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The Malliavin Derivative

12.1 Definition and Basic Properties

In the Brownian motion case we saw that there were several ways of defining the Malliavin derivative:

(1) Either as a *stochastic gradient*, using the concept of directional derivatives, either on the Wiener space as in Appendix A (see Definition A.10) or on the space $\Omega = \mathcal{S}'(\mathbb{R}_0)$ as in Chap. 6 (see Definition 6.1).
(2) Or by means of the *chaos expansion* in terms of iterated integrals with respect to Brownian motion (see Lemma A.20).

In the Brownian motion case, those approaches are equivalent and they lead to "essentially" the same differential operator.

We now consider the pure jump martingale case, when

$$\eta(t) := \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz), \quad t \in [0,T].$$

In this case, it turns out that the two approaches do not give the same operator and it is necessary to make a choice about which gives the most useful derivative concept. For several reasons we choose the approach based on the chaos expansions (see Theorem 9.15). For example, this is a definition that gives us a Clark–Ocone type theorem for compensated Poisson random measures similar to Theorem 4.1 for the Brownian motion, see Theorem 12.16. For the other approach to the Malliavin calculus we refer to [35, 58].

Definition 12.1. *The* stochastic Sobolev space $\mathbb{D}_{1,2}$ *consists of all* $\mathcal{F}_T$*-measurable random variables* $F \in L^2(P)$ *with chaos expansion*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \widetilde{L}^2((\lambda \times \nu)^n),$$

G.Di Nunno et al., *Malliavin Calculus for Lévy Processes with Applications to Finance,*

satisfying the convergence criterion

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty. \tag{12.1}$$

Comparing the aforementioned condition with (10.5) we see that $\mathbb{D}_{1,2}$ is strictly contained in the space of all $\mathcal{F}_T$-measurable random variables in $L^2(P)$.

Definition 12.2. *We define the operator D:*

$$L^2(P) \supset \mathbb{D}_{1,2} \ni F \quad \Longrightarrow \quad DF \in L^2(P\times\lambda\times\nu)$$

by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t,z)), \qquad F \in \mathbb{D}_{1,2}. \tag{12.2}$$

Here $I_{n-1}(f_n(\cdot,t,z))$ means that the $(n-1)$-fold iterated integral of f_n is regarded as a function of its $(n-1)$ first pairs of variables $(t_1,z_1),...,(t_{n-1},z_{n-1})$, while the final pair (t,z) is kept as a parameter. In view of Definition 3.1 for the Brownian motion, it is natural to call $D_{t,z}F$ the Malliavin derivative of F at (t,z).

Note that we indeed have that $DF \in L^2(P\times\lambda\times\nu)$ because

$$\begin{aligned}
\|DF\|^2_{L^2(\lambda\times\nu\times P)} &= \int_0^T \int_{\mathbb{R}_0} E\big[(D_{t,z}F)^2\big]\nu(dz)dt \\
&= \int_0^T \int_{\mathbb{R}_0} \sum_{n=1}^{\infty} n^2(n-1)!\|f_n(\cdot,t,z)\|^2_{L^2((\lambda\times\nu)^{n-1})}\nu(dz)dt \qquad (12.3) \\
&= \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2((\lambda\times\nu)^n)} = \|F\|^2_{\mathbb{D}_{1,2}} < \infty.
\end{aligned}$$

Example 12.3. Choose $F = \int_0^T \int_{\mathbb{R}_0} f(t,z)\tilde{N}(dt,dz)$, with the deterministic integrand $f \in L^2(\lambda\times\nu)$. Then $F = I_1(f)$ and hence

$$D_{t,z}F = I_0(f(\cdot,t,z)) = f(t,z). \tag{12.4}$$

In particular, if $F = \eta(T) := \int_0^T \int_{\mathbb{R}_0} z\tilde{N}(dt,dz)$, then

$$D_{t,z}\eta(T) = z. \tag{12.5}$$

Example 12.4. Let $F = \eta^2(T)$, then by (10.6) we have

$$\eta^2(T) = I_0(f_0) + I_1(f_1) + I_2(f_2),$$

where

$$f_0 = T \int_{\mathbb{R}_0} z^2 \nu(dz)$$
$$f_1(t_1, z_1) = z_1^2$$
$$f_2(t_1, z_1, t_2, z_2) = z_1 z_2.$$

Hence, by (12.2),

$$\begin{aligned} D_{t,z}\eta^2(T) &= z^2 + 2I_1(f_2(\cdot, t, z)) \\ &= z^2 + 2\int_0^T \int_{\mathbb{R}_0} z_1 z \tilde{N}(dt_1, dz_1) \\ &= z^2 + 2\eta(T)z. \end{aligned} \tag{12.6}$$

Since $D_{t,z}\eta(T) = z$ (see (12.5)) we conclude that

$$\begin{aligned} D_{t,z}\eta^2(T) &= 2\eta(T)D_{t,z}\eta(T) + \big(D_{t,z}\eta(T)\big)^2 \\ &= \big(\eta(T) + D_{t,z}\eta(T)\big)^2 - \eta^2(T). \end{aligned} \tag{12.7}$$

This shows that D does not satisfy the usual chain rule of a differential operator. In fact, it illustrates that D is a *difference* operator and not a differential operator.

Example 12.5. Let $F = Y(T)$, as in Example 10.4. Then by (10.7) we have

$$\begin{aligned} D_{t,z}F &= \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)) = \sum_{n=1}^{\infty} \frac{n}{n!}(e^{h(t)z} - 1)I_{n-1}(e^{h(t)z} - 1)^{\otimes(n-1)} \\ &= (e^{h(t)z} - 1)\sum_{n=1}^{\infty} \frac{1}{(n-1)!} I_{n-1}(e^{h(t)z} - 1)^{\otimes(n-1)} \\ &= F(e^{h(t)z} - 1). \end{aligned}$$

Theorem 12.6. Closability of the Malliavin derivative. *Suppose $F \in L^2(P)$ and F_k, $k = 1, 2, ...,$ are in $\mathbb{D}_{1,2}$ and that*

(1) $F_k \longrightarrow F$, $k \to \infty$ in $L^2(P)$
(2) $D_{t,z}F_k$, $k = 1, 2, ...,$ converges in $L^2(P \times \lambda \times \nu)$.

Then $F \in \mathbb{D}_{1,2}$ and

$$D_{t,z}F_k \longrightarrow D_{t,z}F, \quad k \to \infty, \quad \textit{in } L^2(P \times \lambda \times \nu).$$

Proof Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and $F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)})$, $k = 1, 2, ...$ From (1) we know that

$$f_n^{(k)} \longrightarrow f_n, \quad k \to \infty, \quad \text{in } L^2((\lambda \times \nu)^n)$$

for all $n = 0, 1, ...$ Since (2) holds, we deduce that

$$\sum_{n=0}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2((\lambda\times\nu)^n)} = \|D_{t,z}F_k - D_{t,z}F_j\|^2_{L^2(\lambda\times\nu\times P)} \longrightarrow 0, \quad k, j \to \infty.$$

Hence by the Fatou lemma,

$$\begin{aligned}
&\lim_{k\to\infty} \sum_{n=0}^{\infty} nn! \|f_n^{(k)} - f_n\|^2_{L^2((\lambda\times\nu)^n)} \\
&\le \lim_{k\to\infty} \left(\varliminf_{j\to\infty} \sum_{n=0}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2((\lambda\times\nu)^n)} \right) = 0,
\end{aligned}$$

which means that $F \in \mathbb{D}_{1,2}$ and

$$D_{t,z}F_k \longrightarrow D_{t,z}F, \quad k \to \infty, \quad \text{in } L^2(P \times \lambda \times \nu). \quad \square$$

12.2 Chain Rules for Malliavin Derivative

As in Example 12.5, let us consider

$$G_1 = \exp\Big\{ \int_0^T \int_{\mathbb{R}_0} h_1(s) z \tilde{N}(ds, dz) \Big\}, \tag{12.8}$$

with $h_1 \in L^2([0,T])$. Its derivative can be written as

$$D_{t,z}G_1 = G_1(e^{h_1(t)z} - 1). \tag{12.9}$$

Let $\mathbb{D}^{\mathcal{E}}_{1,2}$ denote the set of linear combinations of such exponentials. Now choose $G_2 = \exp\{\int_0^T \int_{\mathbb{R}_0} h_2(t) z \tilde{N}(dt, dz)\} \in \mathbb{D}^{\mathcal{E}}_{1,2}$. Then from the above

$$\begin{aligned}
D_{t,z}(G_1 G_2) &= D_{t,z}\big(\exp\Big\{ \int_0^T \int_{\mathbb{R}_0} (h_1(t) + h_2(t)) z \tilde{N}(dt, dz) \Big\}\big) \\
&= G_1 G_2 \big(e^{(h_1(t)+h_2(t))z} - 1 \big) \\
&= \big(G_1 + G_1(e^{h_1(t)z} - 1) \big)\big(G_2 + G_2(e^{h_2(t)z} - 1) \big) - G_1 G_2 \\
&= \big(G_1 + D_{t,z}G_1 \big)\big(G_2 + D_{t,z}G_2 \big) - G_1 G_2 \\
&= G_1 D_{t,z}G_2 + G_2 D_{t,z}G_1 + D_{t,z}G_1\, D_{t,z}G_2.
\end{aligned}$$

By linearity this continues to hold if we replace G_1 and G_2 by linear combinations F_1, F_2 of such exponentials. This proves the following result.

