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The Wiener–Itô Chaos Expansion

The celebrated Wiener–Itô chaos expansion is fundamental in stochastic analysis. In particular, it plays a crucial role in the Malliavin calculus as it is presented in the sequel. This result which concerns the representation of square integrable random variables in terms of an infinite orthogonal sum was proved in its first version by Wiener in 1938 [227]. Later, in 1951, Itô [120] showed that the expansion could be expressed in terms of *iterated Itô integrals* in the Wiener space setting.

Before we state the theorem we introduce some useful notation and give some auxiliary results.

1.1 Iterated Itô Integrals

Let $W = W(t) = W(\omega, t)$, $\omega \in \Omega$ $t \in [0, T]$ $(T > 0)$, be a one-dimensional Wiener process, or equivalently Brownian motion, on the complete probability space $(\Omega, \mathcal{F}, P)$ such that $W(0) = 0$ P-a.s.

For any t, let $\mathcal{F}_t$ be the σ-algebra generated by $W(s)$, $0 \le s \le t$, augmented by all the P-zero measure events. We denote the corresponding filtration by

$$\mathbb{F} = \{\mathcal{F}_t,\ t \in [0, T]\}. \tag{1.1}$$

Note that this filtration is both left- and right-continuous, that is,

$$\mathcal{F}_t = \lim_{s \nearrow t} \mathcal{F}_s := \sigma\Big\{ \bigcup_{s<t} \mathcal{F}_s \Big\},$$

respectively,

$$\mathcal{F}_t = \lim_{u \searrow t} \mathcal{F}_u := \bigcap_{u>t} \mathcal{F}_u.$$

See, for example, [129] or [207].

Definition 1.1. *A real function* $g : [0,T]^n \to \mathbb{R}$ *is called* symmetric *if*

$$g(t_{\sigma_1}, \ldots, t_{\sigma_n}) = g(t_1, \ldots, t_n) \tag{1.2}$$

for all permutations $\sigma = (\sigma_1, ..., \sigma_n)$ *of* $(1, 2, \ldots, n)$.

Let $L^2([0,T]^n)$ be the standard space of square integrable Borel real functions on $[0,T]^n$ such that

$$\|g\|^2_{L^2([0,T]^n)} := \int_{[0,T]^n} g^2(t_1, \ldots, t_n) dt_1 \cdots dt_n < \infty. \tag{1.3}$$

Let $\widetilde{L}^2([0,T]^n) \subset L^2([0,T]^n)$ be the space of *symmetric* square integrable Borel real functions on $[0,T]^n$. Let us consider the set

$$S_n = \{(t_1, \ldots, t_n) \in [0,T]^n : \;\; 0 \le t_1 \le t_2 \le \cdots \le t_n \le T\}.$$

Note that this set S_n occupies the fraction $\frac{1}{n!}$ of the whole n-dimensional box $[0,T]^n$. Therefore, if $g \in \widetilde{L}^2([0,T]^n)$ then $g_{|S_n} \in L^2(S_n)$ and

$$\|g\|^2_{L^2([0,T]^n)} = n! \int_{S_n} g^2(t_1, \ldots, t_n) dt_1 \ldots dt_n = n! \|g\|^2_{L^2(S_n)}, \tag{1.4}$$

where $\|\cdot\|_{L^2(S_n)}$ denotes the norm induced by $L^2([0,T]^n)$ on $L^2(S_n)$, the space of the square integrable functions on S_n.

If f is a real function on $[0,T]^n$, then its *symmetrization* $\widetilde{f}$ is defined by

$$\widetilde{f}(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \ldots, t_{\sigma_n}), \tag{1.5}$$

where the sum is taken over all permutations σ of $(1, \ldots, n)$. Note that $\widetilde{f} = f$ if and only if f is symmetric.

