# Mathematical Behavioural Finance
# A Mini Course

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# Chapter 3:

# Market Equilibrium and Asset Pricing under RDUT

# Section 1

## An Arrow-Debreu Economy

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- Aggregate endowment is $(e_0, \tilde{e}_1) := \left( \sum_{i=1}^{I} e_{0i}, \sum_{i=1}^{I} \tilde{e}_{1i} \right)$

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- The preference of agent $i$ over $(c_{0i}, \tilde{c}_{0i})$ is represented by

$$V_i(c_{0i}, \tilde{c}_{1i}) = u_{0i}(c_{0i}) + \beta_i \int u_{1i}(\tilde{c}_{1i}) d(w_i \circ \mathrm{P}),$$

where

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- The set of all feasible consumption plans is denoted by $\mathscr{C}$

# Pricing Kernel

- The above economy is denoted by

$$\mathscr{E} := \Big\{(\Omega, \mathcal{F}, \mathrm{P}),\ (e_{0i}, \tilde{e}_{1i})_{i=1}^{I},\ \mathscr{C},\ \big(V_i(c_{0i}, \tilde{c}_{1i})\big)_{i=1}^{I}\Big\}$$

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- A *pricing kernel* (or *state-price density, stochastic discount factor*) is an $\mathcal{F}$-measurable random variable $\tilde{\rho}$, with $\mathrm{P}(\tilde{\rho} > 0) = 1$, $\mathrm{E}[\tilde{\rho}] < \infty$ and $\mathrm{E}[\tilde{\rho}\tilde{e}_1] < \infty$, such that any claim $\tilde{x}$ tomorrow is priced at $\mathrm{E}[\tilde{\rho}\tilde{x}]$ today

# Arrow-Debreu Equilibrium

An *Arrow–Debreu equilibrium* of $\mathscr{E}$ is a collection $\{\tilde{\rho}, (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\}$ consisting of a pricing kernel $\tilde{\rho}$ and a collection $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I$ of feasible consumption plans, that satisfies the following conditions:

Individual optimality : For every $i$, $(c^*_{0i}, \tilde{c}^*_{1i})$ maximises the preference of agent $i$ subject to the budget constraint, that is,

$$V_i(c^*_{0i}, \tilde{c}^*_{1i}) = \max_{(c_{0i}, \tilde{c}_{1i}) \in \mathscr{C}} V_i(c_{0i}, \tilde{c}_{1i})$$

subject to $c_{0i} + \mathrm{E}[\tilde{\rho}\tilde{c}_{1i}] \leq e_{0i} + \mathrm{E}[\tilde{\rho}\tilde{e}_{1i}]$

Market clearing : $\sum_{i=1}^I c^*_{0i} = e_0$ and $\sum_{i=1}^I \tilde{c}^*_{1i} = \tilde{e}_1$

# Literature

- Mainly on CPT economies, and on existence of equilibria
  - Qualitative structures of pricing kernel for both CPT and SP/A economies, assuming existence of equilibrium: Shefrin (2008)
  - Non-existence: De Giorgi, Hens and Riegers (2009), Azevedo and Gottlieb (2010)
  - Under specific asset return distribution: Barberis and Huang (2008)
  - One risky asset: He and Zhou (2011)
- RDUT economy with **convex** weighting function: Carlier and Dana (2008), Dana (2011) – existence

# Standing Assumptions

- Agents have **homogeneous beliefs** $\mathrm{P}$; $(\Omega, \mathcal{F}, \mathrm{P})$ admits no atom.
- For every $i$, $e_{0i} \geq 0$, $\mathrm{P}(\tilde{e}_{1i} \geq 0) = 1$, and $e_{0i} + \mathrm{P}(\tilde{e}_{1i} > 0) > 0$. Moreover, $\tilde{e}_1$ is **atomless**, $\mathrm{P}(\tilde{e}_1 > 0) = 1$, and $e_0 > 0$.
- For every $i$, $u_{0i}, u_{1i} : [0, \infty) \to \mathbb{R}$ are strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfy the **Inada** condition: $u'_{0i}(0+) = u'_{1i}(0+) = \infty$, $u'_{0i}(\infty) = u'_{1i}(\infty) = 0$. Moreover, $u_{1i}(0) = 0$.
- For every $i$, $w_i : [0, 1] \to [0, 1]$ is strictly increasing and continuously differentiable, and satisfies $w_i(0) = 0$, $w_i(1) = 1$.