Theorem 12.7. Product rule. *Let $F, G \in \mathbb{D}^{\mathcal{E}}_{1,2}$. Then $FG \in \mathbb{D}^{\mathcal{E}}_{1,2}$ and*

$$D_{t,z}\big(FG\big) = F D_{t,z}G + G D_{t,z}F + D_{t,z}F D_{t,z}G. \tag{12.10}$$

By induction it follows that if $F \in \mathbb{D}^{\mathcal{E}}_{1,2}$ then

$$D_{t,z}(F^n) = (F + D_{t,z}F)^n - F^n. \tag{12.11}$$

For a related result see also Lemma 6.1 in [172]. For an extension to the so-called normal martingales, see, for example, Proposition 1 in [196] or Proposition 5 in [199].
More generally we have the following result.

Theorem 12.8. Chain rule. *Let $F \in \mathbb{D}_{1,2}$ and let φ be a real continuous function on $\mathbb{R}$. Suppose $\varphi(F) \in L^2(P)$ and $\varphi(F + D_{t,z}F) \in L^2(P \times \lambda \times \nu)$. Then $\varphi(F) \in \mathbb{D}_{1,2}$ and*

$$D_{t,z}\varphi(F) = \varphi(F + D_{t,z}F) - \varphi(F). \tag{12.12}$$

Proof First assume that φ has compact support and $F \in \mathbb{D}^{\mathcal{E}}_{1,2}$. Then

$$\varphi(F) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iyF} \hat{\varphi}(y) dy,$$

where

$$\hat{\varphi}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \varphi(x) dx$$

is the Fourier transform of φ. By (12.11) and Theorem 12.6 we get that

$$\begin{aligned}
D_{t,z}\varphi(F) =& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{1}{n!} (iy)^n \big((F + D_{t,z}F)^n - F\big) \hat{\varphi}(y) dy \\
=& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \big(e^{iy(F + D_{t,z}F)} - e^{iyF}\big) \hat{\varphi}(y) dy \\
=& \varphi(F + D_{t,z}F) - \varphi(F),
\end{aligned}$$

so the result holds in this case. For general $F \in \mathbb{D}_{1,2}$ we proceed by approximation. Choose $F_n \in \mathbb{D}^{\mathcal{E}}_{1,2}$, $n = 1, 2, ...,$ such that $F_n \longrightarrow F$, $n \to \infty$, in $\mathbb{D}_{1,2}$, see (12.1). Then $\varphi(F_n) \longrightarrow \varphi(F)$ in $L^2(P)$ and $\varphi(F_n + D_{t,z}F_n) - \varphi(F_n) \longrightarrow \varphi(F + D_{t,z}F) - \varphi(F)$ in $\mathbb{D}_{1,2}$. Hence the result holds for all $F \in \mathbb{D}_{1,2}$ in the case of φ with compact support. The extension to the case when $\varphi(F) \in L^2(P)$ and $\varphi(F + D_{t,z}F) \in L^2(P \times \lambda \times \nu)$ follows by a similar limit argument. □

Example 12.9. The chain rule (12.12) is useful for the evaluation of Malliavin derivatives. To illustrate this, consider the following:

(1) The derivative of $\eta^2(T)$ is

$$\begin{aligned}
D_{t,z}\eta^2(T) &= \big(\eta(T) + D_{t,z}\eta(T)\big)^2 - \eta^2(T) \\
&= \big(\eta(T) + z\big)^2 - \eta^2(T) = 2\eta(T)z + z^2,
\end{aligned}$$

which is what we found in (12.6).

(2) With

$$G = \exp\Big(\int_0^T \int_{\mathbb{R}_0} h(t) z \tilde{N}(dt, dz)\Big)$$

as in (12.8), the chain rule (12.12) gives

$$\begin{aligned} D_{t,z} G &= \exp\Big(\int_0^T \int_{\mathbb{R}_0} h(t) z \tilde{N}(dt, dz) + h(t) z\Big) - G \\ &= G\big(e^{h(t)z} - 1\big), \end{aligned}$$

which is (12.9).

(3) Let $F = (\eta(T) - K)^+$ be a *European call payoff*, where $K > 0$ is a constant. Then

$$D_{t,z} F = \big(\eta(T) + z - K\big)^+ - \big(\eta(T) - K\big)^+. \tag{12.13}$$

12.3 Malliavin Derivative and Skorohod Integral

In this section we explore the relationship between the Malliavin derivative and the Skorohod integral following the same lines as in the Brownian motion case. We also derive useful rules of calculus.

12.3.1 Skorohod Integral as Adjoint Operator to the Malliavin Derivative

For the following result we can also refer to [29, 54, 69, 172].

Theorem 12.10. Duality formula. *Let $X(t,z)$, $t \in [0,T], z \in \mathbb{R}$, be Skorohod integrable and $F \in \mathbb{D}_{1,2}$. Then*

$$E\left[\int_0^T \int_{\mathbb{R}_0} X(t,z) D_{t,z} F \nu(dz) dt\right] = E\left[F \int_0^T \int_{\mathbb{R}_0} X(t,z) \widetilde{N}(\delta t, dz)\right]. \tag{12.14}$$

Proof The proof of this is the same as the proof of the corresponding result in the Brownian motion case. See Theorem 3.14. □

12.3.2 Integration by Parts and Closability of the Skorohod Integral

The following result is basically Theorem 7.1 in [172], here presented in the setting of Poisson random measures.

Theorem 12.11. Integration by parts. *Let* $X(t, z)$, $t \in [0, T], z \in \mathbb{R}$, *be a Skorohod integrable stochastic process and* $F \in \mathbb{D}_{1,2}$ *such that the product* $X(t, z) \cdot (F + D_{t,z}F)$, $t \in [0, T], z \in \mathbb{R}$, *is Skorohod integrable. Then*

$$F \int_0^T \int_{\mathbb{R}_0} X(t, z)\widetilde{N}(\delta t, dz) \tag{12.15}$$

$$= \int_0^T \int_{\mathbb{R}_0} X(t, z)\big(F + D_{t,z}F\big)\widetilde{N}(\delta t, dz) + \int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\nu(dz)dt.$$

Proof First assume that $F \in D^{\mathcal{E}}_{1,2}$. Let $G \in \mathbb{D}^{\mathcal{E}}_{1,2}$. Then we obtain by Theorem 12.10 and Theorem 12.7

$$E\left[G \int_0^T \int_{\mathbb{R}_0} FX(t, z)\widetilde{N}(\delta t, dz)\right] = E\left[\int_0^T \int_{\mathbb{R}_0} FX(t, z)D_{t,z}G\nu(dz)dt\right]$$

$$= E\left[GF \int_0^T \int_{\mathbb{R}_0} X(t, z)\widetilde{N}(\delta t, dz)\right] - E\left[G \int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\nu(dz)dt\right]$$

$$-E\left[G \int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\widetilde{N}(\delta t, dz)\right]$$

$$= E\Bigg[G\Bigg(F \int_0^T \int_{\mathbb{R}_0} X(t, z)\widetilde{N}(\delta t, dz) - \int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\nu(dz)dt$$

$$- \int_0^T \int_{\mathbb{R}_0} X(t, z)D_{t,z}F\widetilde{N}(\delta t, dz)\Bigg)\Bigg].$$

The proof then follows by a density argument applied to F and G. □

Remark 12.12. Using the Poisson interpretation of Fock space, the formula (12.15) has been shown to be an expression of the multiplication formula for Poisson stochastic integrals. See [125, 220], Proposition 2 and Relation (6) of [197], Definition 7 and Proposition 6 of [201], Proposition 2 of [199], and Proposition 1 of [195]. Moreover, formula (12.15) has been known for some time to quantum probabilitists in identical or close formulations. See Proposition 21.6 and Proposition 21.8 in [188], Proposition 18 in [34], and Relation (5.6) in [7], see also [127].

Theorem 12.13. Closability of the Skorohod integral. *Suppose that* $X_n(t, z)$, $t \in [0, T], z \in \mathbb{R}$, *is a sequence of Skorohod integrable random fields and that the corresponding sequence of integrals*

$$I(X_n) := \int_0^T \int_{\mathbb{R}_0} X_n(t, z)\widetilde{N}(\delta t, dz), \quad n = 1, 2, ...$$

converges in $L^2(P)$. *Moreover, suppose that*

$$\lim_{n\to\infty} X_n = 0 \qquad in \quad L^2(P\times\lambda\times\nu).$$

Then we have

$$\lim_{n\to\infty} I(X_n) = 0 \qquad in \quad L^2(P).$$

Proof By Theorem (12.10) we have that

$$\big(I(X_n), F\big)_{L^2(P)} = \big(X_n, D_{t,z}F\big)_{L^2(P\times\lambda\times\nu)} \longrightarrow 0, \qquad n\to\infty,$$

for all $F \in \mathbb{D}_{1,2}$. Then we conclude that $\lim_{n\to\infty} I(X_n) = 0$ weakly in $L^2(P)$. And since the sequence $I(X_n)$, $n = 1, 2, ...$, is convergent in $L^2(P)$, the result follows. □

Remark 12.14. In view of Theorem 12.13 we can see that if X_n, $n = 1, 2, ...$, is a sequence of Skorohod integrable random fields such that

$$X = \lim_{n\to\infty} X_n \qquad \text{in} \quad L^2(P\times\lambda\times\nu).$$

Then we can define the *Skorohod integral of* X as

$$I(X) := \int_0^T\int_{\mathbb{R}_0} X(t,z)\widetilde{N}(\delta t, dz) = \lim_{n\to\infty}\int_0^T\int_{\mathbb{R}_0} X_n(t,z)\widetilde{N}(\delta t, dz) =: \lim_{n\to\infty} I(X_n),$$

provided that this limit exists in $L^2(P)$.