Example 1.2. The symmetrization of the function

$$f(t_1, t_2) = t_1^2 + t_2 \sin t_1, \quad (t_1, t_2) \in [0,T]^2,$$

is

$$\widetilde{f}(t_1, t_2) = \frac{1}{2}\left[t_1^2 + t_2^2 + t_2 \sin t_1 + t_1 \sin t_2\right], \quad (t_1, t_2) \in [0,T]^2.$$

Definition 1.3. *Let* f *be a deterministic function defined on* S_n $(n \ge 1)$ *such that*

$$\|f\|^2_{L^2(S_n)} := \int_{S_n} f^2(t_1, \ldots, t_n) dt_1 \cdots dt_n < \infty.$$

Then we can define the n-fold iterated Itô integral *as*

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \ldots, t_n) dW(t_1) dW(t_2) \cdots dW(t_{n-1}) dW(t_n). \tag{1.6}$$

Note that at each iteration $i = 1, ..., n$ the corresponding Itô integral with respect to $dW(t_i)$ is well-defined, being the integrand $\int_0^{t_i} \cdots \int_0^{t_2} f(t_1, ..., t_n)$ $dW(t_1)...dW(t_{i-1})$, $t_i \in [0, t_{i+1}]$, a stochastic process that is $\mathbb{F}$-adapted and square integrable with respect to $dP \times dt_i$. Thus, (1.6) is well-defined.

Thanks to the construction of the Itô integral we have that $J_n(f)$ belongs to $L^2(P)$, that is, the space of square integrable random variables. We denote the norm of $X \in L^2(P)$ by

$$\|X\|_{L^2(P)} := \Big(E\left[X^2\right]\Big)^{1/2} = \Big(\int_\Omega X^2(\omega)P(d\omega)\Big)^{1/2}.$$

Applying the Itô isometry iteratively, if $g \in L^2(S_m)$ and $h \in L^2(S_n)$, with $m < n$, we can see that

$$\begin{aligned}
&E\Big[J_m(g)J_n(h)\Big] = E\Big[\Big(\int_0^T \int_0^{s_m} \cdots \int_0^{s_2} g(s_1, \dots, s_m)dW(s_1)\cdots dW(s_m)\Big) \\
&\cdot \Big(\int_0^T \int_0^{s_m} \cdots \int_0^{t_2} h(t_1, \dots, t_{n-m}, s_1, \dots, s_m)dW(t_1)\cdots dW(t_{n-m})dW(s_1)\cdots dW(s_m)\Big)\Big] \\
&= \int_0^T E\Big[\Big(\int_0^{s_m} \cdots \int_0^{s_2} g(s_1, \dots, s_{m-1}, s_m)dW(s_1)\cdots dW(s_{m-1})\Big) \\
&\cdot \Big(\int_0^{s_m} \cdots \int_0^{t_2} h(t_1, \dots, s_{m-1}, s_m)dW(t_1)\cdots dW(s_{m-1})\Big)\Big]ds_m = \dots \\
&= \int_0^T \int_0^{s_m} \cdots \int_0^{s_2} g(s_1, s_2, \dots, s_m)E\Big[\int_0^{s_1} \cdots \int_0^{t_2} h(t_1, \dots, t_{n-m}, s_1, \dots, s_m) \\
&\cdot dW(t_1)\cdots dW(t_{n-m})\Big]ds_1\cdots ds_m = 0
\end{aligned} \tag{1.7}$$

because the expected value of an Itô integral is zero. On the other hand, if both g and h belong to $L^2(S_n)$, then

$$\begin{aligned}
E\big[J_n(g)J_n(h)\big] &= \int_0^T E\Big[\int_0^{s_n} \cdots \int_0^{s_2} g(s_1, \dots, s_n)dW(s_1)\cdots dW(s_{n-1}) \\
&\cdot \int_0^{s_n} \cdots \int_0^{s_2} h(s_1, \dots, s_n)dW(s_1)\cdots dW(s_{n-1})\Big]ds_n = \dots \\
&= \int_0^T \cdots \int_0^{s_2} g(s_1, \dots, s_n)h(s_1, \dots, s_n)ds_1\cdots ds_n = (g, h)_{L^2(S_n)}
\end{aligned} \tag{1.8}$$

We summarize these results as follows.

