Mathematical Behavioural Finance
A Mini Course

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January 2013 Winter School @ Lunteren
Chapter 4:
Portfolio Choice under CPT
1. Formulation of CPT Portfolio Choice Model

2. Divide and Conquer

3. Solutions to GPP and LPP

4. Grand Solution

5. Continuous Time and Time Inconsistency

6. Summary and Further Readings

7. Final Words
Section 1

Formulation of CPT Portfolio Choice Model
Model Primitives

- Present date $t = 0$ and a future date $t = 1$
- Randomness described by $(\Omega, \mathcal{F}, P)$ at $t = 1$
- An atomless pricing kernel $\tilde{\rho}$ so that any future payoff $\tilde{X}$ is evaluated as $E[\tilde{\rho} \tilde{X}]$ at present
- An agent with
  - initial endowment $x_0$ at $t = 0$
  - preference specified by CPT

... wants to choose future consumption (wealth) $\tilde{c}$
Portfolio Choice/Consumption Model under CPT

The model

\[
\max_{\tilde{c}} \quad V(\tilde{c}) = \int_{0}^{\infty} w_+ \left( P \left( u_+ \left( (\tilde{c} - \tilde{B})^+ \right) > x \right) \right) dx \\
- \int_{0}^{\infty} w_- \left( P \left( u_- \left( (\tilde{c} - \tilde{B})^- \right) > x \right) \right) dx \\
\text{subject to} \quad E[\hat{\rho}\hat{c}] \leq x_0, \text{ } \tilde{c} \text{ is bounded below}
\]

(CPT)
The model

\[
\max_{\tilde{c}} \quad V(\tilde{c}) = \int_0^\infty w_+ \left( P \left( u_+ \left( (\tilde{c} - \tilde{B})^+ \right) > x \right) \right) dx \\
- \int_0^\infty w_- \left( P \left( u_- \left( (\tilde{c} - \tilde{B})^- \right) > x \right) \right) dx
\]

subject to \quad \mathbb{E}[\tilde{\rho} \tilde{c}] \leq x_0, \quad \tilde{c} \text{ is bounded below}

(CPT)

\[u_\pm \text{ is assumed to be concave so overall value function} \]
\[u_+ (x) \mathbf{1}_{x \geq 0} - u_- (x) \mathbf{1}_{x < 0} \text{ is } S\text{-shaped}; \quad u_\pm (0) = 0\]
Portfolio Choice/Consumption Model under CPT

- The model

\[
\begin{align*}
\text{Max}_{\tilde{c}} & \quad V(\tilde{c}) = \int_0^\infty w_+ \left( P \left( u_+ \left( (\tilde{c} - \tilde{B})^+ \right) > x \right) \right) dx \\
& - \int_0^\infty w_- \left( P \left( u_- \left( (\tilde{c} - \tilde{B})^- \right) > x \right) \right) dx \\
\text{subject to} & \quad E[\tilde{\rho}\tilde{c}] \leq x_0, \quad \tilde{c} \text{ is bounded below}
\end{align*}
\]

- \( u_\pm \) is assumed to be concave so overall value function
  \( u_+(x)1_{x \geq 0} - u_-(x)1_{x < 0} \) is S-shaped; \( u_\pm(0) = 0 \)

- \( w_\pm \) is in general non-convex/non-concave
Portfolio Choice/Consumption Model under CPT

The model

\[ \max_{\tilde{c}} \quad V(\tilde{c}) = \int_{0}^{\infty} w_+ \left( P \left( u_+ \left( (\tilde{c} - \tilde{B})^+ \right) > x \right) \right) dx \]
\[ - \int_{0}^{\infty} w_- \left( P \left( u_- \left( (\tilde{c} - \tilde{B})^- \right) > x \right) \right) dx \]

subject to \( E[\tilde{\rho} \tilde{c}] \leq x_0, \) \( \tilde{c} \) is bounded below

(CPT)

- \( u_\pm \) is assumed to be concave so overall value function \( u_+(x)1_{x \geq 0} - u_-(x)1_{x < 0} \) is S-shaped; \( u_\pm(0) = 0 \)
- \( w_\pm \) is in general non-convex/non-concave
- \( \tilde{B} = 0 \) without loss of generality
CPT Preference

Write $V(\tilde{c}) = V_+(\tilde{c}^+) - V_-(\tilde{c}^-)$ where

$$V_+(\tilde{c}) := \int_0^\infty w_+ (P (u_+ (\tilde{c}) > x)) \, dx$$

$$V_- (\tilde{c}) := \int_0^\infty w_- (P (u_- (\tilde{c}) > x)) \, dx$$
Mathematical Challenges

- Two difference sources
Mathematical Challenges

- Two difference sources
- Probability weighting and \( S \)-shaped value function
Literature

- Almost none
Almost none

Berkelaar, Kouwenberg and Post (2004): no probability weighting; two-piece power value function
Standing Assumptions

- \( \tilde{\rho} > 0 \) a.s., **atomless**, with \( E[\tilde{\rho}] < +\infty \).