## Section 2

# Individual Optimality

# Individual Consumptions

Consider

$$\begin{array}{ll} \underset{(c_0,\tilde{c}_1)\in\mathscr{C}}{\text{Max}} & V(c_0,\tilde{c}_1) := u_0(c_0) + \beta\int_0^\infty w\left(\mathrm{P}\left(u_1(\tilde{c}_1) > x\right)\right)dx \\ \text{subject to} & c_0 + \mathrm{E}[\tilde{\rho}\tilde{c}_1] \le \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1] \end{array} \tag{1}$$

where $\tilde{\rho}$ is **exogenously** given, atomless, and $\varepsilon_0$ and $\tilde{\varepsilon}_1$ are endowments at $t = 0$ and $t = 1$ respectively

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- Recall the set of quantile functions of nonnegative random variables

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- Problem (1) can be reformulated as

$$\begin{aligned} \underset{c_0 \geq 0,\, G \in \mathbb{G}}{\text{Max}} \quad & U(c_0, G) := u_0(c_0) + \beta \int_0^1 u_1(G(p))d\bar{w}(p) \\ \text{subject to} \quad & c_0 + \int_0^1 F_{\tilde{\rho}}^{-1}(1-p)G(p)dp \leq \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1], \end{aligned} \qquad (2)$$

where $\bar{w}(p) = 1 - w(1-p)$

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  where $\bar{w}(p) = 1 - w(1-p)$

- If $(c_0^*, G^*) \in [0,\infty) \times \mathbb{G}$ solves (2), then $(c_0^*, \tilde{c}_1^*)$, where $\tilde{c}_1^* = G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$, solves (1)

# Lagrange

Step 1. For a fixed Lagrange multiplier $\lambda > 0$, solve

$$\underset{c_0\geq 0,\, G\in\mathbb{G}}{\text{Max}} \quad u_0(c_0) + \beta\int_0^1 u_1(G(p))d\bar{w}(p) - \lambda\left(c_0 + \int_0^1 F_{\tilde{\rho}}^{-1}(1-p)G(p)dp - \varepsilon_0 - \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1]\right).$$

The solution $(c_0^*, G^*)$ implicitly depends on $\lambda$

Step 2. Determine $\lambda$ by

$$c_0^* + \int_0^{1-} F_{\tilde{\rho}}^{-1}(1-p)G^*(p)dp = \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1]$$

Step 3. $\tilde{c}_1^* := G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$

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- We have solved this problem ... provided that $M(z) = \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ satisfies some monotone condition!
- **Difficulty:** Such a condition (or literally any condition) is **not** permitted in our equilibrium problem!

# Calculus of Variation

- Set

$$\mathbb{G}_0 = \{G : [0,1) \to [0,\infty] \,|\, G \in \mathbb{G} \text{ and } G(p) > 0 \text{ for all } p \in (0,1)\}$$

- Calculus of variation shows that solving (3) is equivalent to finding $G \in \mathbb{G}_0$ satisfying

$$\begin{cases} \int_q^1 u_1'(G(p))d\bar{w}(p) - \frac{\lambda}{\beta}\int_q^1 F_{\tilde{\rho}}^{-1}(1-p)dp \le 0 \quad \forall q \in [0,1), \\ \int_0^1 \left(\int_q^{1-} u_1'(G(p))d\bar{w}(p) - \frac{\lambda}{\beta}\int_q^1 F_{\tilde{\rho}}^{-1}(1-p)dp\right) dG(q) = 0 \end{cases} \tag{4}$$