Proposition 1.4. *The following relations hold true:*

$$E[J_m(g)J_n(h)] = \begin{cases} 0 & , \; n \neq m \\ (g,h)_{L^2(S_n)} & , \; n = m \end{cases} \qquad (m,n = 1,2,...), \tag{1.9}$$

where

$$(g,h)_{L^2(S_n)} := \int\limits_{S_n} g(t_1,\dots,t_n)h(t_1,\dots,t_n)dt_1\cdots dt_n$$

is the inner product of $L^2(S_n)$. In particular, we have

$$\|J_n(h)\|_{L^2(P)} = \|h\|_{L^2(S_n)}. \tag{1.10}$$

Remark 1.5. Note that (1.9) also holds for $n = 0$ or $m = 0$ if we define $J_0(g) = g$, when g is a constant, and $(g,h)_{L^2(S_0)} = gh$, when g, h are constants.

Remark 1.6. It is straightforward to see that the n-fold iterated Itô integral

$$L^2(S_n) \ni f \quad \Longrightarrow \quad J_n(f) \in L^2(P)$$

is a linear operator, that is, $J_n(af + bg) = aJ_n(f) + bJ_n(g)$, for $f, g \in L^2(S_n)$ and $a, b \in \mathbb{R}$.

Definition 1.7. *If $g \in \widetilde{L}^2([0,T]^n)$ we define*

$$I_n(g) := \int\limits_{[0,T]^n} g(t_1,\dots,t_n)dW(t_1)\dots dW(t_n) := n!J_n(g). \tag{1.11}$$

We also call n-fold iterated Itô integrals the $I_n(g)$ here above.

Note that from (1.9) and (1.11) we have

$$\begin{aligned}\|I_n(g)\|^2_{L^2(P)} &= E[I_n^2(g)] = E[(n!)^2 J_n^2(g)] \\ &= (n!)^2\|g\|^2_{L^2(S_n)} = n!\|g\|^2_{L^2([0,T]^n)}\end{aligned} \tag{1.12}$$

for all $g \in \widetilde{L}^2([0,T]^n)$. Moreover, if $g \in \widetilde{L}^2([0,T]^m)$ and $h \in \widetilde{L}^2([0,T]^n)$, we have

$$E[I_m(g)I_n(h)] = \begin{cases} 0 & , \; n \neq m \\ (g,h)_{L^2([0,T]^n)} & , \; n = m \end{cases} \qquad (m,n = 1,2,...),$$

with $(g,h)_{L^2([0,T]^n)} = n!(g,h)_{L^2(Sn)}$.

There is a useful formula due to Itô [120] for the computation of the iterated Itô integral. This formula relies on the relationship between Hermite polynomials and the Gaussian distribution density. Recall that the *Hermite polynomials* $h_n(x)$, $x \in \mathbb{R}$, $n = 0, 1, 2, \dots$ are defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}), \qquad n = 0,1,2,\dots, \tag{1.13}$$

Thus, the first Hermite polynomials are

$$h_0(x) = 1,\ h_1(x) = x,\ h_2(x) = x^2 - 1,\ \ h_3(x) = x^3 - 3x,$$
$$h_4(x) = x^4 - 6x^2 + 3,\ h_5(x) = x^5 - 10x^3 + 15x, \ldots .$$

We also recall that the family of Hermite polynomials constitute an orthogonal basis for $L^2(\mathbb{R}, \mu(dx))$ if $\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx$ (see, e.g., [215]).

Proposition 1.8. *If $\xi_1, \xi_2, \ldots$ are orthonormal functions in $L^2([0,T])$, we have that*

$$I_n\big(\xi_1^{\otimes\alpha_1} \hat{\otimes} \cdots \hat{\otimes} \xi_m^{\otimes\alpha_m}\big) = \prod_{k=1}^{m} h_{\alpha_k}\Big(\int_0^T \xi_k(t) W(t)\Big), \tag{1.14}$$

with $\alpha_1 + \cdots + \alpha_m = n$. Here $\otimes$ denotes the tensor power *and $\alpha_k \in \{0, 1, 2, \ldots\}$ for all k.*

See [120]. In general, the *tensor product* $f \otimes g$ of two functions f, g is defined by

$$(f \otimes g)(x_1, x_2) = f(x_1) g(x_2)$$

and the *symmetrized tensor product* $f \hat{\otimes} g$ is the symmetrization of $f \otimes g$. In particular, from (1.14), we have

$$n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} g(t_1) g(t_2) \cdots g(t_n) dW(t_1) \cdots dW(t_n) = \|g\|^n h_n\Big(\frac{\theta}{\|g\|}\Big), \tag{1.15}$$

for the tensor power of $g \in L^2([0,T])$. Here above we have used $\|g\| = \|g\|_{L^2([0,T])}$ and $\theta = \int_0^T g(t) dW(t)$.