- \( u_\pm : [0, \infty) \rightarrow \mathbb{R} \) are strictly increasing, concave, with \( u_\pm(0) = 0 \). Moreover, \( u_+ \) is continuously differentiable on \( (0, \infty) \), strictly concave, and satisfies the Inada condition: \( u'_+(0+) = \infty, \ u'_+(\infty) = 0 \).

- \( w_\pm : [0, 1] \rightarrow [0, 1] \) are strictly increasing and continuously differentiable, and satisfies \( w_\pm(0) = 0, \ w_\pm(1) = 1 \).
Section 2

Divide and Conquer
Our Model (Again)

\[
\begin{align*}
\max_{\tilde{c}} & \quad V(\tilde{c}) = \int_{0}^{\infty} w_+ \left( P \left( u_+ (\tilde{c}^+) > x \right) \right) dx \\
& \quad - \int_{0}^{\infty} w_- \left( P \left( u_- (\tilde{c}^-) > x \right) \right) dx \\
\text{subject to} & \quad E[\tilde{\rho} \tilde{c}] \leq x_0, \quad \tilde{c} \geq 0
\end{align*}
\]

This problem admits a quantile formulation.
Divide and Conquer

We do “divide and conquer”
We do “**divide and conquer**”

- Step 1: divide into two problems: one concerns the **gain** part of $\tilde{c}$ and the other the **loss** part of $\tilde{c}$
We do “divide and conquer”

- Step 1: divide into two problems: one concerns the gain part of $\tilde{c}$ and the other the loss part of $\tilde{c}$
- Step 2: combine them together via solving another problem
Step 1 – Gain Part Problem (GPP)

A problem with parameters \((A, x_+):\)

Max \[ V_+ (\tilde{c}) = \int_0^\infty w_+ \left( P \left( u_+ (\tilde{c}) > x \right) \right) dx \]

subject to \[
\begin{align*}
E[\tilde{\rho} \tilde{c}] &= x_+, \quad \tilde{c} \geq 0 \\
\tilde{c} &= 0 \text{ on } A^C,
\end{align*}
\]

where \(x_+ \geq x_0^+ \geq 0\) and \(A \in \mathcal{F}\) with \(P(A) \leq 1\)

Define its optimal value to be \(v_+ (A, x_+)\)
A problem with parameters \((A, x_+)\):

\[
\begin{align*}
\text{Min} & \quad V_-(\tilde{c}) = \int_0^\infty w_-(P(u_-(\tilde{c}) > x)) \, dx \\
\text{subject to} & \quad \left\{ 
\begin{array}{l}
E[\tilde{\rho}\tilde{c}] = x_+ - x_0, \quad \tilde{c} \geq 0 \\
\tilde{c} = 0 \text{ on } A, \quad \tilde{c} \text{ is bounded}
\end{array} \right. \quad (2)
\end{align*}
\]

where \(x_+ \geq x_0^+\) and \(A \in \mathcal{F}\) with \(P(A) \leq 1\)

- Define its optimal value to be \(v_-(A, x_+)\)
In Step 2 we solve

\[ \text{Max} \quad v_+(A, x_+) - v_-(A, x_+) \]

subject to

\[
\begin{align*}
A \in \mathcal{F}, \quad x_+ &\geq x_0^+, \\
\text{when } P(A) &\neq 0, \\
x_+ &= x_0 \text{ when } P(A) = 1.
\end{align*}
\]
It Works

Theorem

(Jin and Zhou 2008) Given $\tilde{c}^*$, define $A^* := \{\omega : \tilde{c}^* \geq 0\}$ and $x_+^* := E[\tilde{\rho}(\tilde{c}^*)^+]$. Then $\tilde{c}^*$ is optimal for the CPT portfolio choice problem (CPT) iff $(A^*, x_+^*)$ are optimal for Problem (3) and $(X^*)^+$ and $(X^*)^-$ are respectively optimal for Problems (1) and (2) with parameters $(A^*, x_+^*)$.