12.3.3 Fundamental Theorem of Calculus

The following result is basically Theorem 4.2 in [172], here presented for Poisson random measures, see, for example, [64].

Theorem 12.15. Fundamental theorem of calculus. *Let* $X = X(s,y)$, $(s,y) \in [0,T]\times\mathbb{R}_0$, *be a stochastic process such that*

$$E\Big[\int_0^T\int_{\mathbb{R}_0} X^2(s,y)\nu(dy)ds\Big] < \infty.$$

Assume that $X(s,y) \in \mathbb{D}_{1,2}$ *for all* $(s,y) \in [0,T]\times\mathbb{R}_0$, *and that* $D_{t,z}X(\cdot,\cdot)$ *is Skorohod integrable with*

$$E\Big[\int_0^T\int_{\mathbb{R}_0}\Big(\int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\tilde{N}(\delta s, dy)\Big)^2\nu(dz)dt\Big] < \infty.$$

Then

$$\int_0^T\int_{\mathbb{R}_0} X(s,y)\tilde{N}(\delta s, dy) \in \mathbb{D}_{1,2}$$

and

$$D_{t,z}\int_0^T\int_{\mathbb{R}_0} X(s,y)\tilde{N}(\delta s,dy) = \int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\tilde{N}(\delta s,dy) + X(t,z). \tag{12.16}$$

In particular, if $X(s,y) = Y(s)y$*, then*

$$D_{t,z}\int_0^T Y(s)\delta\eta(s) = \int_0^T D_{t,z}Y(s)\delta\eta(s) + zY(t). \tag{12.17}$$

Proof First suppose that

$$X(s,y) = I_n(f_n(\cdot,s,y)),$$

where $f_n(t_1,z_1,...,t_n,z_n,s,y)$ is symmetric with respect to $(t_1,z_1),...,(t_n,z_n)$. By Definition 3.1 we have

$$\int_0^T\int_{\mathbb{R}_0} X(s,y)\widetilde{N}(\delta s,dy) = I_{n+1}(\widehat{f_n}), \tag{12.18}$$

where

$$\widehat{f_n}(t_1,z_1,...,t_n,z_n,t_{n+1},z_{n+1})$$
$$= \frac{1}{n+1}\left[f_n(t_{n+1},z_{n+1},\cdot,t_1,z_1) + ... + f_n(t_{n+1},z_{n+1},\cdot,t_n,z_n)\right.$$
$$\left. + f_n(t_1,z_1,\cdot,t_{n+1},z_{n+1})\right]$$

is the symmetrization of f_n with respect to the variables $(t_1,z_1),...,(t_n,z_n)$, $(t_{n+1},z_{n+1}) = (s,y)$. Therefore, we get

$$D_{t,z}\left(\int_0^T\int_{\mathbb{R}_0} X(s,y)\widetilde{N}(\delta s,dy)\right) = I_n(f_n(t,z,\cdot,t_1,z_1) + ... + f_n(t,z,\cdot,t_n,z_n)$$
$$+ f_n(\cdot,t,z)).$$

On the other hand we see that

$$\int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\widetilde{N}(\delta s,dy) \tag{12.19}$$

$$= \int_0^T\int_{\mathbb{R}_0} nI_{n-1}(f_n(\cdot,t,z,s,y))\widetilde{N}(\delta s,dy) = nI_n(\widehat{f_n}(\cdot,t,z,\cdot)),$$

where

$$\widehat{f_n}(t_1,z_1,...,t_{n-1},t_{n-1},t,z,t_n,z_n) = \frac{1}{n}\left[f_n(t,z,\cdot,t_1,z_1) + ... + f_n(t,z,\cdot,t_n,z_n)\right]$$

is the symmetrization of $f_n(t_1,z_n,...,t_{n-1},z_{n-1},t,z,t_n,z_n)$ with respect to $(t_1,z_1),...,(t_{n-1},z_{n-1})$, $(t_n,z_n) = (s,y)$. A comparison of (12.18) and (12.19) yields formula (12.16).

Next consider the general case

$$X(s,y) = \sum_{n\geq 0} I_n(f_n(\cdot,s,y)).$$

Define

$$X_m(s,y) = \sum_{n=0}^{m} I_n(f_n(\cdot,s,y)),\ m = 1,2,...$$

Then (12.16) holds for X_m. Since

$$\left\| \int_0^T \int_{\mathbb{R}_0} D_{t,z} X_m(s,y)\widetilde{N}(\delta s,dy) - \int_0^T \int_{\mathbb{R}_0} D_{t,z} X(s,y)\widetilde{N}(\delta s,dy) \right\|^2_{L_2(P\times\lambda\times\nu)}$$

$$= \sum_{n\geq m+1} n^2 n! \left\| \widehat{f_n} \right\|^2_{L_2((\lambda\times\nu)^{n+1})} \longrightarrow 0, \qquad m \longrightarrow \infty,$$

the proof follows by the closedness of $D_{t,z}$. □

12.4 The Clark–Ocone Formula

In this section we state and prove a jump diffusion version of the Clark–Ocone formula (see Theorem 4.1). For this result we refer to, for example, [154].

Theorem 12.16. *Let $F \in \mathbb{D}_{1,2}$. Then*

$$F = E[F] + \int_0^T \int_{\mathbb{R}_0} E[D_{t,z}F|\mathcal{F}_t]\tilde{N}(dt,dz), \tag{12.20}$$

where we have chosen a predictable version of the conditional expectation process $E[D_{t,z}F|\mathcal{F}_t]$, $t \geq 0$.

Proof The proof is similar to the one for the Brownian motion case (Theorem 4.1). Let us consider the chaos expansion of $F = \sum_{n=0}^{\infty} I_n(f_n)$, where $f_n \in \tilde{L}^2((\lambda\times\nu)^n)$, $n = 1,2,....$ Then the following equalities hold true:

$$\int_0^T\int_{\mathbb{R}_0} E[D_{t,z}F|\mathcal{F}_t]\tilde{N}(dt,dz) = \int_0^T\int_{\mathbb{R}_0} E\left[\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t,z))|\mathcal{F}_t\right]\tilde{N}(dt,dz)$$

$$= \int_0^T\int_{\mathbb{R}_0} \sum_{n=1}^{\infty} n(n-1)!E[J_{n-1}(f_n(\cdot,t,z))|\mathcal{F}_t]\tilde{N}(dt,dz)$$

$$= \sum_{n=1}^{\infty} n! \int_0^T\int_{\mathbb{R}_0} E\Big[\int_0^T\int_{\mathbb{R}_0}\cdots\int_0^{t_2^-}\int_{\mathbb{R}_0} f_n(t_1,z_1,...,t_{n-1},z_{n-1},t,z)\tilde{N}(dt_1,dz_1)$$

$$\cdots\tilde{N}(dt_{n-1},dz_{n-1})|\mathcal{F}_t\Big]\tilde{N}(dt,dz)$$

$$= \sum_{n=1}^{\infty} n!J_n(f_n) = \sum_{n=1}^{\infty} I_n(f_n) = F - E[F]. \quad \square$$

Remark 12.17. Comparing (12.20) with the Itô representation (9.33), we can see that the difference is that (12.20) provides an explicit formula for the process $\Psi(t,z)$, $t \geq 0, z \in \mathbb{R}_0$.

Example 12.18. Suppose $F \in \mathbb{D}_{1,2}$ has the form $F = \varphi(\eta(T))$ for some continuous real function $\varphi(x)$, $x \in \mathbb{R}$. Then by the Clark–Ocone theorem combined with the Markov property of the process η, we get

$$\begin{aligned}\varphi(\eta(T)) &= E\big[\varphi(\eta(T))\big] + \int_0^T \int_{\mathbb{R}_0} E[\varphi(\eta(T)+z) - \varphi(\eta(T))|\mathcal{F}_t]\tilde{N}(dt,dz)\\ &= E\big[\varphi(\eta(T))\big] + \int_0^T \int_{\mathbb{R}_0} E[\varphi(y+\eta(T-t)+z)\\ &\qquad - \varphi(y+\eta(T-t))]_{|y=\eta(t)}\tilde{N}(dt,dz).\end{aligned} \tag{12.21}$$

12.5 A Combination of Gaussian and Pure Jump Lévy Noises

We now outline how the results of the previous sections can be generalized to the case of combinations of independent Gaussian and pure jump Lévy noise. Let us sketch a framework for treating this combination of noises. Here we follow the ideas in [2], though this work is settled in the white noise framework. The white noise setting will also be treated later in this book (see Chap. 13).

Another approach to deal with the noise generated by general stochastic measures with independent values can be found in [61]. See also [71].

Denote the probability space on which $W = W(t)$, $t \geq 0$, is a Wiener process by $(\Omega_0, \mathcal{F}_T^W, P^W)$ (see Sect. 1.1) and denote the one on which $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is a compensated Poisson random measure by $(\Omega_0, \mathcal{F}_T^{\widetilde{N}}, P^{\widetilde{N}})$ (see Sect. 9.1).