# Equivalent Condition

Previous condition is equivalent to

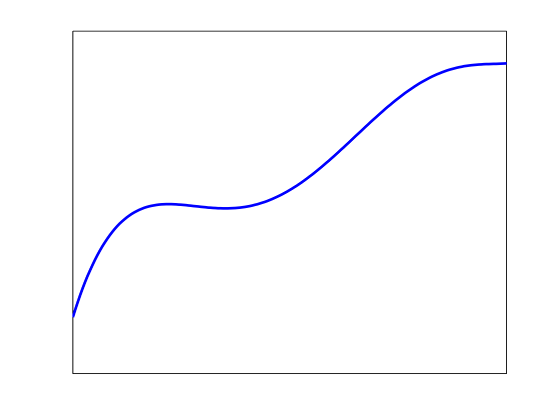
$$\begin{cases} K(q) \geq \dfrac{\lambda}{\beta} N(q) \quad \text{for all } q \in (0,1), \\ K \text{ is affine on } \left\{ q \in (0,1) : K(q) > \dfrac{\lambda}{\beta} N(q) \right\}, \\ K(0) = \dfrac{\lambda}{\beta} N(0),\ K(1-) = N(1-) \end{cases} \tag{5}$$

where

$$\begin{cases} K(q) = -\displaystyle\int_q^1 u_1'(G(\bar{w}^{-1}(p)))dp \\ N(q) = -\displaystyle\int_q^1 F_{\tilde{\rho}}^{-1}(1-\bar{w}^{-1}(p))d\bar{w}^{-1}(p) \end{cases} \tag{6}$$

for all $q \in [0,1)$

# How Concave Envelope is Formed



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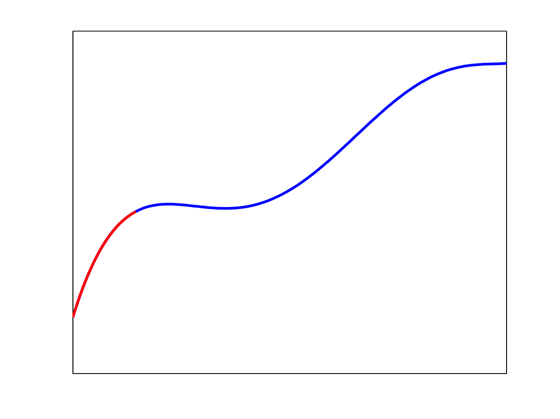

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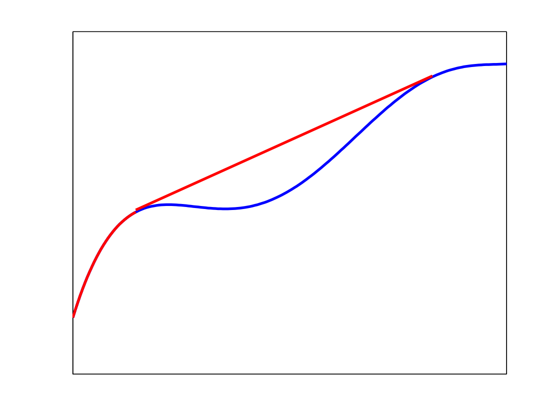

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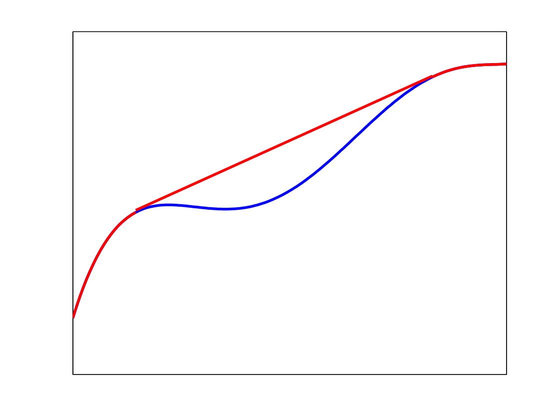

