9

A Short Introduction to Lévy Processes

In this chapter we present some basic definitions and results about Lévy processes. We do not aim at being complete in our presentation, but for a smoother reading, we like to include the notions we constantly refer to in this book. We can address the reader to the recent monographs, for example, [8, 32, 115, 213] for a deeper study of Lévy processes. Our presentation follows the survey given in [183, Chap. 1].

9.1 Basics on Lévy Processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 9.1. *A one-dimensional* Lévy process *is a stochastic process* $\eta = \eta(t)$, $t \geq 0$*:*

$$\eta(t) = \eta(t, \omega), \quad \omega \in \Omega,$$

with the following properties:

(i) $\eta(0) = 0$ *P-a.s.,*

(ii) η *has independent increments, that is, for all* $t > 0$ *and* $h > 0$*, the increment* $\eta(t+h) - \eta(t)$ *is independent of* $\eta(s)$ *for all* $s \leq t$*,*

(iii) η *has stationary increments, that is, for all* $h > 0$ *the increment* $\eta(t+h) - \eta(t)$ *has the same probability law as* $\eta(h)$*,*

(iv) It is stochastically continuous, that is, for every $t \geq 0$ *and* $\varepsilon > 0$ *then* $\lim_{s \to t} P\{|\eta(t) - \eta(s)| > \varepsilon\} = 0$*,*

(v) η *has càdlàg paths, that is, the trajectories are right-continuous with left limits.*

G.Di Nunno et al., *Malliavin Calculus for Lévy Processes with Applications to Finance*,

A stochastic process η satisfying (i)–(iv) is called a *Lévy process in law.*

The *jump of η at time t* is defined by

$$\Delta\eta(t) := \eta(t) - \eta(t^-).$$

Put $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ be the σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset \mathbb{R}_0$. If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we define

$$N(t, U) := \sum_{0 \le s \le t} \chi_U(\Delta\eta(s)), \tag{9.6}$$

that is, the number of jumps of size $\Delta\eta(s) \in U$ for any s in $0 \le s \le t$. Since the paths of η are càdlàg we can see that $N(t, U) < \infty$ for all $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$; see, e.g. [213]. Moreover, (9.6) defines in a natural way a Poisson random measure N on $\mathcal{B}(0, \infty) \times \mathcal{B}(\mathbb{R}_0)$ given by

$$(a, b] \times U \longmapsto N(b, U) - N(a, U), \qquad 0 < a \le b,\ U \in \mathcal{B}(\mathbb{R}_0),$$

and its standard extension. See e.g. [97], [79]. We call this random measure the *jump measure of η*. Its differential form is denoted by $N(dt, dz)$, $t > 0$, $z \in \mathbb{R}_0$.

The *Lévy measure* ν of η is defined by

$$\nu(U) := E\big[N(1, U)\big], \quad U \in \mathcal{B}(\mathbb{R}_0). \tag{9.7}$$

It is important to note that ν does not need to be a finite measure. It can be possible that

$$\int_{\mathbb{R}_0} \min(1, |z|)\nu(dz) = \infty. \tag{9.8}$$

This is the case when the trajectories of η would appear with many jumps of small size, a situation that is of interest in financial modeling (see, e.g., [17, 48, 75, 216] and references therein).

On the contrary, the Lévy measure always satisfies

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty.$$

In fact, a measure ν on $\mathcal{B}(\mathbb{R}_0)$ can be a Lévy measure of some Lévy process η if and only if the condition above holds true. This is due to the following theorem.

Theorem 9.2. The Lévy–Khintchine formula.

(1) Let η be a Lévy process in law. Then

$$E\big[e^{iu\eta(t)}\big] = e^{i\Psi(u)}, \quad u \in \mathbb{R} \quad (i = \sqrt{-1}), \tag{9.9}$$

with the characteristic exponent

$$\Psi(u) := i\alpha u - \frac{1}{2}\sigma^2 u^2 + \int_{|z|<1} \big(e^{iuz} - 1 - iuz\big)\nu(dz) + \int_{|z|\geq 1} \big(e^{iuz} - 1\big)\nu(dz), \tag{9.10}$$

where the parameters $\alpha \in \mathbb{R}$ *and* $\sigma^2 \geq 0$ *are constants and* $\nu = \nu(dz)$, $z \in \mathbb{R}_0$*, is a* σ*-finite measure on* $\mathcal{B}(\mathbb{R}_0)$ *satisfying*

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty. \tag{9.11}$$

It follows that ν *is the Lévy measure of* η.

(2) Conversely, given the constants $\alpha \in \mathbb{R}$ *and* $\sigma^2 \geq 0$ *and the* σ*-finite measure* ν *on* $\mathcal{B}(\mathbb{R}_0)$ *such that* (9.11) *holds, then there exists a process* η *(unique in law) such that* (9.9) *and* (9.10) *hold. The process* η *is a Lévy process in law.*

There always exists a càdlàg version of the above Lévy process in law (see, e.g., [213]), which is a Lévy process, cf. Definition 9.1. Using this càdlàg version, we can give the representation (9.7) of the σ-finite measure ν.