Let $(\Omega_1, \mathcal{F}_T^{(1)}, \mu_1), ..., (\Omega_N, \mathcal{F}_T^{(N)}, \mu_N)$ be N independent copies of $(\Omega_0, \mathcal{F}_T^W, P^W)$ and let $(\Omega_{N+1}, \mathcal{F}_T^{(N+1)}, \mu_{N+1}), ..., (\Omega_{N+R}, \mathcal{F}_T^{(N+R)}, \mu_{N+R})$ be R independent copies of $(\Omega_0, \mathcal{F}_T^{\widetilde{N}}, P^{\widetilde{N}})$, for some $N, R \in \mathbb{N} \cup \{0\}$. We set

$$\Omega = \Omega_1 \times ... \times \Omega_{N+R}, \quad \mathcal{F}_T = \mathcal{F}_T^{(1)} \otimes ... \otimes \mathcal{F}_T^{(N+R)}, \quad P = \mu_1 \times ... \times \mu_{N+R}. \tag{12.22}$$

In the sequel, we call the space $(\Omega, \mathcal{F}, P)$ the *Wiener–Poisson space.*
We can consider the product of the form

$$\mathbb{H}_\alpha(\omega) := \prod_{k=1}^{L} I_{\alpha^{(k)}}(f_{k,\alpha^{(k)}})(\omega_k) \tag{12.23}$$

for any $\alpha \in \mathcal{J}^L$, which is the set of indices of the form $\alpha = (\alpha^{(1)}, ..., \alpha^{(L)})$, with $\alpha^{(k)} = 0, 1, ...,$ for $k = 1, ..., L$. Here $I_{\alpha^{(k)}}(f_{k,\alpha^{(k)}})$ is the $\alpha^{(k)}$-fold iterated

Itô integral with respect to the Wiener process, if $k = 1, ..., N$, or to the compensated Poisson random measure, if $k = N + 1, ..., L$.

The elements $\mathbb{H}_\alpha$, $\alpha \in \mathcal{J}^L$, constitute an orthogonal basis in $L^2(P)$. Any real $\mathcal{F}_T$-measurable random variable $F \in L^2(P)$ can be written as

$$F = \sum_{\alpha \in \mathcal{J}^L} \mathbb{H}_\alpha$$

for an appropriate choice of deterministic symmetric integrands in the iterated Itô integrals.

Definition 12.19. *(1) We say that $F \in \mathbb{D}_{1,2}$ if*

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{k=1}^{N} \sum_{\alpha \in \mathcal{J}^L} \alpha^{(k)} \alpha^{(k)}! \|f_{k,\alpha^{(k)}}\|^2_{L^2([0,T]^{\alpha^{(k)}})} + \sum_{k=N+1}^{L} \sum_{\alpha \in \mathcal{J}^L} \alpha^{(k)} \alpha^{(k)}! \|f_{k,\alpha^{(k)}}\|^2_{L^2(([0,T]\times\mathbb{R}_0)^{\alpha^{(k)}})} < \infty. \tag{12.24}$$

(2) If $F \in \mathbb{D}_{1,2}$, we define the Malliavin derivative *DF of F as the gradient*

$$DF = \big(D_{1,t}F, ..., D_{N,t}F, D_{N+1,t,z}F, ..., D_{L,t,z}F\big), \tag{12.25}$$

where

$$D_{k,t}F = \sum_{\alpha \in \mathcal{J}^L} \alpha^{(k)} \mathbb{H}_{\alpha - \epsilon^{(k)}}(t), \quad t \in [0,T] \quad (k = 1, ..., N),$$

and

$$D_{k,t,z}F = \sum_{\alpha \in \mathcal{J}^L} \alpha^{(k)} \mathbb{H}_{\alpha - \epsilon^{(k)}}(t,z), \quad t \in [0,T], z \in \mathbb{R}_0 \quad (k = N+1, ..., L).$$

Here $\epsilon^{(k)} = (0, ...0, 1, 0, ...0)$ with 1 in the kth position, cf. Definition 3.1 and Definition 12.1.

Based on the same concepts and arguments as in the previous sections, one can show the following Clark–Ocone formula:

Theorem 12.20. Clark–Ocone theorem for combined Gaussian-pure jump Lévy noise. *Let $F \in \mathbb{D}_{1,2}$. Then*

$$F = E[F] + \sum_{k=1}^{N} \int_0^T E\big[D_{k,t}F|\mathcal{F}_t\big] dW_k(t) + \sum_{k=N+1}^{L} \int_0^T \int_{\mathbb{R}_0} E\big[D_{k,t,z}F|\mathcal{F}_t\big] \widetilde{N}_k(dt, dz). \tag{12.26}$$

A generalization of this formula to processes with conditionally independent increments can be found in [228].

Similar to the Brownian motion case one can obtain a Clark–Ocone formula under change of measure for Lévy processes. This was first proved by [113] for random variables in $\mathbb{D}_{1,2}$. Subsequently the result was generalized in [185] to all random variables in $L^2(P)$, by means of white noise theory (see Chap. 13). We present the statement of this generalized version later without proof, and refer to the original papers for more information. We first recall the Girsanov theorem for Lévy processes, as presented in [183] (cf. Problem 9.5). See also [149].

Theorem 12.21. Girsanov theorem for Lévy processes. *Let* $\theta(s,x) \leq 1$, $s \in [0,T]$, $x \in \mathbb{R}_0$ *and* $u(s)$, $s \in [0,T]$, *be* $\mathbb{F}-$*predictable processes such that*

$$\int_0^T \int_{\mathbb{R}_0} \{| \log(1+\theta(s,x)) | + \theta^2(s,x)\}\nu(dx)dt < \infty \quad P\text{-a.e.}, \tag{12.27}$$

$$\int_0^T u^2(s)ds < \infty \quad P - a.e. \tag{12.28}$$

Let

$$\begin{aligned} Z(t) = \exp\Big\{ & -\int_0^t u(s)dW(s) - \int_0^t u^2(s)ds \\ & + \int_0^t \int_{\mathbb{R}_0} \{\log(1-\theta(s,x)) + \theta(s,x)\}\nu(dx)ds \\ & + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta(s,x))\widetilde{N}(ds,dx) \Big\}, \quad t \in [0,T]. \end{aligned}$$

Define a measure Q *on* $\mathcal{F}_T$ *by*

$$dQ(\omega) = Z(\omega,T)dP(\omega).$$

Assume that $Z(T)$ *satisfies the Novikov condition, that is,*

$$\begin{aligned} E\Bigg[\exp\Bigg(\frac{1}{2}\int_0^T u^2(s)ds + \int_0^T \int_{\mathbb{R}_0} \big\{(1-\theta(s,x))\log(1-\theta(s,x)) \\ +\theta(s,x)\big\}\nu(dx)ds\Big)\Big] < \infty. \end{aligned}$$

Then $E[Z(T)] = 1$ *and hence* Q *is a probability measure on* $\mathcal{F}_T$. *Define*

$$\widetilde{N}_Q(dt,dx) = \theta(t,x)\nu(dx)dt + \tilde{N}(dt,dx)$$

and

$$dW_Q(t) = u(t)dt + dW(t).$$

Then $\widetilde{N}_Q(\cdot,\cdot)$ *and* $W_Q(\cdot)$ *are compensated Poisson random measure of* $N(\cdot,\cdot)$ *and Brownian motion under* Q, *respectively.*

In this setting, the Clark–Ocone formula gets the following form. See [113, 185].

Theorem 12.22. Generalized Clark–Ocone theorem under change of measure for Lévy processes. *Let $F \in L^2(P) \cap L^2(Q)$ be $\mathcal{F}_T$-measurable. Assume that u satisfies (4.7), that $\theta \in L^2(P \times \lambda \times \nu)$ and that $(t, x) \to D_{t,x}\theta(s, z)$ is Skorohod integrable for all s, z, with $\delta(D_{t,x}\theta) \in \mathcal{G}^*$. Then the integral representation of F with respect to W_Q and $\widetilde{N}_Q$ is as follows:*

$$F = E_Q[F] + \int_0^T E_Q\big[D_t F - F \int_t^T D_t u(s) dW_Q(s) | \mathcal{F}_t\big] \, dW_Q(t) \\ + \int_0^T \int_{\mathbb{R}_0} E_Q\big[F(\tilde{H} - 1) + \tilde{H} D_{t,x} F | \mathcal{F}_t\big] \, \tilde{N}_Q(dt, dx),$$

where

$$\tilde{H} = \exp\Bigg\{ \int_0^t \int_{\mathbb{R}_0} \big[D_{t,x}\theta(s, z) + \log(1 - \frac{D_{t,x}\theta(s, z)}{1 - \theta(s, z)})(1 - \theta(s, z))\big] \nu(dz) \, ds \\ + \log(1 - \frac{D_{t,x}\theta(s, z)}{1 - \theta(s, z)}) \tilde{N}_Q(ds, dz) \Bigg\}.$$

12.6 Application to Minimal Variance Hedging with Partial Information

Consider a financial market where the unit prices $S_i(t)$, $t \geq 0$, of the assets are as follows:

$$\begin{aligned} \text{risk free asset} \quad & S_0(t) \equiv 1, \quad t \in [0, T], \\ \text{risky assets} \quad & dS_j(t) = \sigma_j(t) dW(t) \\ & + \int_{\mathbb{R}_0^n} \gamma_j(t, z) \tilde{N}(dt, dz), \quad t \in (0, T], \ j = 1, ..., n, \end{aligned}$$

where $\sigma(t) = [\sigma_{i,j}(t)] \in \mathbb{R}^{n \times n}$ and $\gamma(t, z) = [\gamma_{i,j}(t, z)] \in \mathbb{R}^{n \times n}$ are predictable processes, which might depend on $S(s) = (S_1(s), ..., S_n(s))$, $s \in [0, t]$.