Proof. Direct by definitions of maximum/minimum.
Solution Flow

- Solve GPP for any parameter \((A, x_+),\) getting optimal solution \(\tilde{c}_+(A, x_+)\) and optimal value \(v_+(A, x_+)\)
Solution Flow

- Solve GPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_+(A, x_+)\) and optimal value \(v_+(A, x_+)\)
- Solve LPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_-(A, x_+)\) and optimal value \(v_-(A, x_+)\)
Solution Flow

- Solve GPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_+(A, x_+)\) and optimal value \(v_+(A, x_+)\)

- Solve LPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_-(A, x_+)\) and optimal value \(v_-(A, x_+)\)

- Solve Step 2 problem and get optimal \((A^*, x^*_+)\)
Solution Flow

- Solve GPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_+(A, x_+)\) and optimal value \(v_+(A, x_+)\)
- Solve LPP for any parameter \((A, x_+)\), getting optimal solution \(\tilde{c}_-(A, x_+)\) and optimal value \(v_-(A, x_+)\)
- Solve Step 2 problem and get optimal \((A^*, x^*_+)\)
- Then \(\tilde{c}^* := \tilde{c}_+(A^*, x^*_+) - \tilde{c}_-(A^*, x^*_+)\) solves the CPT model
Simplification

Recall Step 2 problem

$$v_+(A, x_+ - v_-(A, x_+)$$

optimisation over a set of random events $A$: hard to handle
Recall Step 2 problem

\[ v_+(A, x_+) - v_-(A, x_+) \]

optimisation over a set of random events \( A \): hard to handle

**Theorem**

*(Jin and Zhou 2008)* For any feasible pair \((A, x_+)\) of Problem (3), there exists \( c \in [\text{essinf } \tilde{\rho}, \text{esssup } \tilde{\rho}] \) such that

\[ \bar{A} := \{ \omega : \tilde{\rho} \leq a \} \text{ satisfies } \]

\[ v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+). \]  
(4)
Recall Step 2 problem

\[ v_+(A, x_+) - v_-(A, x_+) \]

optimisation over a set of *random events* \( A \): hard to handle

**Theorem**

(Jin and Zhou 2008) *For any feasible pair* \( (A, x_+) \) *of Problem (3), there exists* \( c \in [\text{essinf } \tilde{\rho}, \text{esssup } \tilde{\rho}] \) *such that*

\[ \tilde{A} := \{ \omega : \tilde{\rho} \leq a \} \]

satisfies

\[ v_+(\tilde{A}, x_+) - v_-(\tilde{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+). \]  \hspace{1cm} (4)

**Proof.** One needs only to look for \( \tilde{c} = g(\tilde{\rho}) \) where \( g \) is non-increasing. Hence

\[ A = \{ \omega : \tilde{c} \geq 0 \} = \{ \omega : g(\tilde{\rho}) \geq 0 \} = \{ \omega : \tilde{\rho} \leq a \}. \]
Step 2 Problem Rewritten

- Use $v_+(a, x_+)$ and $v_-(a, x_+)$ to denote $v_+(\{\omega : \tilde{\rho} \leq a\}, x_+)$ and $v_-(\{\omega : \tilde{\rho} \leq a\}, x_+)$ respectively.
Use \( v_+(a, x_+) \) and \( v_-(a, x_+) \) to denote \( v_+(\{\omega : \tilde{\rho} \leq a\}, x_+) \) and \( v_-(\{\omega : \tilde{\rho} \leq a\}, x_+) \) respectively.

Problem (3) is equivalent to

\[
\text{Max} \quad v_+(a, x_+) - v_-(a, x_+)
\]

subject to \( \left\{ \begin{array}{l}
\text{essinf} \tilde{\rho} \leq a \leq \text{esssup} \tilde{\rho}, \quad x_+ \geq x_0^+,
\quad x_+ = 0 \text{ when } a = \text{essinf} \tilde{\rho},
\quad x_+ = x_0 \text{ when } a = \text{esssup} \tilde{\rho}
\end{array} \right. \)
Section 3

Solutions to GPP and LPP
GPP

Max $V_+ (\tilde{c}) = \int_0^\infty w_+ (P (u_+ (\tilde{c}) > x)) \, dx$

subject to

$$\begin{cases} E[\tilde{\rho}\tilde{c}] = x_+ \,, \quad \tilde{c} \geq 0 \\ \tilde{c} = 0 \text{ on } A^C \,, \end{cases}$$

where $x_+ \geq x_0^+ \text{ and } A = \{\omega : \tilde{\rho} \leq a\}$ with $\text{essinf} \, \tilde{\rho} \leq a \leq \text{esssup} \, \tilde{\rho}$