We define the *compensated jump measure* $\tilde{N}$, also called the *compensated Poisson random measure*, by

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt. \tag{9.12}$$

For any t, let $\mathcal{F}_t$ be the σ-algebra generated by the random variables $W(s)$ and $\widetilde{N}(ds, dz)$, $z \in \mathbb{R}_0$, $s \leq t$, augmented for all the sets of P-zero probability. Let us equip the given probability space $(\Omega, \mathcal{F}, P)$ with the corresponding filtration

$$\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}.$$

A stochastic process $\theta = \theta(t, z)$, $t \geq 0, z \in \mathbb{R}_0$, is called $\mathbb{F}$-adapted if for all $t \geq 0$ and for all $z \in \mathbb{R}_0$, the random variable $\theta(t, z) = \theta(t, z, \omega)$, $\omega \in \Omega$, is $\mathcal{F}_t$-measurable. For any $\mathbb{F}$-adapted process θ such that

$$E\Big[\int_0^T \int_{\mathbb{R}_0} \theta^2(t, z)\nu(dz)dt\Big] < \infty \quad \text{for some } T > 0, \tag{9.13}$$

we can see that the process

$$M_n(t) := \int_0^t \int_{|z|\geq \frac{1}{n}} \theta(s, z)\tilde{N}(ds, dz), \quad 0 \leq t \leq T,$$

is a martingale in $L^2(P)$ and its limit

$$M(t) := \lim_{n\to\infty} M_n(t) := \int_0^t \int_{\mathbb{R}_0} \theta(s, z)\tilde{N}(ds, dz), \quad 0 \leq t \leq T, \tag{9.14}$$

in $L^2(P)$ is also a martingale. Moreover, we have the Itô isometry

$$E\Big[\Big(\int_0^T \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(dt,dz)\Big)^2\Big] = E\Big[\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\nu(dz)dt\Big]. \tag{9.15}$$

A Wiener process is a special case of a Lévy process. In fact, we have the following general representation theorem (see, e.g., [119, 213].

Theorem 9.3. The Lévy–Itô decomposition theorem. *Let η be a Lévy process. Then $\eta = \eta(t)$, $t \geq 0$, admits the following integral representation*

$$\eta(t) = a_1 t + \sigma W(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds,dz) + \int_0^t \int_{|z|\geq 1} zN(ds,dz) \tag{9.16}$$

for some constants $a_1, \sigma \in \mathbb{R}$. Here $W = W(t)$, $t \geq 0$ ($W(0) = 0$), is a standard Wiener process.

In particular, we can see that if the Lévy process has continuous trajectories, then it is of the form

$$\eta(t) = a_1 t + \sigma W(t), \quad t \geq 0.$$

It can be proved that if

$$E\big[|\eta(t)|^p\big] < \infty \quad \text{for some } p \geq 1,$$

then

$$\int_{|z|\geq 1} |z|^p \nu(dz) < \infty,$$

see [213]. In particular, if we assume that

$$E\big[\eta^2(t)\big] < \infty, \qquad t \geq 0, \tag{9.17}$$

then we have

$$\int_{|z|\geq 1} |z|^2 \nu(dz) < \infty$$

and the representation (9.16) appears as

$$\eta(t) = at + \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz), \tag{9.18}$$

where $a = a_1 + \int_{|z|\geq 1} z\nu(dz)$. A Lévy process of the type above with $\sigma = 0$ is called a *pure jump Lévy process.*

We assume from now on that (9.17) holds and hence that η has the representation (9.18).

Motivated by the representation (9.18), it is natural to consider processes $X = X(t)$, $t \geq 0$, admitting a stochastic integral representation in the form

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz), \tag{9.19}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t,z)$ are predictable processes such that, for all $t > 0$, $z \in \mathbb{R}_0$,

$$\int_0^t \big[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\big] ds < \infty, \qquad P\text{-}a.s. \tag{9.20}$$

This condition implies that the stochastic integrals are well-defined and local martingales. If we strengthened the condition to

$$E\Big[\int_0^t \big[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\big] ds\Big] < \infty,$$

for all $t > 0$, then the corresponding stochastic integrals are martingales.

We call such a process an *Itô–Lévy process*. In analogy with the Brownian motion case, we use the short-hand differential notation

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz); \qquad X(0) = x \tag{9.21}$$

for the processes of type (9.19).

Recall that a *predictable process* is a stochastic process measurable with respect to the σ-algebra generated by

$$A \times (s,u] \times B, \quad A \in \mathcal{F}_s,\ 0 \le s < u,\ B \in \mathcal{B}(\mathbb{R}_0).$$

Moreover, any measurable $\mathbb{F}$-adapted and left-continuous (with respect to t) process is predictable.