Example 1.9. Let $g \equiv 1$ and $n = 3$, then we get

$$6 \int_0^T \int_0^{t_3} \int_0^{t_2} 1\, dW(t_1) dW(t_2) dW(t_3) = T^{3/2} h_3\Big(\frac{W(T)}{T^{1/2}}\Big) = W^3(T) - 3T\, W(T).$$

1.2 The Wiener–Itô Chaos Expansion

Theorem 1.10. The Wiener–Itô chaos expansion. *Let ξ be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \widetilde{L}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n), \tag{1.16}$$

where the convergence is in $L^2(P)$. Moreover, we have the isometry

$$\|\xi\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2([0,T]^n)}. \tag{1.17}$$

Proof By the Itô representation theorem there exists an $\mathbb{F}$-adapted process $\varphi_1(s_1)$, $0 \le s_1 \le T$, such that

$$E\Big[\int_0^T \varphi_1^2(s_1)ds_1\Big] \le E\left[\xi^2\right] \tag{1.18}$$

and

$$\xi = E[\xi] + \int_0^T \varphi_1(s_1)dW(s_1). \tag{1.19}$$

Define

$$g_0 = E[\xi].$$

For almost all $s_1 \le T$ we can apply the Itô representation theorem to $\varphi_1(s_1)$ to conclude that there exists an $\mathbb{F}$-adapted process $\varphi_2(s_2, s_1)$, $0 \le s_2 \le s_1$, such that

$$E\Big[\int_0^{s_1} \varphi_2^2(s_2, s_1)ds_2\Big] \le E[\varphi_1^2(s_1)] < \infty \tag{1.20}$$

and

$$\varphi_1(s_1) = E[\varphi_1(s_1)] + \int_0^{s_1} \varphi_2(s_2, s_1)dW(s_2). \tag{1.21}$$

Substituting (1.21) in (1.19) we get

$$\xi = g_0 + \int_0^T g_1(s_1)dW(s_1) + \int_0^T \int_0^{s_1} \varphi_2(s_2, s_1)dW(s_2)dW(s_1), \tag{1.22}$$

where

$$g_1(s_1) = E[\varphi_1(s_1)].$$

Note that by (1.18), (1.20), and the Itô isometry we have

$$E\Big[\Big(\int_0^T \int_0^{s_1} \varphi_2(s_2, s_1)dW(s_2)dW(s_1)\Big)^2\Big] = \int_0^T \int_0^{s_1} E[\varphi_2^2(s_2, s_1)]ds_2ds_1 \le E[\xi^2].$$

Similarly, for almost all $s_2 \le s_1 \le T$, we apply the Itô representation theorem to $\varphi_2(s_2, s_1)$ and we get an $\mathbb{F}$-adapted process $\varphi_3(s_3, s_2, s_1)$, $0 \le s_3 \le s_2$, such that

$$E\Big[\int_0^{s_2} \varphi_3^2(s_3, s_2, s_1)ds_3\Big] \le E[\varphi_2^2(s_2, s_1)] < \infty \tag{1.23}$$

and

$$\varphi_2(s_2, s_1) = E[\varphi_2(s_2, s_1)] + \int_0^{s_2} \varphi_3(s_3, s_2, s_1)dW(s_3). \tag{1.24}$$