We have solved this problem – RDUT portfolio choice!
Integrability Condition

Impose the integrability condition

\[ E \left[ u_+ \left( (u'_+)^{-1} \left( \frac{\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right) w'_+(F_{\tilde{\rho}}(\tilde{\rho})) \right] < +\infty \]
Integrability Condition

- Impose the integrability condition

$$E \left[ u_+ \left( (u'_+)^{-1} \left( \frac{\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right) w'_+(F_{\tilde{\rho}}(\tilde{\rho})) \right] < +\infty$$

- In the following, we always assume the integrability condition holds
Theorem

(Jin and Zhou 2008) Assume $M(z) = \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ is non-decreasing on $z \in (0, 1)$.

(i) If $x_+ = 0$, then optimal solution of (6) is $\tilde{c}^* = 0$ and $v_+(a, x_+) = 0$.

(ii) If $x_+ > 0$ and $a = \text{essinf } \tilde{\rho}$, then there is no feasible solution to (6) and $v_+(a, x_+) = -\infty$.

(iii) If $x_+ > 0$ and $\text{essinf } \tilde{\rho} < a \leq \text{esssup } \tilde{\rho}$, then optimal solution to (6) is $\tilde{c}^* = (u'_+)^{-1} \left( \frac{\lambda^* \tilde{\rho}}{w'_+ (F_{\tilde{\rho}}(\tilde{\rho}))} \right) 1(\tilde{\rho} \leq a)$ with the optimal value $v_+(a, x_+) = E \left[ u_+ \left( (u'_+)^{-1} \left( \frac{\lambda^* \tilde{\rho}}{w'_+ (F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right) w'_+ (F_{\tilde{\rho}}(\tilde{\rho})) 1(\tilde{\rho} \leq a) \right]$, where $\lambda^*$ is determined by $E(\tilde{\rho} \tilde{c}^*) = x_+$. 
Idea of Proof

- Work on conditional probability space
  \((\Omega \cap A, \mathcal{F} \cap A, P_A := P(\cdot | A))\)
Idea of Proof

- Work on conditional probability space
  \((\Omega \cap A, \mathcal{F} \cap A, P_A := P(\cdot | A))\)
- Revise weighting function

\[
    w_A(x) := w_+(xP(A))/w_+(P(A)), \quad x \in [0, 1]
\]
Idea of Proof

- Work on conditional probability space
  \((\Omega \cap A, \mathcal{F} \cap A, P_A := P(\cdot|A))\)
- Revise weighting function
  \[ w_A(x) := \frac{w_+ (xP(A))}{w_+ (P(A))}, \quad x \in [0, 1] \]
- GPP is rewritten as
  \[
  \max V_+ (\tilde{c}) = w_+ (P(A)) \int_0^\infty w_A \left( P_A \left( u_+ (\tilde{c}) > x \right) \right) dx \\
  \text{subject to} \quad \left\{ \begin{array}{l}
  E_A [\tilde{\rho} \tilde{c}] = x_+/P(A), \quad \tilde{c} \geq 0
  \end{array} \right. 
  \]
Idea of Proof

- Work on conditional probability space
  \((\Omega \cap A, \mathcal{F} \cap A, P_A := P(\cdot | A))\)
- Revise weighting function
  \[ w_A(x) := \frac{w_+(xP(A))}{w_+(P(A))}, \quad x \in [0, 1] \]
- GPP is rewritten as
  \[
  \begin{align*}
  \text{Max} \quad V_+ (\tilde{c}) &= w_+(P(A)) \int_0^\infty w_A \left( P_A (u_+ (\tilde{c}) > x) \right) \, dx \\
  \text{subject to} \quad \{ E_A[\tilde{\rho} \tilde{c}] &= x_+/P(A), \quad \tilde{c} \geq 0 \}
  \end{align*}
  \]
- Apply result in Chapter 2
Min \[ V_-(\tilde{c}) = \int_0^\infty w_-(P(u_-(\tilde{c}) > x)) \, dx \]

subject to \[ \begin{align*}
E[\tilde{\rho}\tilde{c}] &= x_+ - x_0, \quad \tilde{c} \geq 0 \\
\tilde{c} &= 0 \text{ on } A, \quad \tilde{c} \text{ is bounded}
\end{align*} \] (7)

where \( x_+ \geq x_0^+ \) and \( A = \{\omega : \tilde{\rho} \leq a\} \) with \( \text{essinf} \, \tilde{\rho} \leq a \leq \text{esssup} \, \tilde{\rho} \)

