Sample-Path Large Deviations in Credit Risk

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Outline

1. Introduction
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Motivation

- Goal is to characterize loss distribution in large portfolio
- Current techniques focus on single point in time
- Path dependent measures capture more of characteristics
- Consider events: \( \exists t : L(t) > \zeta(t) \) or \( \forall t : L(t) < \xi(t) \)
Model and Notation

- Model loss in portfolio consisting of \( n \) obligors
- Companies identically and independently distributed
- Separately model the default time \( \tau \) and loss given default \( U \)
- Assume \( \tau \) and \( U \) are independent
Model and Notation (2)

- Loss process given by

\[ L_n(t) = \sum_{i=1}^{n} U_i Z_i(t) \]

\[ Z_i(t) = \mathbb{I}_{\{\tau_i \leq t\}} \]

Where \( U_i \sim U \) and \( \tau_i \sim \tau \)

- Consider the loss process on time grid \( \{1, 2, \ldots, N\} \).

- The distribution of the default times given by

\[ p_i := \mathbb{P}(\tau = i) \]

\[ F_i := \sum_{j=1}^{i} p_j \]
Large Deviation Principle

- Let \((\mathcal{X}, d)\) be a metric space.
- Let \(\{\mu_n\}\) be a sequence of measures on Borel sets of \(\mathcal{X}\).
- Study behavior of \(\{\mu_n\}\) as \(n \to \infty\).
- Large Deviation Principle states exponential upper and lower bounds.

**Definition (Rate Function)**

A Rate Function is a lower semicontinuous mapping \(I : \mathcal{X} \to [0, \infty]\). This means that for all \(\alpha \in [0, \infty)\) the set \(\{x \mid I(x) \leq \alpha\}\) is a closed subset of \(\mathcal{X}\).
Definition (Large Deviation Principle)

We say that \( \{\mu_n\} \) satisfies the Large Deviation Principle (LDP) with rate function \( I(\cdot) \) if

(i) *(Upper bound)* for all closed \( F \subseteq \mathcal{X} \)

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(F) \leq - \inf_{x \in F} I(x)
\]

(ii) *(Lower bound)* for all open \( G \subseteq \mathcal{X} \)

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(G) \geq - \inf_{x \in G} I(x)
\]
Definition (Large Deviation Principle (continued))

We say that a family of random variables $X = \{X_n\}$, with values in $\mathcal{X}$, satisfies a large deviation principle with rate function $I_X(\cdot)$ iff the laws $\{\mu^X_n\}$ satisfy a large deviation principle with rate function $I_X(\cdot)$.

Definition (Fenchel-Legendre Transform)

Let $X$ be a random variable. The Fenchel-Legendre Transform is given by

$$\Lambda^+_X(x) = \sup_{\theta} (\theta x - \Lambda_X(\theta))$$

where $\Lambda_X$ is the logarithmic moment generating function of $X$

$$\Lambda_X(\theta) = \log \left( \mathbb{E} e^{\theta X} \right)$$
Cramér’s Theorem

Theorem (Cramér)

Let \( \{X_i\} \) be i.i.d. sequence of random variables and let \( \mu_n \) be the law of the average \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i \). Then \( \{\mu_n\} \) satisfies an LDP with rate function \( \Lambda_X^*(\cdot) \).

Example (Loss Process)

For any \( T > 0 \), the average loss process \( L_n(T)/n \) satisfies a Large Deviation Principle, where the rate function is given by the Legendre-Fenchel transform of the variable \( U_{Z(T)} = U_{\{\tau \leq T\}} \), so

\[
I(x) = \Lambda_{U_{Z(T)}}^*(x)
\]
Additional Notation

- Finite time grid $T_N = \{t_1 < t_2 < \cdots < t_N\}$, or for simplicity $T_N = \{1, 2, \ldots, N\}$

- Space of all nonnegative and nondecreasing functions on $T_N$:
  \[
  S = \{ f : T_N \to \mathbb{R}^+ | 0 \leq f_i \leq f_{i+1}, \text{ for } i < N \}
  \]

