}^{n} \mathbb{P}(\theta_i X > x)
\]

\[
\sim F(x) \sum_{i=1}^{n} \mathbb{E}[\theta_i^{\alpha}].
\]

Holds even uniformly for \( n = 1, 2, \ldots \); see Wang (2005).
Example: Stop-loss premium and Value-at-Risk [1]

Let \( \tilde{S}_n = \sum_{i=1}^{n} \theta_i X_i \). Then

- Stop-loss premium:
  \[
  \mathbb{E}[(\tilde{S}_n - d)_+] \approx \sum_{i=1}^{n} \mathbb{E}[(\theta_i X - d)_+].
  \]

- VaR:
  \[
  \inf\{s : F_{\tilde{S}_n}(s) \geq p\} \approx \inf\left\{s : \sum_{i=1}^{n} F_{\theta_i X}(s) \leq 1 - p\right\}.
  \]
Example: Stop-loss premium and Value-at-Risk [2]

Furthermore, let \( F_X \in \mathcal{R}_{-\alpha} \) for some \( \alpha > 0 \). Then

- Stop-loss premium:
  \[
  \mathbb{E}[(\tilde{S}_n - d)_+] \approx \mathbb{E}[(X - d)_+] \sum_{i=1}^{n} \mathbb{E}[\theta_i^\alpha].
  \]

- VaR:
  \[
  \inf \{ s : F_{\tilde{S}_n}(s) \geq p \} \approx \\
  \inf \left\{ s : \overline{F}_X(s) \sum_{i=1}^{n} \mathbb{E}[\theta_i^\alpha] \leq 1 - p \right\}.
  \]
Example: Stop-loss premium and Value-at-Risk [3]

- \( \theta_n = Y_1 \cdots Y_n \), i.i.d.: \( \mathbb{E}[\theta_n^\alpha] = \mathbb{E}[Y^\alpha]^n \).

- \( (\theta_1, \ldots, \theta_n) \overset{d}{=} \text{LE}_n(\mu_n, \Sigma_n, \phi): \mathbb{E}[\theta_n^\alpha] \) is explicit; see e.g., Fang, Kotz & Ng (1990) and Owen & Rabinovitch (1983).

- \( (\theta_1, \ldots, \theta_n) \overset{d}{=} \text{LNVMM}_n(\mu_n, \beta_n, \Sigma_n, G): \mathbb{E}[\theta_n^\alpha] \) is explicit; see e.g., Barndorff-Nielsen (1997).
## A numerical illustration

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*Notes:* “Real” versus approximate values of stop-loss premiums and quantiles for Pareto losses and i.i.d. lognormal stochastic discount factors. Fixed parameter values: $n = 5, \alpha = 1.5, \mu = -0.04, \sigma = 0.10$ and 5,000,000 simulations.

Analytic approximations!
References [1]


References [2]