This is a minimisation problem!
A General Problem

\[ \text{Min} \quad \tilde{c} \]
\[ \int_{0}^{\infty} w \left( \mathbb{P} \left( u(\tilde{c}) > x \right) \right) dx \]
subject to \[ E[\tilde{\rho} \tilde{c}] \geq x_0, \quad \tilde{c} \geq 0 \]

(G)
Hardy–Littlewood Inequality (Again)

Lemma

(Jin and Zhou 2008) We have that \( \tilde{c}^* := G(F_{\tilde{\rho}}(\tilde{\rho})) \) solves
\[
\max_{\tilde{c}' \sim \tilde{c}} E[\tilde{\rho}\tilde{c}'],
\]
where \( G \) is quantile of \( \tilde{c} \). If in addition
\[-\infty < E[\tilde{\rho}\tilde{c}^*] < +\infty, \]
then \( \tilde{c}^* \) is the unique optimal solution.

Hardy, Littlewood and Pòlya (1952), Dybvig (1988)
Quantile Formulation

The quantile formulation of $(G)$ is:

$\min_{G \in \mathcal{G}} \quad U(G(\cdot)) := \int_0^1 u(G(z))w'(1 - z)dz$

subject to $\int_0^1 F_{\tilde{\rho}}^{-1}(z)G(z)dz \geq x_0$  \hspace{1cm} (Q)
To minimise a concave functional: “wrong” direction!
Combinatorial Optimisation in Function Spaces

- To **minimise** a **concave** functional: “wrong” direction!
- ... which originates from $S$-shaped value function
Combinatorial Optimisation in Function Spaces

- To **minimise** a **concave** functional: “wrong” direction!
- ... which originates from *S*-shaped value function
- Solution must have a very different structure compared with the maximisation counterpart
Combinatorial Optimisation in Function Spaces

- To **minimise** a **concave** functional: “wrong” direction!
- ... which originates from $S$-shaped value function
- Solution must have a very different structure compared with the maximisation counterpart
- Lagrange fails (positive duality gap)
To **minimise** a **concave** functional: “wrong” direction!

- ... which originates from $S$-shaped value function
- Solution must have a very different structure compared with the maximisation counterpart
- Lagrange fails (positive duality gap)
- Solution should be a “corner point solution”: essentially a combinatorial optimisation in an infinite dimensional space
Characterising Corner Point Solutions

**Proposition**

*(Jin and Zhou 2008)* Assume \( u(\cdot) \) is strictly concave at 0. Then the optimal solution to (Q), if it exists, must be in the form

\[
G^*(z) = q(b)1_{(b,1)}(z), \quad z \in [0,1), \text{ with some } b \in [0,1) \text{ and }
\]

\[
q(b) := \frac{a}{E[\tilde{\rho}1_{\{F_{\tilde{\rho}}(\tilde{\rho}) > b\}}]}. \quad \text{Moreover, in this case, the optimal solution is}
\]

\[
\tilde{c}^* = G^*(F_{\tilde{\rho}}(\tilde{\rho})).
\]

- One only needs to find an optimal **number** \( b \in [0,1) \).
Characterising Corner Point Solutions

**Proposition**

*Jin and Zhou 2008* Assume $u(\cdot)$ is strictly concave at 0. Then the optimal solution to (Q), if it exists, must be in the form

$$G^*(z) = q(b)1_{(b,1)}(z), z \in [0,1), \text{ with some } b \in [0,1) \text{ and }$$

$$q(b) := \frac{a}{E[\hat{\rho}1_{\{F_{\hat{\rho}}(\hat{\rho})>b\}}]}.$$  

Moreover, in this case, the optimal solution is

$$\tilde{c}^* = G^*(F_{\hat{\rho}}(\hat{\rho})).$$

- One only needs to find an optimal **number** $b \in [0,1)$
- ... which motivates introduction of the following problem

$$\min_b f(b) := \int_0^1 u(G(z))w'(1-z)dz$$

subject to

$$G(\cdot) = \frac{a}{E[\rho1_{\{F_{\bar{\rho}}(\bar{\rho})>b\}}]}1_{(b,1]}(\cdot), \quad 0 \leq b < 1.$$
Solving (G)

**Theorem**

*(Jin and Zhou 2008)* Assume $u(\cdot)$ is strictly concave at 0. Then (G) admits an optimal solution if and only if the following problem

$$\min_{0 \leq b < \text{esssup} \tilde{\rho}} u \left( \frac{x_0}{E[\tilde{\rho}1(\tilde{\rho} > b)]} \right) w(P(\tilde{\rho} > b))$$

admits an optimal solution $b^*$, in which case the optimal solution to (G) is

$$\tilde{c}^* = \frac{x_0}{E[\tilde{\rho}1(\tilde{\rho} > b^*)]} 1(\tilde{\rho} > b^*).$$
Solutions to LPP

**Theorem**

*(Jin and Zhou 2008)* Assume $u(\cdot)$ is strictly concave at 0.