Let $\mathbb{E} = \{\mathcal{E}_t, \ t \geq 0\}$ be a given filtration such that

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \in [0, T].$$

We think of $\mathcal{E}_t$ as the information available to an agent at time t.

Definition 12.23. *A predictable process $\varphi = \varphi(t)$, $t \in [0, T]$, is called admissible if*

(1) $\varphi(t)$ *is* $\mathbb{E}$*-adapted*
(2) $E\Big[\sum_{j=1}^n \int_0^T \varphi_j^2(t) \big(\sum_{i=1}^n \sigma_{i,j}^2(t) + \int_{\mathbb{R}_0} \gamma_{i,j}^2(t, z) \nu(dz) \big) dt \Big] < \infty.$

The set of all $\mathbb{E}$-admissible portfolios is denoted by $\mathcal{A}_{\mathbb{E}}$.

We now pose the following question: given a claim $F \in L^2(\mathcal{F}_T)$, how close can we get to F at time T by hedging with portfolios? If we consider closeness in terms of variance, the precise formulation of this question is the following: *given $F \in L^2(\mathcal{F}_T)$ find $\varphi^* \in \mathcal{A}_{\mathbb{E}}$ such that*

$$\inf_{\varphi \in \mathcal{A}_{\mathbb{E}}} E\Big[\Big(F - E[F] - \int_0^T \varphi(t)dS(t)\Big)^2\Big] = E\Big[\Big(F - E[F] - \int_0^T \varphi^*(t)dS(t)\Big)^2\Big]. \tag{12.29}$$

Such a portfolio φ^* is called a *partial information minimal variance portfolio.*

We use Malliavin calculus to obtain explicit formulae for such portfolios φ^*. We refer to [25] for the following result. See also [161] and [69] for an extension to the white noise setting and [59, 60] for market models driven by general martingales and random fields.

Theorem 12.24. *Suppose $F \in \mathbb{D}_{1,2}$. Then the partial information minimal variance portfolio $\varphi^* \in \mathcal{A}_{\mathbb{E}}$ for F is given by*

$$\varphi^*(t) = Q^{-1}(t)R(t), \quad t \in [0,T]. \tag{12.30}$$

Here $Q(t) \in \mathbb{R}^{n\times n}$ has components

$$Q_{ik}(t) = E\big[N_{ik}(t)|\mathcal{E}_t\big],$$

where

$$N_{ik}(t) = \sum_{j=1}^n \sigma_{ij}(t)\sigma_{jk}(t) + \int_{\mathbb{R}_0} \gamma_{ij}(t,z)\gamma_{jk}(t,z)\nu_j(dz) \qquad (i,j = 1,..,n).$$

The matrix $Q^{-1}(t)$ is the inverse of $Q(t)$ (if it exists). The vector $R(t) \in \mathbb{R}^n$ has components

$$R_i(t) = E\big[M_i|\mathcal{E}_t\big],$$

where

$$M_i(t) = \sum_{j=1}^n \sigma_{ij}(t)E\big[D_{j,t}F|\mathcal{F}_t\big] + \int_{\mathbb{R}_0} \gamma_{ij}(t,z)E\big[D_{j,t,z}F|\mathcal{F}_t\big]\nu_j(dz).$$

Moreover, $D_{j,t}F$, $t \in [0,T]$, denotes the Malliavin derivative with respect to W_j and $D_{j,t,z}F$, $t \in [0,T]$, $z \in \mathbb{R}_0$, stands for the Malliavin derivative with respect to $\tilde{N}_j$.

Proof Let φ_j^* be as above and define

$$\begin{aligned}\hat{F} &:= E[F] + \sum_{j=1}^n \int_0^T \varphi_j^*(t)dS_j(t) \\ &= E[F] + \sum_{j=1}^n \int_0^T \varphi_j^*(t)\sigma_j(t)dW_j(t) + \int_0^T \int_{\mathbb{R}_0} \varphi_j^*(t)\gamma_j(t,z)\tilde{N}_j(dt,dz).\end{aligned}$$

To prove the statements it is enough to show that

$$E\big[(F-\hat{F})G\big]=0$$

for all $G \in L^2(P)$, $\mathcal{F}_T$-measurable of the form

$$\begin{aligned} G &:= E[G]+\sum_{j=1}^{n}\int_0^T \psi_j(t)dS_j(t) \\ &= E[G]+\sum_{j=1}^{n}\int_0^T \psi_j(t)\sigma_j(t)dW_j(t)+\int_0^T\int_{\mathbb{R}_0}\psi_j(t)\gamma_j(t,z)\tilde{N}_j(t,z), \end{aligned}$$

with $\psi \in \mathcal{A}_{\mathbb{E}}$. By the Clark–Ocone theorem (Theorem 12.16) we have

$$F=E[F]+\sum_{j=1}^{n}\int_0^T E\big[D_{j,t}F|\mathcal{F}_t\big]dW_j(t)+\int_0^T\int_{\mathbb{R}_0}E\big[D_{j,t,z}F|\mathcal{F}_t\big]\tilde{N}_j(t,z).$$

This gives

$$\begin{aligned} E\big[(F-\hat{F})G\big] &= E\Big[\Big(\sum_{j=1}^{n}\int_0^T E\big[D_{j,t}F|\mathcal{F}_t\big]dW_j(t)-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T\varphi_j^*(t)\sigma_{jk}(t)dW_k(t) \\ &\quad +\sum_{j=1}^{n}\int_0^T\int_{\mathbb{R}_0}E\big[D_{j,t,z}F|\mathcal{F}_t\big]\tilde{N}_j(dt,dz)-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T\int_{\mathbb{R}_0}\varphi_j^*(t)\gamma_{jk}(t,z)\tilde{N}_k(dt,dz)\Big) \\ &\quad \cdot\Big(\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T\psi_j(t)\sigma_{jk}(t)dW_k(t)+\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T\int_{\mathbb{R}_0}\psi_j(t)\gamma_{jk}(t,z)\tilde{N}_k(dt,dz)\Big)\Big] \\ &= E\Big[\sum_{j=1}^{n}\int_0^T\Big(E\big[D_{j,t}F|\mathcal{F}_t\big]-\sum_{k=1}^{n}\varphi_k^*(t)\sigma_{kj}(t)\Big)\cdot\Big(\sum_{k=1}^{n}\psi_k(t)\sigma_{kj}(t)\Big)dt \\ &\quad +\sum_{j=1}^{n}\int_0^T\int_{\mathbb{R}_0}\Big(E[D_{j,t,z}F|\mathcal{F}_t]-\sum_{k=1}^{n}\varphi_k^*(t)\gamma_{kj}(t,z)\Big)\cdot\Big(\sum_{k=1}^{n}\psi_k(t)\gamma_{kj}(t,z)\Big)\nu_j(dz)dt \\ &= E\Big[\sum_{j=1}^{n}\int_0^T\sum_{k=1}^{n}\psi_k(t)\big(\sigma_{kl}(t)E\big[D_{j,t}F|\mathcal{F}_t\big]-\sum_{i=1}^{n}\varphi_i^*(t)\sigma_{ij}(t)\sigma_{kj}(t) \\ &\quad +\int_{\mathbb{R}_0}\big(\gamma_{kj}(t,z)E\big[D_{j,t,z}F|\mathcal{F}_t\big]-\sum_{i=1}^{n}\varphi_i^*(t)\gamma_{ij}(t,z)\gamma_{kj}(t,z)\big)\nu_j(dz)\big]dt\Big] \\ &= E\Big[\sum_{k=1}^{n}\int_0^T\psi_k(t)L_k(t)dt\Big]=0, \end{aligned}$$

where

$$\begin{aligned} L_k(t) &= \sum_{j=1}^{n}\Big(\sigma_{kj}(t)E\big[D_{j,t}F|\mathcal{F}_t\big]-\sum_{i=1}^{n}\varphi_i^*(t)\sigma_{ij}(t)\sigma_{kj}(t) \\ &\quad +\int_{\mathbb{R}_0}\Big[\gamma_{kj}(t,z)E\big[D_{j,t,z}F|\mathcal{F}_t\big]-\sum_{i=1}^{n}\varphi_i^*(t)\gamma_{ij}(t,z)\gamma_{kj}(t,z)\Big]\nu(dz)\Big). \end{aligned}$$

This holds for all $\psi \in \mathcal{A}_{\mathbb{E}}$ if and only if

$$E\big[L_k(t)|\mathcal{E}_t\big] = 0, \quad t \in [0,T], \quad k = 1, ..., n.$$

We can write

$$L_k(t) = M_k(t) - \sum_{i=1}^{n} \varphi_i^*(t) N_{ik}(t),$$

where

$$M_k(t) = \sum_{k=1}^{n} \big(\sigma_{kj}(t) E\big[D_{j,t}F|\mathcal{F}_t\big] + \int_{\mathbb{R}_0} \gamma_{kj}(t,z) E\big[D_{j,t,z}F|\mathcal{F}_t\big]\nu_j\big)$$

and

$$N_{ik}(t) = \sum_{j=1}^{n} \big(\sigma_{ij}(t)\sigma_{kj}(t) + \int_{\mathbb{R}_0} \gamma_{ij}(t,z)\gamma_{kj}(t,z)\nu_j(dz)\big).$$