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- $G^*(q) = (u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'(1 - w(1-q))\right)$
- $\tilde{c}_1^* = G^*(1 - F_{\tilde{\rho}}(\tilde{\rho})) = (u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$

# Complete/Explicit Solution to Individual Consumption

## Theorem

**(Xia and Zhou 2012)** *Assume that* $\tilde{\rho} > 0$ *a.s., atomless, with* $E[\tilde{\rho}] < +\infty$. *Then the optimal consumption plan is given by*

$$\begin{cases} c_0^* = (u_0')^{-1}(\lambda) \\ \tilde{c}_1^* = (u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right), \end{cases}$$

*where* $\lambda$ *is determined by*

$$(u_0')^{-1}(\lambda) + \mathbb{E}\left[\tilde{\rho}(u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = \varepsilon_0 + \mathbb{E}[\tilde{\rho}\tilde{\varepsilon}].$$

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- It recovers one of the results in Chapter 2!

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- If there exists $\varepsilon > 0$ such that

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- "Fear causes consumption insurance" (see Chapter 2)

## Section 3

# Representative RDUT Agent

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where $\lambda_i^*$ satisfies

$$(u_{0i}')^{-1}(\lambda_i^*) + \mathrm{E}\left[\tilde{\rho}(u_{1i}')^{-1}\left(\frac{\lambda_i^*}{\beta_i}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = e_{0i} + \mathrm{E}[\tilde{\rho}\tilde{e}_{1i}]$$

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- Aggregate consumption is

$$c_0^* = \sum_{i=1}^{I}(u_{0i}')^{-1}(\lambda_i^*),\ \tilde{c}_1^* = \sum_{i=1}^{I}(u_{1i}')^{-1}\left(\frac{\lambda_i^*}{\beta_i}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)$$

# A Representative Agent

- For $\lambda_1 > 0, \ldots, \lambda_I > 0$, set $\lambda = (\lambda_1, \ldots, \lambda_I)$ and
$$h_{0\lambda}(y) := \sum_{i=1}^{I} (u'_{0i})^{-1}(\lambda_i y), \; h_{1\lambda}(y) := \sum_{i=1}^{I} (u'_{1i})^{-1}\left(\frac{\lambda_i y}{\beta_i}\right)$$
- Define $u_{t\lambda}(x) = \int_0^x h_{t\lambda}^{-1}(z)dz$, $t = 0, 1$
- Then
$$c_0^* = (u'_{0\lambda^*})^{-1}(1), \; \tilde{c}_1^* = (u'_{1\lambda^*})^{-1}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$$
- Consider an **RDUT** agent, indexed by $\lambda^*$, whose preference is
$$V_{\lambda^*}(c_0, \tilde{c}_1) := u_{0\lambda^*}(c_0) + \int u_{1\lambda^*}(\tilde{c}_1)d(w \circ \mathrm{P}) \qquad (7)$$
and whose endowment is the aggregate endowment $(e_0, \tilde{e}_1)$
- This representative agent's optimal consumption plan is the aggregate consumption plan

# What's Next – Idea

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- Use existing results for EUT economy

# Section 4

## Asset Pricing

# Explicit Expression of Pricing Kernel

**Theorem**

**(Xia and Zhou 2012)** *If there exists an equilibrium of economy $\mathscr{E}$ where the pricing kernel $\tilde{\rho}$ is atomless and $\lambda^*$ is the corresponding Lagrange vector, then*

$$\boxed{\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))\frac{u'_{1\lambda^*}(\tilde{e}_1)}{u'_{0\lambda^*}(e_0)} \quad a.s..} \tag{8}$$

**Idea of proof**. Market clearing – $\tilde{e}_1 = \tilde{c}_1^* = (u'_{1\lambda^*})^{-1}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$ – manipulate quantiles (see also next slide)

# Endogenous Monotonicity

- A simple fact: if $\tilde{Y} = f(\tilde{Z})$ for a non-increasing and left-continuous function $f$ and $\tilde{Z} \sim U(0,1)$, then $G_{\tilde{Y}}(p) = f(1-p)$ (prove it!)