(i) If $a = \text{esssup} \tilde{\rho}$ and $x_+ = x_0$, then optimal solution of (7) is $\tilde{c}^* = 0$ and $v_-(a, x_+) = 0$.

(ii) If $a = \text{esssup} \tilde{\rho}$ and $x_+ \neq x_0$, then there is no feasible solution to (7) and $v_-(a, x_+) = +\infty$.

(iii) If $\text{essinf} \tilde{\rho} \leq a < \text{esssup} \tilde{\rho}$, then

$$v_-(a, x_+) = \inf_{b \in [a, \text{esssup} \tilde{\rho}]} u_-(\frac{x_+ - x_0}{E[\tilde{\rho} \mathbb{1}_{(\tilde{\rho} > b)}]}) w_- (1 - F_{\tilde{\rho}}(b)).$$

Moreover, Problem (7) admits an optimal solution $\tilde{c}^*$ iff the following problem

$$\min_{b \in [a, \text{esssup} \tilde{\rho}]} u_-(\frac{x_+ - x_0}{E[\tilde{\rho} \mathbb{1}_{(\tilde{\rho} > b)}]}) w_- (1 - F_{\tilde{\rho}}(b))$$

admits an optimal solution $b^*$, in which case $\tilde{c}^* = \frac{x_+ - x_0}{E[\tilde{\rho} \mathbb{1}_{(\tilde{\rho} > b^*)}]} \mathbb{1}_{\hat{\rho} > b^*}$.
Section 4

Grand Solution
Consider a mathematical programme in $(a, x_+)$:

$$\text{Max}_{(a,x_+)} E \left[ u_+ \left( (u'_+)^{-1} \left( \frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F\tilde{\rho}(\tilde{\rho}))} \right) \right) w'_+(F\tilde{\rho}(\tilde{\rho}))1(\tilde{\rho} \leq a) \right]$$

$$-u_-(\frac{x_+-x_0}{E[\tilde{\rho}1_{\tilde{\rho}>a}]} \bigg) w_-(1 - F(a))$$

subject to

$$\begin{cases} 
\text{essinf} \tilde{\rho} \leq a \leq \text{esssup} \tilde{\rho}, & x_+ \geq x_0^+, \\
x_+ = 0 \text{ when } a = \text{essinf} \tilde{\rho}, & x_+ = x_0 \text{ when } a = \text{esssup} \tilde{\rho}, \\
\end{cases}$$

$$\text{(MP)}$$

where $\lambda(a, x_+)$ satisfies

$$E \left[ (u'_+)^{-1} \left( \frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F\tilde{\rho}(\tilde{\rho}))} \right) \tilde{\rho}1(\tilde{\rho} \leq a) \right] = x_+$$
Theorem

(Jin and Zhou 2008) Assume $u_-(\cdot)$ is strictly concave at 0 and $M$ is non-decreasing. Let $(a^*, x^*)$ solves (MP). Then the optimal solution to (CPT) is

$$\tilde{c}^* = \left[ (u'_+)^{-1} \left( \frac{\lambda \tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right] 1(\tilde{\rho} \leq a^*) - \left[ \frac{x^* - x_0}{E[\tilde{\rho} 1(\tilde{\rho} > a^*)]} \right] \mathbb{1}(\tilde{\rho} > a^*).$$
Interpretations and Implications

\[ \tilde{c}^* = \left( (u'_+)^{-1} \left( \frac{\lambda \tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right) \mathbf{1}(\tilde{\rho} \leq a^*) - \left[ \frac{x^* - x_0}{E[\tilde{\rho} \mathbf{1}(\tilde{\rho} > a^*)]} \right] \mathbf{1}(\tilde{\rho} > a^*) \]
Interpretations and Implications

\[ \tilde{c}^* = \left[ \left( u'_+ \right)^{-1} \left( \frac{\lambda \tilde{\rho}}{w'_+ (F_{\tilde{\rho}}(\tilde{\rho}))} \right) \right] \mathbf{1}(\tilde{\rho} \leq a^*) - \left[ \frac{x^*_+ - x_0}{E[\tilde{\rho} \mathbf{1}(\tilde{\rho} > a^*)]} \right] \mathbf{1}(\tilde{\rho} > a^*) \]