9.2 The Itô Formula

The following result is fundamental in the stochastic calculus of Lévy processes.

Theorem 9.4. The one-dimensional Itô formula. *Let $X = X(t)$, $t \ge 0$, be the Itô–Lévy process given by* (9.19) *and let $f : (0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $C^{1,2}((0,\infty) \times \mathbb{R})$ and define*

$$Y(t) := f(t, X(t)), \qquad t \ge 0.$$

Then the process $Y = Y(t)$, $t \ge 0$, is also an Itô–Lévy process and its differential form is given by

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))\alpha(t)dt + \frac{\partial f}{\partial x}(t,X(t))\beta(t)dW(t) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\beta^2(t)dt + \int_{\mathbb{R}_0} \Big[f(t,X(t)+\gamma(t,z)) - f(t,X(t)) \\ &- \frac{\partial f}{\partial x}(t,X(t))\gamma(t,z)\Big]\nu(dz)dt + \int_{\mathbb{R}_0} \Big[f(t,X(t^-)+\gamma(t,z)) - f(t,X(t^-))\Big]\tilde{N}(dt,dz). \end{aligned} \tag{9.22}$$

In the multidimensional case we are given a J-dimensional Brownian motion $W(t) = (W_1(t), ..., W_J(t))^T$, $t \geq 0$, and K independent compensated Poisson random measures $\tilde{N}(dt, dz) = (\tilde{N}_1(dt, dz_1), ..., \tilde{N}_K(dt, dz_K))^T$, $t \geq 0$, $z = (z_1, ..., z_K) \in (\mathbb{R}_0)^K$, and n Itô–Lévy processes of the form

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{(\mathbb{R}_0)^K} \gamma(t, z)\tilde{N}(dt, dz), \quad t \geq 0,$$

that is,

$$dX_i(t) = \alpha_i(t)dt + \sum_{j=1}^{J} \beta_{ij}(t)dW_j(t) + \sum_{k=1}^{K} \int_{\mathbb{R}_0} \gamma_{ik}(t, z_k)\tilde{N}_k(dt, dz_k), \quad i = 1, ..., n. \tag{9.23}$$

With this notation we have the following result.

Theorem 9.5. The multidimensional Itô formula. *Let $X = X(t)$, $t \geq 0$, be an n-dimensional Itô–Lévy process of the form* (9.23). *Let $f : (0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function in $C^{1,2}((0, \infty) \times \mathbb{R}^n)$ and define*

$$Y(t) := f(t, X(t)), \qquad t \geq 0.$$

Then the process $Y = Y(t)$, $t \geq 0$, is a one-dimensional Itô–Lévy process and its differential form is given by

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X(t))\alpha_i(t)dt \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{J} \frac{\partial f}{\partial x_i}(t, X(t))\beta_{ij}(t)dW_j(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{J} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X(t))(\beta\beta^T)_{ij}(t)dt \\ &+ \sum_{k=1}^{K} \int_{\mathbb{R}_0} \Big[f(t, X(t) + \gamma^{(k)}(t, z)) - f(t, X(t)) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} f(t, X(t))\gamma_{ik}(t, z)\Big]\nu_k(dz_k)dt \\ &+ \sum_{k=1}^{K} \int_{\mathbb{R}_0} \Big[f(t, X(t^-) + \gamma^{(k)}(t, z)) - f(t, X(t^-))\Big]\tilde{N}_k(dt, dz_k), \end{aligned} \tag{9.24}$$

where $\gamma^{(k)}$ is the column number k of the $n \times K$ matrix $\gamma = \big[\gamma_{ik}\big]$.

Example 9.6. **The generalized geometric Lévy process.** Consider the one-dimensional stochastic differential equation for the càdlàg process $Z = Z(t)$, $t \geq 0$:

$$\begin{cases} dZ(t) = Z(t^-)\Big[\alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz)\Big], & t > 0, \\ Z(0) = z_0 > 0. \end{cases} \tag{9.25}$$

Here $\alpha(t)$, $\beta(t)$, and $\gamma(t, z)$, $t \geq 0$, $z \in \mathbb{R}_0$, are given predictable processes with $\gamma(t, z) > -1$, for almost all $(t, z) \in [0, \infty) \times \mathbb{R}_0$ and for all $0 < t < \infty$

$$\int_0^t \left[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s,z)\nu(dz)\right] ds < \infty \quad P\text{-}a.s.,$$

cf. (9.20). We claim that the solution of this equation is

$$Z(t) = z_0 e^{X(t)}, \quad t \geq 0, \tag{9.26}$$

where

$$\begin{aligned} X(t) = &\int_0^t \Big[\alpha(s) - \frac{1}{2}\beta^2(s) + \int_{\mathbb{R}_0} \big[\log(1+\gamma(s,z)) - \gamma(s,z)\big]\nu(dz)\Big] ds \\ &+ \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \log(1+\gamma(s,z))\tilde{N}(ds,dz). \end{aligned} \tag{9.27}$$