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- $M$ is non-decreasing!

# Interpretations

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- Pricing kernel is a **weighted** marginal rate of substitution between initial and end-of-period consumption
- The weight is $w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))$
- An inverse-S shaped weighting $w$ leads to a **premium** when evaluating assets in both very high and very low future consumption states

# Implied Utility Function

- Define $u_w$ by

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- $u_w$: *implied utility function*

# Implied Relative Risk Aversion

- Implied relative index of risk aversion

$$R^w(x) := -\frac{xu''_w(x)}{u'_w(x)} = -\frac{xu''_{1\lambda^*}(x)}{u'_{1\lambda^*}(x)} + \frac{xw''(1-F_{\tilde{e}_1}(x))}{w'(1-F_{\tilde{e}_1}(x))}f_{\tilde{e}_1}(x) \tag{9}$$

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- It represents **overall** degree of risk-aversion (or risk-loving) of RDUT agent, combining outcome utility and probability weighting

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- $\tilde{\rho}^\diamond$ is pricing kernel under the above EUT economy iff $\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))\tilde{\rho}^\diamond$ is the pricing kernel under RDUT economy
- The two economies have exactly the same pricing formulae and individual consumption plans

# Existence of Equilibria

## Theorem

**(Xia and Zhou 2012)** *If* $\Psi_\lambda(p) \equiv w'(p)\, u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right)$ *is strictly increasing for any* $\lambda$*, and*

$$\begin{cases} \mathrm{E}[w'(1-F_{\tilde{e}_1}(\tilde{e}_1))u_{1i}(\tilde{e}_1)] < \infty \\ \mathrm{E}\left[w'(1-F_{\tilde{e}_1}(\tilde{e}_1))u'_{1i}\left(\dfrac{\tilde{e}_1}{I}\right)\right] < \infty \end{cases}$$

*for all* $i = 1, \ldots, I$*, then there exists an Arrow-Debreu equilibrium of economy* $\mathscr{E}$ *where the pricing kernel is atomless. If in addition*

$$-\frac{c u''_{1i}(c)}{u'_{1i}(c)} \leq 1 \text{ for all } i = 1, \ldots, I \text{ and } c > 0,$$

*then the equilibrium is unique.*

# Monotonicity of $\Psi_\lambda$

- It is defined through model primitives:
  $\Psi_\lambda(p) = w'(p)\, u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right)$
- Monotonicity of $\Psi_\lambda$ for any $\lambda$ requires a **concave** implied utility function for any initial distribution of the wealth.
- Automatically satisfied when $w$ is convex
- Possibly satisfied when $w$ is concave or inverse-S shaped

# Monotonicity of $\Psi_\lambda$: An Example

**Example.** Take $w(p) = p^{1-\alpha}$ where $\alpha \in (0,1)$, $u_{1\lambda}(c) = \frac{c^{1-\beta}}{1-\beta}$ where $\beta \in (0,1)$, and $\tilde{e}_1$ follows the Parato distribution

$$F_{\tilde{e}_1}(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\gamma & x \geq x_m \\ 0 & x < x_m. \end{cases}$$

In this case

$$\Psi_\lambda(p) = w'(p) u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right) = (1-\alpha) x_m^{-\beta} p^{\frac{\beta}{\gamma}-\alpha}.$$

This is a strictly increasing function if and only if $\alpha < \frac{\beta}{\gamma}$.