Therefore, we conclude that, for $k = 1, ..., n$,

$$E\big[M_k(t)|\mathcal{E}_t\big] - \sum_{i=1}^{n} \varphi_i^*(t) E\big[N_{ik}(t)|\mathcal{E}_t\big] = 0$$

or

$$Q(t)\varphi^*(t) = R(t),$$

where

$$Q \in \mathbb{R}^{n \times n}, \quad Q_{ik}(t) = E\big[N_{ik}(t)|\mathcal{E}_t\big], \quad i, k = 1, ..., n,$$

and

$$R(t) \in \mathbb{R}^n, \quad R_i(t) = E\big[M_i(t)|\mathcal{E}_t\big], \quad i = 1, ..., n.$$

The solution of this equation is

$$\varphi^*(t) = Q^{-1}(t)R(t), \quad t \geq 0,$$

which completes the proof. □

Corollary 12.25. *(a) Suppose $n = 1$ and $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \geq 0$. Then*

$$\varphi^*(t) = \frac{E\big[\sigma(t)E\big[D_tF|\mathcal{F}_t\big] + \int_{\mathbb{R}_0} \gamma(t,z)E\big[D_{t,z}F|\mathcal{F}_t\big]\nu(dz)\big|\mathcal{E}_t\big]}{E\big[\sigma^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz)|\mathcal{E}_t\big]}. \tag{12.31}$$

(b) Suppose $n = 1$, σ and γ are $\mathbb{E}$-predictable. Then

$$\varphi^*(t) = \frac{\sigma(t)E\big[D_tF|\mathcal{E}_t\big] + \int_{\mathbb{R}_0} \gamma(t,z)E\big[D_{t,z}F|\mathcal{E}_t\big]\nu(dz)}{\sigma^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz)}. \tag{12.32}$$

Example 12.26. (1) Suppose $n = 1$ and

$$dS(t) = dW(t) + \int_{\mathbb{R}_0} z\tilde{N}(dt, dz).$$

What is the closest hedge to $F := \int_0^T \int_{\mathbb{R}_0} z\tilde{N}(dt, dz)$ in terms of minimal variance? Since $D_t F \equiv 0$, $t \geq 0$, and $D_{t,z}F = z = E\big[D_{t,z}F|\mathcal{F}_t\big]$, $t \geq 0$, $z \in \mathbb{R}_0$, we get

$$\varphi^*(t) = \big(1 + \int_{\mathbb{R}_0} z^2\nu(dz)\big)^{-1} \int_{\mathbb{R}_0} z^2\nu(dz), \quad t \in [0, T]. \tag{12.33}$$

We see that this process is actually constant.

(2) Suppose $n = 1$ and $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \geq 0$, and

$$dS(t) = d\eta(t) = \int_{\mathbb{R}_0} z\tilde{N}(dt, dz).$$

Let us consider $F = S^2(T) = \eta^2(T)$. Then

$$D_{t,z}F = \big(\eta(T) + D_{t,z}\eta(T)\big)^2 - \eta^2(T) = 2\eta(T)z + z^2.$$

Hence the minimal variance portfolio is

$$\begin{aligned}\varphi^*(t) &= \big(\int_{\mathbb{R}_0} z^2\nu(dz)\big)^{-1} \int_{\mathbb{R}_0} \big(2z^2 E\big[S(T)|\mathcal{E}_t\big] + z^3\big)\nu(dz)\\ &= 2E\big[S(t)|\mathcal{E}_t\big] + \frac{\int_{\mathbb{R}_0} z^3\nu(dz)}{\int_{\mathbb{R}_0} z^2\nu(dz)}.\end{aligned} \tag{12.34}$$

(3) Suppose $n = 1$ and $\mathcal{E}_t = \mathcal{F}_t$, $t \geq 0$, and

$$dS(t) = d\eta(t) = \int_{\mathbb{R}_0} z\tilde{N}(dt, dz).$$

Let us consider $F = S^2(T) = \eta^2(T)$. Then, since S is a martingale with respect to $\mathbb{F}$, we get

$$\varphi^*(t) = 2\eta(t^-) + \frac{\int_{\mathbb{R}_0} z^3\nu(dz)}{\int_{\mathbb{R}_0} z^2\nu(dz)}. \tag{12.35}$$

The closest hedge $\hat{F}$ in this case is therefore given by

$$\hat{F} - E[F] = \int_0^T 2\eta(t^-)d\eta(t) + \int_0^T \int_{\mathbb{R}_0} z \frac{\int_{\mathbb{R}_0} \zeta^3\nu(d\zeta)}{\int_{\mathbb{R}_0} \zeta^2\nu(d\zeta)}\tilde{N}(dt, dz). \tag{12.36}$$

If we compare this to (9.34), that is,

$$F - E[F] = \int_0^T 2\eta(t^-)d\eta(t) + \int_0^T \int_{\mathbb{R}_0} z^2 \tilde{N}(dt, dz),$$

we see that the closest hedge to the non-replicable claim

$$G := \int_0^T \int_{\mathbb{R}_0} z^2 \tilde{N}(dt, dz)$$

is the replicable claim

$$\hat{G} := \int_0^T z \frac{\int_{\mathbb{R}_0} \zeta^3 \nu(d\zeta)}{\int_{\mathbb{R}_0} \zeta^2 \nu(d\zeta)} \tilde{N}(dt, dz) = \int_0^T \frac{\int_{\mathbb{R}_0} \zeta^3 \nu(d\zeta)}{\int_{\mathbb{R}_0} \zeta^2 \nu(d\zeta)} d\eta(t).$$

(4) Suppose $n = 1$ and

$$dS(t) = \int_{\mathbb{R}_0} z \tilde{N}(dt, dz)$$

and

$$F := \int_0^T \int_{\mathbb{R}_0} z^n \tilde{N}(dt, dz).$$

Then the minimal variance portfolio

$$\varphi^*(t) = \Big(\int_{\mathbb{R}_0} z^2 \nu(dz) \Big)^{-1} \int_{\mathbb{R}_0} z^{n+1} \nu(dz), \quad t \in [0, T],$$

is constant.

(5) Suppose $n = 1$, $\mathcal{E}_t = \mathcal{F}_t$, $t \geq 0$, and

$$dS(t) = S(t^-) \int_{\mathbb{R}_0} z \tilde{N}(dt, dz).$$

Let us consider $F = S^2(T)$. In this case

$$D_{t,z} F = \big(S(T) + D_{t,z} S(T) \big)^2 - S^2(T).$$

Since $S(T) = S(0) \exp U(T)$, where

$$U(T) := \int_0^T \int_{\mathbb{R}_0} \big(\log(1+z) - z \big) \nu(dz) ds + \int_0^T \int_{\mathbb{R}_0} \log(1+z) \tilde{N}(ds, dz),$$

then we have

$$\begin{aligned} D_{t,z} S(T) &= S(0) \exp\{U + D_{t,z} U)\} - S(0) \exp\{U\} \\ &= S(0) \exp U \exp\{\log(1+z) - 1\} \\ &= S(0) \exp U z = S(T) z. \end{aligned}$$

Hence

$$D_{t,z}F = \big(S(T) + S(T)z\big)^2 - S^2(T) = 2S^2(T)z + S^2(T)z^2$$

and the minimal variance hedging portfolio is

$$\varphi^*(t) = \frac{\int_{\mathbb{R}_0} S^2(T)\big(2z^2 + z^3\big)\nu(dz)}{\int_{\mathbb{R}_0} z^2\nu(dz)}.$$

(6) Suppose $n = 1$, $\mathcal{E}_t = \mathcal{F}_t$, $t \geq 0$, and

$$dS(t) = \int_{\mathbb{R}_0} z\tilde{N}(dt, dz).$$

Consider the digital claim

$$F := \chi_{[k,\infty)}(S(T)).$$

The claim F may not belong to $\mathbb{D}_{1,2}$. Then an extended version of Theorem 12.24 can be applied (see Chap. 13). In this case we have

$$D_{t,z}F = \chi_{[k,\infty)}(S(T) + z) - \chi_{[k,\infty)}(S(T)),$$

which yields

$$\begin{aligned}\varphi^*(t) &= \Big(\int_{\mathbb{R}_0} z^2\nu(dz)\Big)^{-1}\int_{\mathbb{R}_0} zE\big[D_{t,z}F|\mathcal{F}_t\big]\nu(dz)\\ &= \Big(\int_{\mathbb{R}_0} z^2\nu(dz)\Big)^{-1}\int_{\mathbb{R}_0} zE\big[\chi_{[k,\infty)}(S(T-t) + z)\\ &\quad - \chi_{[k,\infty)}(S(T-t))|S(t)\big]\nu(dz).\end{aligned}$$

12.7 Computation of "Greeks" in the Case of Jump Diffusions

In Sect. 4.4 it has been demonstrated how Malliavin calculus can be employed to compute option price sensitivities, commonly referred to as the "greeks," for asset price processes modeled by stochastic differential equations driven by a Wiener process. The greeks are in a sense "risk measures", which are used by investors on financial markets to hedge their positions. These greeks measure changes of contract prices with respect to parameters in the underlying model. Roughly speaking, greeks are derivatives with respect to a parameter θ of a risk-neutral price, that is, for example, of the form

$$\frac{\partial}{\partial\theta}E\left[\phi(S(T))\right],$$

where $\phi(S(T))$ is the payoff function and $S(T)$ is the underlying asset, which depends on θ.

The Malliavin approach of [81] for the calculation of greeks has proven to be numerically effective and in many respects sometimes better than other tools such as, the finite difference or the likelihood ratio method [43]. This technique is especially useful if it comes to handling discontinuous payoffs and path dependent options. See for example, [45] for more information and the references therein.