To see this we apply the one-dimensional Itô formula to $Y(t) = f(t, X(t))$, $t \geq 0$, with $f(t,x) = z_0 e^x$ and $X(t)$, as given in (9.27). Then we obtain

$$\begin{aligned} dY(t) = &z_0 e^{X(t)} \Big[\Big(\alpha(t) - \frac{1}{2}\beta^2(t) + \int_{\mathbb{R}_0} \big[\log(1+\gamma(t,z)) - \gamma(t,z)\big]\nu(dz)\Big)dt + \beta(t)dW(t)\Big] \\ &+ z_0 e^{X(t)} \frac{1}{2}\beta^2(t)dt + \int_{\mathbb{R}_0} z_0 \big[e^{X(t)+\log(1+\gamma(t,z))} - e^{X(t)} - e^{X(t)}\log(1+\gamma(t,z))\big]\nu(dz)dt \\ &+ \int_{\mathbb{R}_0} z_0 \big[e^{X(t^-)+\log(1+\gamma(t,z))} - e^{X(t^-)}\big]\tilde{N}(dt,dz) \\ = &Y(t^-)\Big[\alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)\Big] \end{aligned}$$

as required.

Example 9.7. **The quadratic covariation process.** Let

$$dX_i(t) = \int_{\mathbb{R}_0} \gamma_{i1}(t,z)\tilde{N}_1(dt,dz) + \int_{\mathbb{R}_0} \gamma_{i2}(t,z)\tilde{N}_2(dt,dz), \quad i = 1,2 \tag{9.28}$$

be two pure jump Lévy processes (see (9.23) with $\beta_{ij} = 0$. Define

$$Y(t) = X_1(t)X_2(t), \quad t \geq 0.$$

Then by the two-dimensional Itô formula we have

$$\begin{aligned} dY(t) = &\sum_{k=1,2} \int_{\mathbb{R}_0} \big[(X_1(t) + \gamma_{1k}(t,z_k))(X_2(t) + \gamma_{2k}(t,z_k)) - X_1(t)X_2(t) \\ &- \gamma_{1k}(t,z_k)X_2(t) - \gamma_{2k}(t,z_k)X_1(t)\big]\nu_k(dz_k)dt \\ &+ \sum_{k=1,2} \int_{\mathbb{R}_0} \big[(X_1(t^-) + \gamma_{1k}(t,z_k))(X_2(t^-) + \gamma_{2k}(t,z_k)) - X_1(t^-)X_2(t^-)\big]\tilde{N}_k(dt,dz_k) \\ = &\sum_{k=1,2} \int_{\mathbb{R}_0} \gamma_{1k}(t,z_k)\gamma_{2k}(t,z_k)\nu_k(dz_k)dt \\ &+ \sum_{k=1,2} \int_{\mathbb{R}_0} \big[\gamma_{1k}(t,z_k)X_2(t^-) + \gamma_{2k}(t,z_k)X_1(t^-) + \gamma_{1k}(t,z_k)\gamma_{2k}(t,z_k)\big]\tilde{N}_k(dt,dz_k) \\ = &X_1(t^-)dX_2(t) + X_2(t^-)dX_1(t) + \sum_{k=1,2} \int_{\mathbb{R}_0} \gamma_{1k}(t,z_k)\gamma_{2k}(t,z_k)N_k(dt,dz_k). \end{aligned}$$

We define the *quadratic covariation process* $[X_1, X_2](t)$, $t \geq 0$, of the processes X_1 and X_2 as

$$d[X_1, X_2](t) := d(X_1(t)X_2(t)) - X_1(t^-)dX_2(t) - X_2(t^-)dX_1(t). \tag{9.29}$$

Hence, for the processes X_1, X_2 given in (9.28), we have that

$$d[X_1, X_2](t) = \sum_{k=1,2} \int_{\mathbb{R}_0} \gamma_{1k}(t, z_k)\gamma_{2k}(t, z_k)N_k(dt, dz_k). \tag{9.30}$$

In particular, note that

$$E\big[X_1(t)X_2(t)\big] = X_1(0)X_2(0) + \sum_{k=1,2} \int_0^t \int_{\mathbb{R}_0} \gamma_{1k}(s, z_k)\gamma_{2k}(s, z_k)\nu_k(dz_k)ds. \tag{9.31}$$

9.3 The Itô Representation Theorem for Pure Jump Lévy Processes

We now proceed to prove the Itô representation theorem for Lévy processes. Since we already know the representation theorem in the continuous case, that is, $\tilde{N} \equiv 0$ (see [179] and Problem 1.4), we concentrate on the pure jump case in this section. We assume that

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz), \quad t \geq 0, \tag{9.32}$$

that is, $a = \sigma = 0$ in (9.18).