Substituting (1.24) in (1.22) we get

$$\begin{aligned}\xi = g_0 &+ \int_0^T g_1(s_1)dW(s_1) + \int_0^T \int_0^{s_1} g_2(s_2, s_1)dW(s_2)dW(s_1) \\ &+ \int_0^T \int_0^{s_1} \int_0^{s_2} \varphi_3(s_3, s_2, s_1)dW(s_3)dW(s_2)dW(s_1),\end{aligned}$$

where

$$g_2(s_2, s_1) = E[\varphi_2(s_2, s_1)], \qquad 0 \le s_2 \le s_1 \le T.$$

By (1.18), (1.20), (1.23), and the Itô isometry we have

$$E\Big[\Big(\int_0^T \int_0^{s_1} \int_0^{s_2} \varphi_3(s_3, s_2, s_1)dW(s_3)dW(s_2)dW(s_1)\Big)^2\Big] \le E\left[\xi^2\right].$$

By iterating this procedure we obtain after n steps a process $\varphi_{n+1}(t_1, t_2, \ldots, t_{n+1})$, $0 \le t_1 \le t_2 \le \cdots \le t_{n+1} \le T$, and $n+1$ deterministic functions $g_0, g_1, \ldots, g_n$, with g_0 constant and g_k defined on S_k for $1 \le k \le n$, such that

$$\xi = \sum_{k=0}^{n} J_k(g_k) + \int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)},$$

where

$$\int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)} := \int_0^T \int_0^{t_{n+1}} \cdots \int_0^{t_2} \varphi_{n+1}(t_1, \ldots, t_{n+1})dW(t_1) \cdots dW(t_{n+1})$$

is the $(n+1)$-fold iterated integral of φ_{n+1}. Moreover,

$$E\Big[\Big(\int_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)}\Big)^2\Big] \le E\left[\xi^2\right].$$

In particular, the family

$$\psi_{n+1} := \int\limits_{S_{n+1}} \varphi_{n+1} dW^{\otimes(n+1)}, \qquad n = 1, 2, \ldots$$

is bounded in $L^2(P)$ and, from the Itô isometry,

$$(\psi_{n+1}, J_k(f_k))_{L^2(P)} = 0 \tag{1.25}$$

for $k \leq n$, $f_k \in L^2([0,T]^k)$. Hence we have

$$\|\xi\|^2_{L^2(P)} = \sum_{k=0}^{n} \|J_k(g_k)\|^2_{L^2(P)} + \|\psi_{n+1}\|^2_{L^2(P)}.$$

In particular,

$$\sum_{k=0}^{n} \|J_k(g_k)\|^2_{L^2(P)} < \infty, \; n = 1, 2, ...$$

and therefore $\sum\limits_{k=0}^{\infty} J_k(g_k)$ is convergent in $L^2(P)$. Hence

$$\lim_{n\to\infty} \psi_{n+1} =: \psi$$

exists in $L^2(P)$. But by (1.25) we have

$$(J_k(f_k), \psi)_{L^2(P)} = 0$$

for all k and for all $f_k \in L^2([0,T]^k)$. In particular, by (1.15) this implies that

$$E\Big[h_k\big(\frac{\theta}{\|g\|}\big) \cdot \psi\Big] = 0$$

for all $g \in L^2([0,T])$ and for all $k \geq 0$, where $\theta = \int\limits_0^T g(t)dW(t)$. But then, from the definition of the Hermite polynomials,

$$E[\theta^k \cdot \psi] = 0$$

for all $k \geq 0$, which again implies that

$$E[\exp\theta \cdot \psi] = \sum_{k=0}^{\infty} \frac{1}{k!} E[\theta^k \cdot \psi] = 0.$$

Since the family

$$\{\exp\theta : \quad g \in L^2([0,T])\}$$

is total in $L^2(P)$ (see [179, Lemma 4.3.2]), we conclude that $\psi = 0$. Hence, we conclude

$$\xi = \sum_{k=0}^{\infty} J_k(g_k) \tag{1.26}$$

and

$$\|\xi\|^2_{L^2(P)} = \sum_{k=0}^{\infty} \|J_k(g_k)\|^2_{L^2(P)}. \tag{1.27}$$

Finally, to obtain (1.16)–(1.17) we proceed as follows. The function g_n is defined only on S_n, but we can extend g_n to $[0,T]^n$ by putting

$$g_n(t_1,\dots,t_n) = 0, \qquad (t_1,\dots,t_n) \in [0,T]^n \setminus S_n.$$

Now define $f_n := \widetilde{g}_n$ to be the symmetrization of g_n - cf. (1.5). Then