## Section 5

# CCAPM and Interest Rate

# Consumption-Based CAPM

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- A rank-dependent consumption-based CAPM (CCAPM):

$$\boxed{\bar{r} - r_f \approx \left[\alpha + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0\right] \mathbf{Cov}(\tilde{g}, \tilde{r})}$$

where $\alpha := -\frac{e_0 u''_{1\lambda^*}(e_0)}{u'_{1\lambda^*}(e_0)}$ and $f_{\tilde{e}_1}$ is density function of $\tilde{e}_1$

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- Classical EUT based CCAPM: $\bar{r} - r_f \approx \alpha \mathbf{Cov}(\tilde{g}, \tilde{r})$

# Prices and Expected Consumption Growth

- Again $\bar{r} - r_f \approx \left[\alpha + \frac{w''(1-F_{\tilde{e}_1}(e_0))}{w'(1-F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0)e_0\right] \mathbf{Cov}(\tilde{g}, \tilde{r})$

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- Recall $1 - F_{\tilde{e}_1}(e_0) = P(\tilde{e}_1 > e_0)$
- The subjective expectation (or belief) on general consumption growth should be priced in for individual assets

# Consumption-Based Real Interest

- A rank-dependent consumption-based real interest rate formula:

$$1 + r_f \approx \frac{1}{\beta w'(1 - F_{\tilde{e}_1}(e_0))} \left[ 1 + \alpha \bar{g} + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right]$$

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- Classical EUT based real interest rate theory: $1 + r_f \approx \frac{1+\alpha\bar{g}}{\beta}$

# Section 6

## Equity Premium and Risk-Free Rate Puzzles

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  - Subsequent empirical studies have confirmed that this puzzle is robust across different time periods and different countries
- Risk-free rate puzzle (Weil 1989): observed risk-free rate is too low to be explainable by classical CCAPM

# Economic Data 1889–1978 (Mehra and Prescott 1985)

| | Consumption growth | | riskless return | | equity premium | | S&P 500 return | |
|---|---|---|---|---|---|---|---|---|
| Periods | Mean | S.D. | Mean | S.D. | Mean | S.D. | Mean | S.D. |
| **1889–1978** | **1.83** | **3.57** | **0.80** | **5.67** | **6.18** | **16.67** | **6.98** | **16.54** |
| 1889–1898 | 2.30 | 4.90 | 5.80 | 3.23 | 1.78 | 11.57 | 7.58 | 10.02 |
| 1899–1908 | 2.55 | 5.31 | 2.62 | 2.59 | 5.08 | 16.86 | 7.71 | 17.21 |
| 1909–1918 | 0.44 | 3.07 | -1.63 | 9.02 | 1.49 | 9.18 | -0.14 | 12.81 |
| 1919–1928 | 3.00 | 3.97 | 4.30 | 6.61 | 14.64 | 15.94 | 18.94 | 16.18 |
| 1929–1938 | -0.25 | 5.28 | 2.39 | 6.50 | 0.18 | 31.63 | 2.56 | 27.90 |
| 1939–1948 | 2.19 | 2.52 | -5.82 | 4.05 | 8.89 | 14.23 | 3.07 | 14.67 |
| 1949–1958 | 1.48 | 1.00 | -0.81 | 1.89 | 18.30 | 13.20 | 17.49 | 13.08 |
| 1959–1968 | 2.37 | 1.00 | 1.07 | 0.64 | 4.50 | 10.17 | 5.58 | 10.59 |
| 1969–1978 | 2.41 | 1.40 | -0.72 | 2.06 | 0.75 | 11.64 | 0.03 | 13.11 |

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- A measure of 30 means indifference between a gamble equally likely to pay \$50,000 or \$100,000 and a certain payoff of \$51,209
- No human is *that* risk averse

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- Hence $1 - F_{\tilde{e}_1}(e_0) = \mathrm{P}(\tilde{e}_1 > e_0)$ lies in the convex domain of $w$
- Expected rate of return provided by our model is larger than that by EUT