The following representation theorem was first proved by Itô [121]. Here we follow the presentation given in [154]. The proof is based on two lemmata.

Let us consider the filtration $\mathbb{F}$ of the σ-algebras $\mathcal{F}_t$ generated by $\eta(s)$, $s \leq t$ $(t \geq 0)$.

Lemma 9.8. *The set of all random variables of the form*

$$\big\{f(\eta(t_1), ..., \eta(t_n)) : \; t_i \in [0, T],\, i = 1, ..., n;\; f \in C_0^\infty(\mathbb{R}^n),\, n = 1, 2, ...\big\}$$

is dense in the subspace $L^2(\mathcal{F}_T, P) \subset L^2(P)$ *of* $\mathcal{F}_T$*-measurable square integrable random variables.*

Proof The proof follows the same argument as in Lemma 4.3.1 in [179]. See also, for example, [154]. □

Lemma 9.9. *The linear span of all the so-called Wick/Doléans–Dade exponentials*

$$\exp\Big\{ \int_0^T \int_{\mathbb{R}_0} h(t)z\chi_{[0,R]}(z)\tilde{N}(dt, dz) - \int_0^T \int_{\mathbb{R}_0} \big[e^{h(t)z\chi_{[0,R]}(z)} - 1 - h(t)z\chi_{[0,R]}(z)\big]\nu(dz)dt\Big\}, \quad h \in C(0, T),\; R > 0,$$

is dense in $L^2(\mathcal{F}_T, P)$.

Proof The proof follows the same argument as in Lemma 4.3.2 in [179]. See also, for example, [154]. □

We are ready now for the first main result of this section. Note that the representation is in terms of a stochastic integral with respect to $\tilde{N}$ and *not* with respect to η.

Theorem 9.10. The Itô representation theorem. *Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable. Then there exists a unique predictable process $\Psi = \Psi(t,z)$, $t \geq 0, z \in \mathbb{R}_0$, such that*

$$E\Big[\int_0^T \int_{\mathbb{R}_0} \Psi^2(t,z)\nu(dz)dt\Big] < \infty$$

for which we have

$$F = E[F] + \int_0^T \int_{\mathbb{R}_0} \Psi(t,z)\tilde{N}(dt,dz). \tag{9.33}$$

Proof First assume that $F = Y(T)$, where

$$Y(t) = \exp\Big\{ \int_0^t \int_{\mathbb{R}_0} h(s)z\chi_{[0,R]}(z)\tilde{N}(ds,dz) - \int_0^t \int_{\mathbb{R}_0} \big[e^{h(s)z\chi_{[0,R]}(z)} - 1 - h(s)z\chi_{[0,R]}(z)\big]\nu(dz)ds\Big\}, \quad t \in [0,T],$$

for some $h \in C(0,\infty)$, that is, F is a Wick/Doléans–Dade exponential. Then by the Itô formula (cf. Theorem 9.4 and see Problem 9.2)

$$dY(t) = Y(t^-)\int_{\mathbb{R}_0} \big[e^{h(t)z\chi_{[0,R]}(z)} - 1\big]\tilde{N}(dt,dz).$$

Therefore,

$$F = Y(T) = Y(0) + \int_0^T 1dY(t) = 1 + \int_0^T \int_{\mathbb{R}_0} Y(t^-)\big[e^{h(t)z\chi_{[0,R]}(z)} - 1\big]\tilde{N}(dt,dz).$$

So for this F the representation (9.33) holds with

$$\Psi(t,z) = Y(t^-)\big[e^{h(t)z\chi_{[0,R]}(z)} - 1\big].$$

Note that

$$E\big[Y^2(T)\big] = 1 + E\Big[\int_0^T \int_{\mathbb{R}_0} Y^2(t^-)\big(e^{h(t)z\chi_{[0,R]}(z)} - 1\big)^2\nu(dz)dt\Big].$$

If $F \in L^2(\mathcal{F}_T, P)$ (i.e., an $\mathcal{F}_T$-measurable random variable in $L^2(P)$) is arbitrary, we can choose a sequence F_n of linear combinations of Wick/Doléan–Dade exponentials such that $F_n \longrightarrow F$ in $L^2(P)$. See Lemma 9.9. Then we have

$$F_n = E[F_n] + \int_0^T \int_{\mathbb{R}_0} \Psi_n(t,z)\tilde{N}(dt,dz),$$

for all $n = 1, 2, ...$, where

$$E[F_n^2] = (E[F_n])^2 + E\Big[\int_0^T \int_{\mathbb{R}_0} \Psi_n^2(t,z)\nu(dz)dt\Big] < \infty.$$