$$I_n(f_n) = n!J_n(f_n) = n!J_n(\widetilde{g}_n) = J_n(g_n)$$

and (1.16) and (1.17) follow from (1.26) and (1.27), respectively. □

Example 1.11. What is the Wiener–Itô expansion of $\xi = W^2(T)$? From (1.15) we get

$$2\int_0^T \int_0^{t_2} 1\, dW(t_1)dW(t_2) = Th_2\Big(\frac{W(T)}{T^{1/2}}\Big) = W^2(T) - T,$$

and therefore

$$\xi = W^2(T) = T + I_2(1).$$

Example 1.12. Note that for a fixed $t \in (0,T)$ we have

$$\int_0^T \int_0^{t_2} \chi_{\{t_1<t<t_2\}}(t_1,t_2)dW(t_1)dW(t_2) = \int_t^T W(t)dW(t_2) = W(t)\big(W(T) - W(t)\big).$$

Hence, if we put

$$\xi = W(t)(W(T) - W(t)), \qquad g(t_1,t_2) = \chi_{\{t_1<t<t_2\}}$$

we can see that

$$\xi = J_2(g) = 2J_2(\widetilde{g}\,) = I_2(f_2),$$

where

$$f_2(t_1,t_2) = \widetilde{g}(t_1,t_2) = \frac{1}{2}\big(\chi_{\{t_1<t<t_2\}} + \chi_{\{t_2<t<t_1\}}\big).$$

Here and in the sequel we denote the *indicator function* by

$$\chi = \chi_A(x) = \chi_{\{x\in A\}} := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

1.3 Exercises

Problem 1.1. (*) Let $h_n(x)$, $n = 0, 1, 2, \ldots$, be the Hermite polynomials defined in (1.13).

(a) Prove that

$$\exp\{tx - \frac{t^2}{2}\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x).$$

[*Hint.* Write $\exp\{tx - \frac{t^2}{2}\} = \exp\{\frac{1}{2}x^2\} \cdot \exp\{-\frac{1}{2}(x-t)^2\}$ and apply the Taylor formula on the last factor.]

(b) Show that if $\lambda > 0$ then

$$\exp\{tx - \frac{t^2\lambda}{2}\} = \sum_{n=0}^{\infty} \frac{t^n \lambda^{\frac{n}{2}}}{n!} h_n(\frac{x}{\sqrt{\lambda}}).$$

(c) Let $g \in L^2([0,T])$. Put

$$\theta = \int_0^T g(s)dW(s).$$

Show that

$$\exp\Big\{\int_0^T g(s)dW(s) - \frac{1}{2}\|g\|^2\Big\} = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n(\frac{\theta}{\|g\|}),$$

where $\|g\| = \|g\|_{L^2([0,T])}$.

(d) Let $t \in [0,T]$. Show that $\exp\{W(t) - \frac{1}{2}t\} = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n(\frac{W(t)}{\sqrt{t}})$.

Problem 1.2. Let ξ and ζ be F_T-measurable random variables in $L^2(P)$ with Wiener–Itô chaos expansions $\xi = \sum_{n=0}^{\infty} I_n(f_n)$ and $\zeta = \sum_{n=0}^{\infty} I_n(g_n)$, respectively. Prove that the chaos expansion of the sum $\xi + \zeta = \sum_{n=0}^{\infty} I_n(h_n)$ is such that $h_n = f_n + g_n$ for all $n = 1, 2, \ldots$

Problem 1.3. (*) Find the Wiener–Itô chaos expansion of the following random variables:

(a) $\xi = W(t)$, where $t \in [0,T]$ is fixed,

(b) $\xi = \int_0^T g(s)dW(s)$, where $g \in L^2([0,T])$,

(c) $\xi = W^2(t)$, where $t \in [0,T]$ is fixed,

(d) $\xi = \exp\{\int_0^T g(s)dW(s)\}$, where $g \in L^2([0,T])$ [*Hint.* Use (1.15).],

(e) $\xi = \int_0^T g(s)W(s)ds$, where $g \in L^2([0,T])$.