- Future world divided by “good” states (where you have gains) and “bad” ones (losses), completely determined by whether \( \tilde{\rho} \leq a^* \) or \( \tilde{\rho} > a^* \).
Interpretations and Implications

\[ \tilde{c}^* = \left[ (u'_+)^{-1} \left( \frac{\lambda \tilde{\rho}}{w'_+ (F_{\tilde{\rho}}(\tilde{\rho})))} \right) \right] 1(\tilde{\rho} \leq a^*) - \left[ \frac{x^*_+ - x_0}{E[\tilde{\rho} 1(\tilde{\rho} > a^*)]} \right] 1(\tilde{\rho} > a^*) \]

- Future world divided by “good” states (where you have gains) and “bad” ones (losses), completely determined by whether \( \tilde{\rho} \leq a^* \) or \( \tilde{\rho} > a^* \)

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- Agent not only invests in stocks, but also generally takes a leverage to do so
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- Agent not only invests in stocks, but also generally takes a leverage to do so

- Optimal strategy is a gambling policy, betting on the good states while accepting a known loss on the bad
Section 5

Continuous Time and Time Inconsistency
A Continuous-Time Economy

- An economy in which \( m + 1 \) securities traded continuously.
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- Market randomness described by a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) along with an \( \mathbb{R}^m \)-valued, \( \mathcal{F}_t \)-adapted standard Brownian motion \( W(t) = (W^1(t), \ldots, W^m(t))' \) with \( \{\mathcal{F}_t\}_{t \geq 0} \) generated by \( W(\cdot) \)
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- A bond whose price process \( S_0(t) \) satisfies

\[
    dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = s_0
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\]
- \( m \) stocks whose price processes \( S_1(t), \ldots, S_m(t) \) satisfy stochastic differential equation (SDE)
  \[
dS_i(t) = S_i(t) \left( \mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t) \right); \quad S_i(0) = s_i
\]
Tame Portfolios

Let

\[ \sigma(t) := (\sigma_{ij}(t))_{m \times m} \]
\[ B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))' \]
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- An \( \mathcal{F}_t \)-progressively measurable process

\[ \pi(t) = (\pi_1(t), \cdots, \pi_m(t))' \] represents a (monetary) portfolio, where \( \pi_i(t) \) is the capital amount invested in stock \( i \) at \( t \).
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- A portfolio $\pi(\cdot)$ is admissible if
  \[ \int_0^T |\sigma(t)'\pi(t)|^2 dt < +\infty, \int_0^T |B(t)'\pi(t)| dt < +\infty, \text{ a.s.} \]
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An \( \mathcal{F}_t \)-progressively measurable process \( \pi(t) = (\pi_1(t), \cdots, \pi_m(t))' \) represents a (monetary) portfolio, where \( \pi_i(t) \) is the capital amount invested in stock \( i \) at \( t \).

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An agent has an initial endowment \( x_0 \).
Wealth Equation

Wealth process $x(\cdot)$ follows the *wealth equation*

\[
\begin{aligned}
\left\{
\begin{array}{ll}
dx(t) &= [r(t)x(t) + B(t)'\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\
x(0) &= x_0
\end{array}
\right.
\end{aligned}
\]
Wealth Equation

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\end{cases}$$

- An admissible portfolio $\pi(\cdot)$ is called tame if the corresponding wealth process $x(\cdot)$ is uniformly lower bounded
Market Assumptions:

(i) There exists $k \in \mathbb{R}$ such that $\int_0^T r(t) dt \geq k$,

(ii) $\int_0^T \left[ \sum_{i=1}^m |b_i(t)| + \sum_{i,j=1}^m |\sigma_{ij}(t)|^2 \right] dt < +\infty$,

(iii) Rank $\left( \sigma(t) \right) = m$, $t \in [0, T]$,

(iv) There exists an $\mathbb{R}^m$-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$
Pricing Kernel

Define

\[ \rho(t) := \exp \left\{ - \int_0^t \left[ r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds - \int_0^t \theta(s)' dW(s) \right\} \]
Define

\[ \rho(t) := \exp \left\{ - \int_0^t \left[ r(s) + \frac{1}{2}|\theta(s)|^2 \right] \, ds - \int_0^t \theta(s)' \, dW(s) \right\} \]

Denote \( \tilde{\rho} := \rho(T) \)
Define

$$\rho(t) := \exp \left\{ - \int_0^t \left[ r(s) + \frac{1}{2} |\theta(s)|^2 \right] ds - \int_0^t \theta(s)' dW(s) \right\}$$