Then by the Itô isometry we have the expression

$$E\big[(F_m - F_n)^2\big] = \big(E[F_m - F_n]\big)^2 + E\Big[\int_0^T \int_{\mathbb{R}_0} \big(\Psi_m(t,z) - \Psi_n(t,z)\big)^2 \nu(dz)dt\Big],$$

which vanishes for $m, n \to \infty$. Therefore, Ψ_n, $n = 1, 2, ...$, is a Cauchy sequence in $L^2(P \times \lambda \times \nu)$; hence, it converges to a limit $\Psi \in L^2(P \times \lambda \times \nu)$. This yields (9.33), in fact

$$\begin{aligned} F = \lim_{n\to\infty} F_n = \lim\nolimits_{n\to\infty} \Big\{EF_n + \textstyle\int_0^T \int_{\mathbb{R}_0} \Psi_n(t,z)\tilde{N}(dt,dz)\Big\} \\ = EF + \textstyle\int_0^T \int_{\mathbb{R}_0} \Psi(t,z)\tilde{N}(dt,dz). \end{aligned}$$

The uniqueness is given by the convergence in L^2-spaces and the Itô isometry.□

Example 9.11. Choose $F = \eta^2(T)$. To find the representation (9.33) for F we define

$$Y(t) = \eta^2(t) = \Big(\int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz)\Big)^2, \quad t \in [0,T].$$

By the Itô formula

$$\begin{aligned} d\eta^2(t) &= \int_{\mathbb{R}_0} \big[(\eta(t) + z)^2 - \eta^2(t) - 2\eta(t)z\big]\nu(dz)dt \\ &\quad + \int_{\mathbb{R}_0} \big[(\eta(t^-) + z)^2 - \eta^2(t^-)\big]\tilde{N}(dt,dz) \\ &= \int_{\mathbb{R}_0} z^2\nu(dz)dt + \int_{\mathbb{R}_0} \big[2\eta(t^-) + z\big]z\tilde{N}(dt,dz) \end{aligned}$$

(see Problem 9.2). Hence we get

$$\eta^2(T) = T\int_{\mathbb{R}_0} z^2\nu(dz) + \int_0^T \int_{\mathbb{R}_0} \big[2\eta(t^-) + z\big]z\tilde{N}(dt,dz). \tag{9.34}$$

Remark 9.12. Note that it is *not* possible to write

$$\eta^2(T) = E\big[\eta^2(T)\big] + \int_0^T \varphi(t)d\eta(t) \tag{9.35}$$

for some predictable process φ. In fact, the representation (9.35) is equivalent to

$$\eta^2(T) = E\big[\eta^2(T)\big] + \int_0^T \int_{\mathbb{R}_0} \varphi(t) z \tilde{N}(dt, dz),$$

which contradicts (9.34), in view of the uniqueness of the Itô stochastic integral representation.

9.4 Application to Finance: Replicability

The fact that the representation (9.35) is not possible has a special interpretation in finance: it means that the *claim* $F = \eta^2(T)$ is not *replicable* in a certain financial market driven by the Lévy process η. We now make this more precise.

Consider a *securities market* with two kinds of investment possibilities:

- A *risk free asset* with a price per unit fixed as
$$S_0(t) = 1, \qquad t \geq 0, \tag{9.36}$$
- n *risky assets* with prices $S_i(t)$, $t \geq 0$ $(i = 1, ..., n)$, given by
$$dS_i(t) = S_i(t^-)\Big[\sigma_i(t) dW(t) + \int_{\mathbb{R}_0} \gamma_i(t, z) \tilde{N}(dt, dz)\Big], \tag{9.37}$$
where $W(t) = \big(W_1(t), ..., W_n(t)\big)^T$, $t \geq 0$, is an n-dimensional Wiener process and $\tilde{N}(dt, dz) = \big(\tilde{N}_1(dt, dz), ..., \tilde{N}_n(dt, dz)\big)^T$, $t \geq 0$, $z \in \mathbb{R}_0$, corresponds to n independent compensated Poisson random measures. The parameters $\sigma_i(t) = \big(\sigma_{i1}(t), ..., \sigma_{in}(t)\big)$, $t \geq 0$ $(i = 1, ..., n)$, and $\gamma_i(t, z) = \big(\gamma_{i1}(t, z), ..., \gamma_{in}(t, z)\big)$, $t \geq 0$, $z \in \mathbb{R}_0$ $(i = 1, ..., n)$, are predictable processes that satisfy
$$E\Big[\sum_{i,j=1}^{n} \int_0^T \big[\sigma_{ij}^2(t) + \int_{\mathbb{R}_0} \gamma_{ij}^2(t, z)\nu(dz)\big] dt\Big] < \infty. \tag{9.38}$$