In this section we wish to extend the method of [81] as presented in Sect. 4.4 to the case of Itô jump diffusions. The general idea is to take the Malliavin derivative in the direction of the Wiener process on the Wiener–Poisson space, see Sect. 12.5. This enables us to stay in the framework of a variational calculus for the Wiener process without major changes. However, it should be mentioned that the pure jump case cannot be treated by this approach in the same way, since the Malliavin derivative with respect to the jump component is a difference operator in the sense of Theorem 12.8.

We remark that there are several authors in the literature dealing with jump diffusion models, see, for example, [15, 54, 55].

Rather than striving for the most general setting, we want to illustrate the basic ideas of this method by considering asset prices described by the Barndorff–Nielsen and Shephard model (see [18]). See also [28].

12.7.1 The Barndorff–Nielsen and Shephard Model

Adopting the notation of Sect. 12.5 we assume that the one-dimensional Wiener process $W(t)$ and the compensated Poisson random measure $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is constructed on the Wiener–Poisson probability space $(\Omega, \mathcal{F}, P)$ given by

$$(\Omega, \mathcal{F}, P) = (\Omega_0 \times \Omega_0, \mathcal{F}_T^W \otimes \mathcal{F}_T^{\widetilde{N}}, P^W \times P^{\widetilde{N}}),$$

where P^W is the Gaussian and $P^{\widetilde{N}}$ is the white noise Lévy measure on the Schwartz distribution space $\Omega_0 = \mathcal{S}'(\mathbb{R})$. As before, let us denote by D_t and $D_{t,z}$ the Malliavin derivatives in the direction of the Wiener process and the Poisson random measure, respectively. The operator D_t can be defined on the Hilbert space

$$\widetilde{\mathbb{D}}_{1,2}, \tag{12.37}$$

which is the closure of a suitable space of smooth random variables (e.g., the linear span of basis elements $\mathbb{H}_\alpha$ as given in Sect. 12.5) with respect to the semi-norm

$$\|F\|_{1,2} := \left(E\left[F^2\right] + E\left[\int_0^T |D_t F|^2 \, dt\right]\right)^{1/2}.$$

It can be seen from this construction that the results obtained in Chap. 3 still hold for D_t on the Wiener–Poisson space $(\Omega, \mathcal{F}, P)$. So, for example, in this setting the chain rule for D_t reads

$$D_t\phi(F_1,...,F_m) = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\phi(F_1,...,F_m)D_tF_j \quad \lambda \times P^W \times P^{\widetilde{N}} - \text{a.e.}, \quad (12.38)$$

if $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a bounded continuously differentiable function and $(F_1,...,F_m)$ is a random vector with components in $\widetilde{\mathbb{D}}_{1,2}$. See Theorem 3.5.

In this section, we want to briefly discuss the Barndorff–Nielsen and Shephard (BNS) model which was introduced in [18]. This model exhibits nice features and can, for example, be used to fit it to high-frequency stock price data.

In the following, we consider a financial market consisting of a single risk-free asset and a risky asset (stock). Further, let us assume that the stock price $S(t)$ is defined on the Wiener–Poisson space $(\Omega, \mathcal{F}, P)$ and given by

$$S(t) = S^x(t) = x\exp(X(t)), 0 \le t \le T, \quad (12.39)$$

where

$$dX(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t) + \rho dZ(\lambda t), X(0) = 0 \quad (12.40)$$

with stochastic volatility $\sigma^2(t)$ given by the Lévy–Ornstein–Uhlenbeck (OU) process

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t), \sigma^2(0) > 0. \quad (12.41)$$

Here $Z(t)$ is a "background driving" Lévy process given by a *subordinator*, that is, a nondecreasing Lévy process. Such a process has the representation

$$Z(t) = mt + \int_0^t \int_{\mathbb{R}_0} zN(ds, dz) \quad (12.42)$$

for a constant $m \ge 0$ with Lévy measure ν such that $\text{supp}(\nu) \subseteq (0, \infty)$. Further, the law of $Z(t),\ 0 \le t \le T$, is completely determined by its cumulant generating function

$$\kappa(\alpha) := \log\left(E\left[\exp(\alpha Z(1))\right]\right), \quad (12.43)$$

(see, e.g., [32]). The processes $W(t)$ and $Z(t)$ are independent. In addition, $r > 0$ is the constant market interest rate and the constants $\lambda > 0, \rho \le 0$ stand for the mean-reversion rate of the stochastic volatility and the leverage effect of the (log-) price process, respectively. Moreover, μ and β are constant parameters.

Using the Itô formula (Theorem 9.4) one finds that the volatility process in (12.41) has the explicit form

$$\sigma^2(t) = \sigma^2(0)e^{-\lambda t} + \int_0^t e^{\lambda(s-t)}dZ(\lambda s), 0 \le t \le T. \quad (12.44)$$

Throughout the rest of this section we require that the subordinator $Z(t)$ has no drift (i.e., $m = 0$) in (12.42) and that ν has a density w with respect to the Lebsegue measure. The latter implies that

$$\kappa(\alpha) = \int_{\mathbb{R}_0} (e^{\alpha z} - 1)\nu(dz) = \int_{(0,\infty)} (e^{\alpha z} - 1)w(z)dz. \tag{12.45}$$

See [32]. Define

$$\widehat{\alpha} = \sup\{\alpha \in \mathbb{R} : \kappa(\alpha) < \infty\}.$$

In addition, we assume that

$$\widehat{\alpha} > \max\{0, 2\lambda^{-1}(1 + \beta + \rho)\} \text{ and } \lim_{\alpha \longrightarrow \widehat{\alpha}} \kappa(\alpha) = \infty. \tag{12.46}$$

The last condition ensures the square integrability of the asset process $S(t)$ and the existence of an invariant distribution of the volatility process. See [167].

As mentioned the greeks are derivatives of the expected (discounted) payoff under a risk neutral measure Q. However, a measure change from the real world measure to Q might result in a dynamics being different from the BNS model (12.40) and (12.41). Therefore, we are interested in "structure preserving" risk neutral measures Q, which transform BNS models into BNS models with possibly different parameters and Lévy measure for $Z(t)$. It turns out (see [167]) that the risk neutral dynamics of the BNS model under such measures takes the general form

$$dX(t) = (r - \lambda\kappa(\rho) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t) + \rho dZ(\lambda t) \tag{12.47}$$

and

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t), \sigma^2(0) > 0. \tag{12.48}$$

From now on we confine ourselves to the risk neutral dynamics of the BNS model given by (12.47) and (12.48), for which conditions (12.45) and (12.46) are satisfied. In what follows we replace Q by our white noise measure P (since we only deal with probability laws under expectations).

12.7.2 Malliavin Weights for "Greeks"

In the sequel we want to compute the Malliavin weights for the delta and gamma of an option. To this end we need the following auxiliary results:

Lemma 12.27. *Suppose that F^θ is a real valued random variable, which depends on a parameter $\theta \in \mathbb{R}$. Further require that the mapping $\theta \longmapsto F^\theta(\omega)$ is continuously differentiable in $[a, b]$ ω-a.e. and that*

$$E\left[\sup_{a \le \theta \le b} \left|\frac{\partial}{\partial\theta} F^\theta\right|\right] < \infty.$$

Then $\theta \longmapsto E\left[F^\theta\right]$ is differentiable in (a, b), and for $\theta \in (a, b)$ we have

$$\frac{\partial}{\partial\theta} E\left[F^\theta\right] = E\left[\frac{\partial}{\partial\theta} F^\theta\right].$$

Proof The result follows from the mean value theorem and dominated convergence. The details are left to the reader. □

Adopting the notation in [28], we denote by $L^2(S)$ the space of locally integrable functions $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that

$$E\left[\phi^2(S(t_1), ..., S(t_m))\right] < \infty. \tag{12.49}$$

Our asset price process depends on the parameters $\theta = x,\ r,\ \rho$, and $\sigma^2(0)$. In the following we write $S(\cdot) = S^\theta(\cdot)$ to indicate this dependency.

Lemma 12.28. *Let $\theta \longmapsto \pi^\theta$ be a process such that $\theta \longmapsto \psi(\theta) := \left\|\pi^\theta\right\|_{L^2(P)}$ is locally bounded. Further assume that*

$$\frac{\partial}{\partial\theta} E\left[\phi(S^\theta(t_1), ..., S^\theta(t_m))\right] = E\left[\phi(S^\theta(t_1), ..., S^\theta(t_m))\pi^\theta\right] \tag{12.50}$$

is valid for all $\phi \in C_c^\infty(\mathbb{R}^m)$ (i.e., ϕ is an infinitely differentiable function with compact support). Then relation (12.50) also holds for all $\phi \in L^2(S)$.