- Topology on induced by supremum norm $\|f\| = \max_i |f_i|$

- Space of all probability measures on $T_N$
  \[
  \Phi = \left\{ \varphi \in \mathbb{R}^N | \sum_{i=1}^{N} \varphi_i = 1, \varphi_i \geq 0, i \leq N \right\}
  \]
Sample-Path Large Deviation Principle

**Theorem**

Let $\Lambda_U(\theta) < \infty$ for all $\theta$. Then the path of the average loss process $L_n(\cdot)/n$, on the points $\{1, 2, \ldots, N\}$, satisfies a Large Deviation Principle with rate function $I_{U,p}$. Here, for $x \in S$, $I_{U,p}$ is given by

$$I_{U,p} = \inf_{\varphi \in \Phi} \sum_{i=1}^{N} \varphi_i \left( \log \left( \frac{\varphi_i}{p_i} \right) + \Lambda_U^* \left( \frac{\Delta x_i}{\varphi_i} \right) \right)$$

with $\Delta x_i = x_i - x_{i-1}$ and $x_0 = 0$.

**Remarks:**

- Decompose influence of default times and losses given default
- Optimizing $\varphi$ can be interpreted as most like loss distribution, given path of $L_n(\cdot)/n$ is close to $x$
Example 1

Example

Let the loss amount $U$ have finite support on $[0, u]$. Then $\Lambda_U(\theta) < \infty$ for all $\theta$ as

$$\Lambda_U(\theta) = \log \left( \mathbb{E} e^{\theta U} \right) \leq \theta u < \infty$$

So the loss process $L_n(\cdot)/n$ satisfies the sample-path LDP.

In practice loss amounts are finite, thus any realistic model for the loss distribution satisfies the sample path LDP.
Example 2

Assume loss amount $U$ is measured in units $u > 0$, e.g. $u, 2u, \ldots$. Assume that $U$ has Poisson-like distribution with parameter $\lambda$, such that for $i = 1, 2, \ldots$

$$\mathbb{P}(U = (i + 1)u) = e^{-\lambda} \frac{\lambda^i}{i!}$$

Then then $\Lambda_U$ is given by

$$\Lambda_U(\theta) = \theta u + \lambda \left( e^{\theta u} - 1 \right)$$

which is finite for all $\theta$, showing that the sample-path LDP is satisfied for a distribution with infinite support.
Remarks and Extensions

- Sample-path LDP is valid for wide range of distributions
- Assumptions not realistic, e.g. independent and identical distributions
- In practice defaults clearly not independent
- Different types of obligors can be distinguished
- Finite grid might be too restrictive
Dependent Defaults

- Relax assumption that obligors are independent
- Use so-called (factor) copula approach
- Conditional on a factor $Y$, the default times and loss amounts are independent
- Apply theorem conditional on realization of $Y$, yielding conditional decay rate $r_y$

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{1}{n} L_n(\cdot) \in A \mid Y = y \right) = r_y$$

- When $Y$ has finite outcomes, say in $\mathcal{Y}$, the unconditional decay rate $r$ is given as $r = \max \{ r_y \mid y \in \mathcal{Y} \}$
Different Types

- Relax assumption that obligors are identically distributed
- Assume that there are $m$ different classes, default ratings for example
- Each class makes up fraction $a_i$ of portfolio
- Split loss process $L_n$ into $m$ sub-loss processes, and condition on realizations, which gives rate function

$$I_{U,p,m}(x) = \inf_{\varphi \in \Phi^m} \inf_{v \in V_x} \sum_{j=1}^{m} \sum_{i=1}^{N} a_i \varphi_i^j \left( \log \left( \frac{\varphi_i^j}{p_i^j} \right) + \Lambda^*_U \left( \frac{v_i^j}{a_i \varphi_i^j} \right) \right)$$

$$V_x = \left\{ v \in \mathbb{R}_+^{m \times N} \mid \sum_{j=1}^{m} v_i^j = \Delta x_i \text{ for all } i \leq N \right\}$$

$$\Phi^m = \Phi \times \ldots \times \Phi, \ (m \text{ times})$$
Extend Finite Grid