A random variable $F \in L^2(\mathcal{F}_T, P)$ represents a financial *claim* (or T-*claim*). The claim F is *replicable* if there exists a predictable process $\varphi(t) = \big(\varphi_1(t), ..., \varphi_n(t)\big)^T$, $t \geq 0$, such that

$$\sum_{i=1}^{n} \int_0^T \varphi_i^2(t) S_i^2(t^-) \Big[\sum_{j=1}^{n} \sigma_{ij}^2(t) + \int_{\mathbb{R}_0} \gamma_{ij}^2(t, z)\nu(dz)\Big] dt < \infty \tag{9.39}$$

and

$$F = E[F] + \sum_{i=1}^{n} \int_0^T \varphi_i(t) dS_i(t). \tag{9.40}$$

If this is the case, then φ is a *replicating portfolio* for the claim F. See Sect. 4.3.

Let $\mathcal{A}$ denote the set of all predictable processes satisfying (9.39). These constitute the set of *admissible portfolios* in this context.

Let us now consider the general Lévy process $\eta(t)$, $t \geq 0$:

$$d\eta(t) = dW(t) + \int_{\mathbb{R}_0} z\tilde{N}(dt, dz).$$

Theorem 9.13. *Let $F \in L^2(\mathcal{F}_T, P)$ be a claim on the market* (9.36)–(9.37)*, with stochastic integral representation of the form*

$$F = E[F] + \int_0^T \alpha(t)dW(t) + \int_0^T \int_{\mathbb{R}_0} \beta(t,z)\tilde{N}(dt,dz). \tag{9.41}$$

Then F is replicable if and only if the integrands $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ have the form

$$\alpha_j(t) = \sum_{i=1}^n \varphi_i(t)S_i(t)\sigma_{ij}(t), \qquad t \geq 0, \tag{9.42}$$

$$\beta_j(t,z) = \sum_{i=1}^n \varphi_i(t)S_i(t^-)\gamma_{ij}(t,z), \qquad t \geq 0, z \in \mathbb{R}_0 \qquad (j = 1, ..., n), \tag{9.43}$$

for some process $\varphi \in \mathcal{A}$. In this case φ is a replicating portfolio for F.

Proof First assume that F is replicable with replicating portfolio φ. Then clearly

$$\begin{aligned} F - E[F] &= \sum_{i=1}^n \int_0^T \varphi_i(t)dS_i(t) \\ &= \int_0^T \sum_{i=1}^n \varphi_i(t)S_i(t)\sigma_i(t)dW(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \sum_{i=1}^n \varphi_i(t)S_i(t^-)\gamma_i(t,z)\tilde{N}(dt,dz). \end{aligned}$$

In view of the uniqueness, comparing this with the representation (9.41), we obtain the results (9.42)–(9.43). The argument works both ways, so that if (9.42)–(9.43) hold, then the above computation lead to representation (9.40) and the claim F is replicable. □

We refer to [25, 29] for further arguments on the aforementioned result.

Remark 9.14. Note that any $F \in L^2(\mathcal{F}_T, P)$ admits representation in form (9.41) (see, e.g., [71, 121]).

Example 9.15. Returning to Example 9.11 we can see that in a securities market model of the type:

- A risk free asset with price per unit $S_0(t) = 1$, $t \geq 0$,
- One risky asset with price per unit $S_1(t)$, $t \geq 0$, given by

$$dS_1(t) = S_i(t^-) \int_{\mathbb{R}_0} z\tilde{N}(dt, dz), \tag{9.44}$$

the claim $F = \eta^2(T)$ is *not* replicable since its representation is given by (9.34):

$$\eta^2(T) = T \int_{\mathbb{R}_0} z^2 \nu(dz) + \int_0^T \int_{\mathbb{R}_0} (2\eta(t^-) + z) z\tilde{N}(dt, dz),$$

with $\beta(t, z) = (2\eta(t^-) + z)z$, $t \geq 0$, $z \in \mathbb{R}_0$, which is not of the form (9.43).

Financial markets in which all claims are replicable are called *complete*. Otherwise the market is called *incomplete*. The market model presented here above is an example of an incomplete market.

9.5 Exercises

Problem 9.1. (*) Let $X(t)$, $t \geq 0$, be a one-dimensional Itô–Lévy process:

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz).$$

Use the Itô formula to express $dY(t) = df(X(t)$ in the standard form:

$$dY(t) = a(t)dt + b(t)dW(t) + \int_{\mathbb{R}_0} c(t, z)\tilde{N}(dt, dz)$$

in the following cases:

(a) $Y(t) = X^2(t)$, $t \geq 0$,
(b) $Y(t) = \exp\{X(t)\}$, $t \geq 0$,
(c) $Y(t) = \cos X(t)$, $t \geq 0$.