Proof Let ϕ be a bounded function. Then there exists a uniformly bounded sequence of functions $\phi_k, k \geq 1$ satisfying (12.50) such that

$$\phi_k \longrightarrow \phi \text{ a.e.}$$

Using transition probability densities in connection with $X(t)$ in (12.47) one finds that

$$\phi_k(S^\theta(t_1), ..., S^\theta(t_m)) \longrightarrow \phi(S^\theta(t_1), ..., S^\theta(t_m))$$

in $L^2(P)$ uniformly on compact sets. Define

$$u(\theta) = E\left[\phi(S^\theta(t_1), ..., S^\theta(t_m))\right] \text{ and } u_k(\theta) = E\left[\phi_k(S^\theta(t_1), ..., S^\theta(t_m))\right].$$

As above one verifies that $u_k(\theta) \longrightarrow u(\theta)$ for all θ. Further let

$$f(\theta) := E\left[\phi(S^\theta(t_1), ..., S^\theta(t_m))\pi^\theta\right].$$

By the Cauchy–Schwartz inequality we get

$$\left|\frac{\partial}{\partial\theta} u_k(\theta) - f(\theta)\right| \leq \epsilon_k(\theta)\psi(\theta),$$

where

$$\epsilon_k(\theta) = \left(E\left[\left(\phi_k(S^\theta(t_1), ..., S^\theta(t_m)) - \phi(S^\theta(t_1), ..., S^\theta(t_m))\right)^2\right]\right)^{1/2}.$$

Since $\theta \longmapsto \psi(\theta)$ is locally bounded, it follows that

$$\frac{\partial}{\partial\theta} u_k(\theta) \longrightarrow f(\theta) \text{ as } k \longrightarrow \infty$$

uniformly on compact sets. Hence, ϕ also fulfills (12.50). So (12.50) is valid for all bounded measurable functions. The general case finally follows from a truncation argument. □

Let us remark that $L^2(S)$ contains important options as, for example, the call option.

We are coming to the main result that is due to [28]:

Theorem 12.29. *Let $\phi \in L^2(S)$ and let $a \in L^2([0,T])$ be an adapted process such that*

$$\int_0^{t_i} a(t)dt = 1 \quad P - a.e.$$

for all $i = 1, ..., m$. Then
(1) The delta of the option is given by

$$\frac{\partial}{\partial x} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\right] = E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\pi^{\Delta}\right],$$

where the Malliavin weight π^{Δ} is given by

$$\pi^{\Delta} = \int_0^T \frac{a(t)}{x\sigma(t)} dW(t).$$

(2) The gamma of the option is given by

$$\frac{\partial^2}{\partial x^2} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\right] = E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\pi^{\Gamma}\right],$$

where the Malliavin weight π^{Γ} has the form

$$\pi^{\Gamma} = \left(\pi^{\Delta}\right)^2 - \frac{1}{x}\pi^{\Delta} - \frac{1}{x^2}\int_0^T \left(\frac{a(t)}{\sigma(t)}\right)^2 dt.$$

Proof It is easily seen that $\theta \longmapsto S^{\theta}$ is pathwise differentiable (with exception of the boundary values $x = 0$ and $\sigma^2(0) = 0$) for the different parameters $\theta = x, r, \rho, \sigma^2(0)$. Further, one checks that the assumptions of Lemma 12.27 and Lemma 12.28 are satisfied. So it remains to verify relation (12.50) for $\phi \in C_0^{\infty}(\mathbb{R}^m)$.
(1) Using Lemma 12.27, we find

$$\begin{aligned}
&\frac{\partial}{\partial x} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\right] \\
&= E\left[e^{-rT}\frac{\partial}{\partial x}\phi(S^x(t_1), ..., S^x(t_m))\right] \\
&= E\left[e^{-rT}\sum_{i=1}^{m_i}\phi_{x_i}(S^x(t_1), ..., S^x(t_m))\frac{\partial}{\partial x}S^x(t_i)\right] \\
&= E\left[e^{-rT}\sum_{i=1}^{m_i}\phi_{x_i}(S^x(t_1), ..., S^x(t_m))\frac{1}{x}S^x(t_i)\right].
\end{aligned}$$

By applying the chain rule (Theorem 3.5 or (12.38)) and the fundamental theorem of stochastic calculus (Theorem 3.18) in the direction of $W(t)$ we obtain

$$D_t S^x(t_i) = \sigma(t) S^x(t_i) \chi_{[0,t_i]}(t).$$

Since $\int_0^{t_i} a(t)dt = 1$, we get

$$\int_0^T \frac{a(t)}{x\sigma(t)} D_t S^x(t_i) dt = \frac{1}{x} S^x(t_i).$$

Hence,

$$\begin{aligned}&\frac{\partial}{\partial x} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\right] \\ &= E\left[e^{-rT}\int_0^T \sum_{i=1}^{m_i} \phi_{x_i}(S^x(t_1), ..., S^x(t_m)) \frac{a(t)}{x\sigma(t)} D_t S^x(t_i) dt\right].\end{aligned}$$

Then the chain rule (12.38) yields

$$\frac{\partial}{\partial x} E\Big[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\Big] = e^{-rT} E\left[\int_0^T \sum_{i=1}^{m_i} D_t \phi(S^x(t_1), ..., S^x(t_m)) \frac{a(t)}{x\sigma(t)} dt\right].$$

Finally, the duality formula (Theorem 3.14) gives the Malliavin weight $\pi^\Delta = \int_0^T \frac{a(t)}{x\sigma(t)} dW(t)$.
(2) Define $F^x = \int_0^T \frac{a(t)}{x\sigma(t)} dW(t)$. Then $\frac{\partial}{\partial x} F^x = -\frac{1}{x} F^x$. Thus

$$\begin{aligned}&\frac{\partial^2}{\partial x^2} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m))\right] \\ &= \frac{\partial}{\partial x} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m)) F^x\right] \\ &= -\frac{1}{x} E\left[e^{-rT}\phi(S^x(t_1), ..., S^x(t_m)) F^x\right] \\ &\quad + E\left[e^{-rT}\sum_{i=1}^{m_i} \phi_{x_i}(S^x(t_1), ..., S^x(t_m)) \frac{1}{x} S^x(t_i) F^x\right].\end{aligned} \tag{12.51}$$

Repeated use of the arguments of (1) gives

$$\begin{aligned}&E\left[e^{-rT}\sum_{i=1}^{m_i} \phi_{x_i}(S^x(t_1), ..., S^x(t_m)) \frac{1}{x} S^x(t_i) F^x\right] \\ &= E\left[e^{-rT}\int_0^T \sum_{i=1}^{m_i} D_t \phi(S^x(t_1), ..., S^x(t_m)) \frac{a(t)}{x\sigma(t)} F^x dt\right] \\ &= E\left[e^{-rT}\sum_{i=1}^{m_i} D_t \phi(S^x(t_1), ..., S^x(t_m)) \delta\left(\frac{a(\cdot)}{x\sigma(\cdot)} F^x\right)\right].\end{aligned}$$

Finally, noting that $D_t F^x = \frac{a(t)}{x\sigma(t)}$, it follows from the integration by parts formula (Theorem 3.15) that

$$\begin{aligned}\delta\left(\frac{a(\cdot)}{x\sigma(\cdot)}F^x\right) &= F^x\int_0^T \frac{a(t)}{x\sigma(t)}dW(t) - \int_0^T\left(\frac{a(t)}{x\sigma(t)}\right)^2 dt \\ &= (F^x)^2 - \int_0^T\left(\frac{a(t)}{x\sigma(t)}\right)^2 dt,\end{aligned}$$

which, in connection with (12.52), gives the proof. □

12.8 Exercises

Problem 12.1. (*) Let

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dz, dz), \quad t \in [0,T].$$

Use Definition 12.2 and the chaos expansions found in Problem 10.1 to compute the following:

(a) $D_{t,z}\eta^3(T)$, $(t,z) \in [0,T]\times\mathbb{R}_0$,
(b) $D_{t,z}\exp\eta(T)$, $(t,z) \in [0,T]\times\mathbb{R}_0$.

Problem 12.2. (*) Compute the Malliavin derivatives in Problem 12.1 by using the chain rule (see Theorem 12.8) together with (12.5).

Problem 12.3. Let the process $X(t)$, $t \in [0,T]$, be the geometric Lévy process

$$dX(t) = X(t^-)[\alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0}\gamma(t,z)\tilde{N}(dt,dz)],$$

where the involved coefficients are deterministic. Find $D_{t,z}X(T)$ for $t \le T$.

Problem 12.4. Use the integration by parts formula (Theorem 12.11) to compute the Skorohod integrals

$$\int_0^T F\delta\eta(t) = \int_0^T\int_{\mathbb{R}_0} Fz\widetilde{N}(\delta t, dz)$$

in the following cases:

(a) $F = \eta(T)$
(b) $F = \eta^2(T)$
(c) $F = \eta^3(T)$
(d) $F = \exp\{\eta(T)\}$
(e) $F = \int_0^T g(s)d\eta(s)$, where $g \in L^2([0,T])$.

Problem 12.5. Solve Problem 9.6 using the Clark–Ocone theorem.

Problem 12.6. Consider the following market

$$\text{risk free asset:} \qquad dS_0(t) = 0, \quad S_0(0) = 1$$
$$\text{risky asset:} \qquad dS_1(t) = S_1(t^-) \int_{\mathbb{R}_0} z\tilde{N}(dt, dz), \quad S_1(0) > 0,$$

where $z > -1$ ν-a.e. Find the closest hedge in terms of minimal variance for the following claims:

(a) $F = S_1^2(T)$,
(b) $F = \exp\{\lambda S_1(T)\}$, with $\lambda \in \mathbb{R}$ constant.

Problem 12.7. Consider the claim $F = \eta^3(T)$ in the Bachelier–Lévy market

$$\text{risk free asset:} \qquad dS_0(t) = 0, \quad S_0(0) = 1$$
$$\text{risky asset:} \qquad dS_1(t) = \int_{\mathbb{R}_0} z\tilde{N}(dt, dz), \quad S_1(0) = 0.$$

(a) Is the claim replicable in this market?
(b) If not, what is the closest hedge in terms of minimal variance?