- Extend current grid \( \{1, 2, \ldots, N\} \) to \( \mathbb{N} \)
- Expected rate function \( I_{U,p,\infty} \):
  \[
  I_{U,p,\infty}(x) = \inf_{\varphi \in \Phi_{\infty}} \sum_{i=1}^{\infty} \varphi_i \left( \log \left( \frac{\varphi_i}{\rho_i} \right) + \Lambda^*_U \left( \frac{\Delta x_i}{\varphi_i} \right) \right)
  \]
- Extend from grid \( \{1, 2, \ldots, N\} \) to interval \( [0, N] \)
- Expected rate function \( I_{U,p,[0,N]} \):
  \[
  I_{U,p,[0,N]}(x) := \inf_{\varphi \in \mathcal{M}} \int_0^N \varphi(t) \left( \log \left( \frac{\varphi(t)}{p(t)} \right) + \Lambda^*_U \left( \frac{x'(t)}{\varphi(t)} \right) \right) dt
  \]
The sample-path LDP provides bounds for the exponential decay rate.

It does not provide exact expression for \( \mathbb{P} \left( \frac{1}{n} L_n(\cdot) \in A \right) \).

For certain events it is possible to obtain exact expression, resulting in expressions like

\[
\mathbb{P} \left( \frac{1}{n} L_n(\cdot) \in A \right) = \frac{C e^{-I_L n}}{\sqrt{n}} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

For certain constants \( C \) and \( I_L \).
Bahadur-Rao Theorem

- The exact asymptotic results depend on Bahadur-Rao theorem

**Theorem (Bahadur-Rao)**

Let \( X_i \) be an i.i.d. real valued sequence of random variables. Then we have

\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq q \right) = \frac{e^{-n \Lambda^*_X(q)} C_{X,q}}{\sqrt{n}} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

\[
C_{X,q} = \frac{1}{\sigma \sqrt{2\pi \Lambda''_X(\sigma)}}
\]

\[
\Lambda'_X(\sigma) = q
\]
Crossing a Barrier

- Work on the infinite time grid \( T = \mathbb{N} \)

- Consider the event that at some point in time \( t \) the loss is above some threshold \( \zeta(t) \)

\[
\left\{ \exists t \in T \mid \frac{1}{n} L_n(t) > \zeta(t) \right\}
\]

- Need that \( \zeta(t) > \mathbb{E}[U] \mathbb{P}(\tau \leq t) \)

- Determine loss path quantiles
Crossing a Barrier(2)

Theorem

Assume that there exists unique \( t^* \in T \) such that

\[
I_{UZ}(t^*) = \min_{t \in T} I_{UZ}(t),
\]

and assume that

\[
\liminf_{t \to \infty} \frac{I_{UZ}(t)}{\log t} > 0,
\]

where \( I_{UZ}(t) = \sup_{\theta} \{ \theta \zeta(t) - \Lambda_{UZ}(t)(\theta) \} = \Lambda_{UZ}^*(t)(\zeta(t)) \). Then

\[
P \left( \exists t \in T \text{ s.t. } \frac{1}{n} L_n(t) > \zeta(t) \right) = \frac{e^{-nI_{UZ}(t^*)} C^*}{\sqrt{n}} \left( 1 + O \left( \frac{1}{n} \right) \right)
\]

Where \( \sigma^* \) is such that \( \Lambda'_{UZ}(t^*)(\sigma^*) = \zeta(t^*) \). The constant \( C^* \) follows from the Bahadur-Rao theorem, with \( C^* = C_{UZ}(t^*),\zeta(t^*) \).
Remarks