Problem 1.4. (*) The *Itô representation theorem* states that if $F \in L^2(P)$ is $\mathcal{F}_T$-measurable, then there exists a unique $\mathbb{F}$-adapted process $\varphi = \varphi(t), 0 \leq t \leq T$, such that

$$F = E[F] + \int_0^T \varphi(t) dW(t).$$

This result only provides the *existence* of the integrand φ, but from the point of view of applications it is important also to be able to find the integrand φ more explicitly. This can be achieved, for example, by the *Clark–Ocone formula* (see Chap. 4), which says that, under some suitable conditions,

$$\varphi(t) = E[D_t F | \mathcal{F}_t], \qquad 0 \leq t \leq T,$$

where $D_t F$ is the *Malliavin derivative* of F. We discuss this topic later in the book. However, for certain random variables F it is possible to find φ directly, by using the Itô formula. For example, find φ when

(a) $F = W^2(T)$
(b) $F = \exp\{W(T)\}$
(c) $F = \int_0^T W(t) dt$
(d) $F = W^3(T)$
(e) $F = \cos W(T)$ [*Hint.* Check that $N(t) := e^{\frac{1}{2}t} \cos W(t)$, $t \in [0, T]$, is a martingale.]

Problem 1.5. (*) This exercise is based on [108]. Suppose the function F of Problem 1.4 has the form

$$F = f(X(T)),$$

where $X = X(t)$, $t \in [0, T]$, is an Itô diffusion given by

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t); \qquad X(0) = x \in \mathbb{R}.$$

Here $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are given Lipschitz continuous functions of at most linear growth, so there exists a unique strong solution $X(t) = X^x(t)$, $t \in [0, T]$. Then there is a useful formula for the process φ in the Itô representation theorem. This formula is achieved as follows. If g is a real function such that

$$E[|g(X^x(t))|] < \infty,$$

then we define

$$u(t, x) := P_t g(x) := E[g(X^x(t))], \qquad t \in [0, T], \quad x \in \mathbb{R}.$$

Suppose that there exists $\delta > 0$ such that

$$|\sigma(x)| \geq \delta \quad \text{for all } x \in \mathbb{R}. \tag{1.28}$$

Then $u(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}$$

(this is the Kolmogorov backward equation, see, for example, [74, Volume 1, Theorem 5.11, p. 162 and Volume 2, Theorem 13.18, p. 53], [177, Theorem 8.1] for details on this issue).

(a) Use the Itô formula for the process

$$Y(t) = g(t, X(t)), \quad t \in [0,T], \qquad \text{with} \quad g(t,x) = P_{T-t}f(x)$$

to show that

$$f(X(T)) = P_T f(x) + \int_0^T \Big[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\Big]_{|\xi=X(t)} dW(t), \tag{1.29}$$

for all $f \in C^2(\mathbb{R})$. In other words, with the notation of Problem 1.4, we have shown that if $F = f(X(T))$, then

$$E[F] = P_T f(x) \quad \text{and} \quad \varphi(t) = \Big[\sigma(\xi)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)\Big]_{|\xi=X(t)}. \tag{1.30}$$

(b) Use (1.30) to compute $E[F]$ and find φ in the Itô representation of the following random variables:
(b.1) $F = W^2(T)$
(b.2) $F = W^3(T)$
(b.3) $F = X(T)$, where $X(t)$, $t \in [0,T]$, is the geometric Brownian motion, that is,

$$dX(t) = \rho X(t)dt + \alpha X(t)dW(t); \quad X(0) = x \in \mathbb{R} \qquad (\rho, \alpha \text{ constants}).$$

(c) Extend formula (1.30) to the case when $X(t) \in \mathbb{R}^n$, $t \in [0,T]$, and $f : \mathbb{R}^n \to \mathbb{R}$. In this case, condition (1.28) must be replaced by the *uniform ellipticity* condition

$$\eta^T \sigma^T(x)\sigma(x)\eta \geq \delta|\eta|^2 \quad \text{for all } x \in \mathbb{R}^n, \eta \in \mathbb{R}^n, \tag{1.31}$$

where $\sigma^T(x)$ denotes the transposed of the $m \times n$-matrix $\sigma(x)$.