# Our Explanation (Cont'd)

- Recall
$$1+r_f \approx \frac{1}{\beta w'(1-F_{\tilde{e}_1}(e_0))}\left[1+\alpha\bar{g}+\frac{w''(1-F_{\tilde{e}_1}(e_0))}{w'(1-F_{\tilde{e}_1}(e_0))}f_{\tilde{e}_1}(e_0)e_0\bar{g}\right]$$

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- We have argued $1 - F_{\tilde{e}_1}(e_0)$ is normally close to 1
- Therefore, for an inverse-S shaped $w$, $w'(1 - F_{\tilde{e}_1}(e_0))$ will be larger than one
- Our interest rate model indicates that an appropriate $w$ can render a lower risk-free rate than EUT model

# Our Explanation (Cont'd)

- Recall
  $1 + r_f \approx \frac{1}{\beta w'(1-F_{\tilde{e}_1}(e_0))} \left[ 1 + \alpha \bar{g} + \frac{w''(1-F_{\tilde{e}_1}(e_0))}{w'(1-F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right]$
- We have argued $1 - F_{\tilde{e}_1}(e_0)$ is normally close to 1
- Therefore, for an inverse-S shaped $w$, $w'(1 - F_{\tilde{e}_1}(e_0))$ will be larger than one
- Our interest rate model indicates that an appropriate $w$ can render a lower risk-free rate than EUT model
- The presence of a suitable probability weighting function will *simultaneously* increase equity premium and decrease risk-free rate under RDUT, diminishing the gap seen under EUT

# Economic Data 1889–1978 (Mehra and Prescott 1985)

| | Consumption growth | | riskless return | | equity premium | | S&P 500 return | |
|---|---|---|---|---|---|---|---|---|
| Periods | Mean | S.D. | Mean | S.D. | Mean | S.D. | Mean | S.D. |
| **1889–1978** | **1.83** | **3.57** | **0.80** | **5.67** | **6.18** | **16.67** | **6.98** | **16.54** |
| 1889–1898 | 2.30 | 4.90 | 5.80 | 3.23 | 1.78 | 11.57 | 7.58 | 10.02 |
| 1899–1908 | 2.55 | 5.31 | 2.62 | 2.59 | 5.08 | 16.86 | 7.71 | 17.21 |
| 1909–1918 | 0.44 | 3.07 | -1.63 | 9.02 | 1.49 | 9.18 | -0.14 | 12.81 |
| 1919–1928 | 3.00 | 3.97 | 4.30 | 6.61 | 14.64 | 15.94 | 18.94 | 16.18 |
| 1929–1938 | -0.25 | 5.28 | 2.39 | 6.50 | 0.18 | 31.63 | 2.56 | 27.90 |
| 1939–1948 | 2.19 | 2.52 | -5.82 | 4.05 | 8.89 | 14.23 | 3.07 | 14.67 |
| 1949–1958 | 1.48 | 1.00 | -0.81 | 1.89 | 18.30 | 13.20 | 17.49 | 13.08 |
| 1959–1968 | 2.37 | 1.00 | 1.07 | 0.64 | 4.50 | 10.17 | 5.58 | 10.59 |
| 1969–1978 | 2.41 | 1.40 | -0.72 | 2.06 | 0.75 | 11.64 | 0.03 | 13.11 |

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- It requires only a sufficiently large value of $\beta w'(1 - F_{\tilde{e}_1}(e_0))$ – explainable by a proper inverse-S shaped $w$

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- Corresponding premium, $0.18\%$, is lowest in Table 1
- In general, at times when most people believe that economy is in a downturn, expected rate of return provided by RDUT is smaller than that provided by EUT model
- Hence we should investigate asset pricing by differentiating periods of economic growth from those of economic depression

## Section 7

# Summary and Further Readings

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- Asset prices not only depend upon level of risk aversion and beta, but also upon agents' belief on economic growth
- Probability weighting may offer a new way of thinking in explaining many economic phenomena

# Essential Readings


- H. Shefrin. *A Behavioral Approach to Asset Pricing (2nd Edition)*, Elsevier, Amsterdam, 2008.
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