- Same type of decay rate as in Bahadur-Rao theorem
- Clearly it holds that
  \[ \mathbb{P} \left( \exists t \in T \text{ s.t. } \frac{1}{n} L_n(t) > \zeta(t) \right) \geq \sup_{t \in T} \mathbb{P} \left( \frac{1}{n} L_n(t) > \zeta(t) \right). \]
- The theorem shows that this bound is tight
- The maximizing \( t^\star \) dominates the contributions. So given the extreme event occurs, it will, with overwhelming probability, happen at time \( t^\star \)
- Relaxing the uniqueness requirements yields similar decay rate, but we lack a clean expression for the proportionality constant
- The second assumption makes sure that we can ignore the 'upper tail'
Lemma

The condition

\[
\lim_{t \to \infty} \inf \frac{I_{UZ}(t)}{\log t} > 0
\]

is satisfied, when

\[
\Lambda^*_U(x)/x \to \infty
\]
\[
\lim_{t} \inf \frac{\zeta(t)}{\log t} > 0
\]

Remarks

- Condition only depends on distribution of losses, and not of default times
- First condition holds quite general
Second condition follows from

\[ \Lambda_{UZ}(t)(\theta) = \log \mathbb{P}(\tau \leq t) \mathbb{E}[e^{\theta U}] + \mathbb{P}(\tau > t) \]

\[ \leq \log \mathbb{E}[e^{\theta U}] \]

\[ l_{UZ}(t) = \Lambda^*_{UZ(t^*)}(\zeta(t)) \]

\[ \geq \Lambda^*_U(\zeta(t)) = \sup_{\theta} \left( \theta \zeta(t) - \log \mathbb{E}[e^{\theta U}] \right) \]

\[ \liminf_{t \to \infty} \frac{\Lambda^*_U(\zeta(t))}{\log t} = \liminf_{t \to \infty} \frac{\Lambda^*_U(\zeta(t)) \zeta(t)}{\log t} > 0 \]
Large Increments of Loss Process

- Look at increments of the average loss process
  \[ \frac{1}{n} (L_n(t) - L_n(s)), \text{ for } s < t, \text{ exceeding a threshold } \xi(s, t) \]
- Need that \( \xi(s, t) > \mathbb{E}[U] (\mathbb{P}(\tau \leq t) - \mathbb{P}(\tau \leq s)) \)

Assumptions

- There is a unique \( s^* < t^* \in T \) such that
  \[ I_{UZ}(s^*, t^*) = \min_{s < t} I_{UZ}(s, t), \]
- Write \( I_{UZ}(s, t) = \sup_\theta (\theta \xi(s, t) - \Lambda_U(Z(t) - Z(s))(\theta)) = \Lambda^*_U(Z(t) - Z(s))(\xi(s, t)) \). and let
  \[ \inf_{s \in T} \liminf_{t \to \infty} \frac{I_{UZ}(s, t)}{\log t} > 0, \]
Large Increments of Loss Process(2)

Theorem

Under these assumptions

$$\mathbb{P} \left( \exists s < t : \frac{1}{n} (L_n(t) - L_n(s)) > \xi(s, t) \right)$$

$$= \frac{e^{-n l_{UZ}(s^*, t^*)} C^*}{\sqrt{n}} \left( 1 + O \left( \frac{1}{n} \right) \right),$$

where $\sigma^*$ is such that $N'_{U(Z(t^*)-Z(s^*))}(\sigma^*) = \xi(s^*, t^*)$. The constant $C^*$ follows from the Bahadur-Rao theorem, with $C^* = C_{U(Z(t^*)-Z(s^*)), \xi(s^*,t^*)}$. 
Remarks

- Result is very similar to result for crossing a barrier

- Conditions look quite restrictive and difficult to check

- However, the following is sufficient

\[ \liminf_{t \to \infty} \frac{\xi(s, t)}{\log t} > 0 \]

For the latter assumption
Conclusions

- Established sample-path LDP for the average loss process $L_n(t)/n$
- Shown how results can be extended
- Future research to formally prove the extensions
- Established the exact asymptotic behavior of the probability of ever crossing a barrier
- Established the exact asymptotic behavior of probability that loss increments cross a certain barrier