Problem 9.2. (*) Let $h \in L^2([0, T])$ be a càglàd real function. Define

$$X(t) := \int_0^t \int_{\mathbb{R}_0} h(s)z\tilde{N}(ds, dz) - \int_0^t \int_{\mathbb{R}_0} \big(e^{h(s)z} - 1 - h(s)z\big)\nu(dz)ds$$

and put

$$Y(t) = \exp\{X(t)\}, \quad t \in [0, T].$$

Show that

$$dY(t) = Y(t^-) \int_{\mathbb{R}_0} \big(e^{h(t)z} - 1\big)\tilde{N}(dt, dz).$$

In particular, $Y(t)$, $t \in [0, T]$, is a local martingale.

Problem 9.3. Use the Itô formula to solve the following stochastic differential equations.

(a) The *Lévy–Ornstein–Uhlenbeck process*:

$$dY(t) = \rho(t)Y(t)dt + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz), \quad t \in [0,T],$$

where $\rho \in L^1([0,T])$ and γ is a predictable process satisfying (9.20).

(b) A multiplicative noise dynamics:

$$dY(t) = \alpha(t)dt + Y(t^-)\int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz), \quad t \in [0,T],$$

where α, γ are predictable processes satisfying (9.20).

Problem 9.4. (a) **Integration by parts.** Let

$$dX_i(t) = \alpha_i(t)dt + \beta_i(t)dW(t) + \int_{\mathbb{R}_0} \gamma_i(t,z)\tilde{N}(dt,dz) \quad (i = 1,2),$$

be two Itô–Lévy processes. Use the two-dimensional Itô formula to show that

$$\begin{aligned} d\big(X_1(t)X_2(t)\big) = X_1(t^-)dX_2(t) + X_2(t^-)dX_1(t) + \beta_1(t)\beta_2(t)dt \\ + \textstyle\int_{\mathbb{R}_0} \gamma_1(t,z)\gamma_2(t,z)N(dt,dz). \end{aligned}$$

(b) **The Itô–Lévy isometry**. In the aforementioned processes, choose $\alpha_i = X_i(0) = 0$ for $i = 1,2$ and assume that $E[X_i^2(t)] < \infty$ for $i = 1,2$. Show that

$$E\big[X_1(t)X_2(t)\big] = E\Big[\int_0^T \Big(\beta_1(t)\beta_2(t) + \int_{\mathbb{R}_0} \gamma_1(t,z)\gamma_2(t,z)\nu(dz)\Big)dt\Big].$$

Problem 9.5. The Girsanov theorem. Let $u = u(t)$, $t \in [0,T]$, and $\theta(t,z) \leq 1$, $t \in [0,T], z \in \mathbb{R}_0$, be predictable processes such that the process

$$\begin{aligned} Z(t) := \exp\Big\{ &-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t u^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,z))\tilde{N}(ds,dz) \\ &+ \int^t \int_{\mathbb{R}_0} (\log(1-\theta(s,z)) + \theta(s,z))\nu(dz)ds\Big), \quad t \in [0,T], \end{aligned}$$

exists and satisfies $E[Z(T)] = 1$.

(a) Show that

$$dZ(t) = Z(t^-)\big[-u(t)dW(t) - \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(dt,dz)\big].$$

Thus, Z is a local martingale.

(b) Let

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz), \quad t \in [0,T],$$

be an Itô–Lévy process. Find $d\big(Z(t)X(t)\big)$. In particular, note that if u and θ are chosen such that

$$\beta(t)u(t) + \int_{\mathbb{R}_0} \gamma(t,z)\theta(t,z)\nu(dz) = \alpha(t),$$

then $Z(t)X(t)$, $t \in [0,T]$, is a local martingale.

(c) Use the Bayes rule to show that if we define the probability measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ = Z(T)dP,$$

then the process X is a local martingale with respect to Q. This is a version of the Girsanov theorem for Lévy processes.

Problem 9.6. (*) Let

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dz,dz), \quad t \in [0,T].$$

Find the integrand ψ for the stochastic integral representation

$$F = E[F] + \int_0^T \int_{\mathbb{R}_0} \psi(t,z)\tilde{N}(dt,dz)$$

of the following random variables:

(a) $F = \int_0^T \eta(t)dt$
(b) $F = \eta^3(T)$
(c) $F = \exp\{\eta(T)\}$
(d) $F = \cos\eta(T)$ [*Hint.* If $x \in \mathbb{R}$ and $i = \sqrt{-1}$, then $e^{ix} = \cos x + i\sin x$].

Moreover, in each case above, decide if F is replicable in the following Bachelier–Lévy type market model on the time interval $[0,T]$:

$$\begin{aligned} \text{risk free asset:} \quad & dS_0(t) = 0, \quad S_0(0) = 1 \\ \text{risky asset:} \quad & dS_1(t) = d\eta(t), \quad S_1(0) \in \mathbb{R}. \end{aligned}$$