Denote $\tilde{\rho} := \rho(T)$

Assume that $\tilde{\rho}$ is atomless
Continuous-Time Portfolio Choice under EUT

\[
\text{Max } \quad E[u(x(T))] \\
\text{subject to } \quad (x(\cdot), \pi(\cdot)) : \text{ tame and admissible pair (9)}
\]

where \( u \) is a concave utility function satisfying the usual assumptions
Let $v$ be the value function corresponding to (9): $v(t, x)$ is the optimal value of (9) if the initial time is $t$ (instead of 0) and the initial budget is $x$ (instead of $x_0$)
Forward Approach: Dynamic Programming

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- *Time (dynamic) consistency:* $E(\tilde{c}|\mathcal{F}_t) = E[E(\tilde{c}|\mathcal{F}_s)|\mathcal{F}_t]$ for all $t < s$. 

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- \textit{Time (dynamic) consistency}: 
  \[
  E(\tilde{c}|\mathcal{F}_t) = E[E(\tilde{c}|\mathcal{F}_s)|\mathcal{F}_t] \\
  \forall t < s
  \]

- \( v \) satisfies the \textit{Hamilton–Jacobi–Bellman (HJB) equation}:
  \[
  \begin{cases}
  v_t + \sup_{\pi \in \mathbb{R}^m} \left( \frac{1}{2} \pi' \sigma \sigma' \pi v_{xx} + B \pi v_x \right) + rxv_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\
  v(T, x) = u(x)
  \end{cases}
  \]  
  \tag{10}
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\]  

- Verification theorem: optimal portfolio

\[
\pi^*(t, x) = - (\sigma(t))^{-1} \theta(t) \frac{v_x(t, x)}{v_{xx}(t, x)}
\]  

(11)
Backward Approach: Replication

One solves first a static optimization problem in terms of terminal wealth, \( \tilde{c} \):

\[
\begin{align*}
\text{Max} & \quad E[u(\tilde{c})] \\
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\end{align*}
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- Solve \textit{backward stochastic differential equation (BSDE)} in $(x^*(\cdot), z^*(\cdot))$:

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- Setting \( \pi^*(t) = (\sigma(t)')^{-1}z^*(t) \) and \((x^*(\cdot), \pi^*(\cdot))\) is optimal pair
Time Inconsistency under Probability Weighting

- Choquet expectation
\[ \hat{E}[\tilde{X}] = \int \tilde{X} d(w \circ P) = \int_0^\infty w(P(\tilde{X} > x)) \, dx \]
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- How to define “conditional Choquet expectation”??
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- Consider a weak notion of “optimality” - equilibrium portfolio in other settings (Ekeland and Pirvu 2008, Hu, Jin and Zhou 2012, Bjork, Murgoci and Zhou 2012)
Replication: Pre-Committed Strategies

- Solve a static optimisation problem (with probability weighting) in terms of terminal wealth
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Replication: Pre-Committed Strategies

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- Such a problem has been solved by our approach developed.
- Find a dynamic portfolio replicating the obtained optimal terminal wealth.
- Such a portfolio is an optimal *pre-committed* strategy (Jin and Zhou 2008, He and Zhou 2011).
Summary and Further Readings
Summary

- Portfolio choice under CPT - probability weighting and S-shaped value function
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- Technical challenges
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- Combinatorial optimisation in infinite dimension
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- Inherent time inconsistency for continuous-time behavioural problems
Essential Readings


Other Readings

Final Words
Two Revolutions in Finance

- Finance ultimately deals with **interplay** between market risk and human judgement
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- History of financial theory over the last 50 years characterised by two revolutions.
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  - Neoclassical (maximising) finance starting 1960s: *Expected utility maximisation, CAPM, efficient market theory, option pricing*
  - Behavioural finance starting 1980s: *Cumulative prospect theory, SP/A theory, regret and self-control, heuristics and biases*
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  - A relatively new field that attempts to explain how and why emotions and cognitive errors influence investors and create stock market anomalies such as bubbles and crashes
  - It seeks to explore the consistency and predictability in human flaws so that such flaws can be avoided or even exploited for profit
Do We Need Both?

- Foundations of the two
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  - Neoclassical finance: Rationality (correct beliefs on information, risk aversion) – A normative theory
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- **Do we need both? *Absolutely yes!***
  - Neoclassical finance tells what people *ought* to do
  - Behavioural finance tells what people *actually* do
  - Robert Shiller (2006), “the two ... have always been intertwined, and some of the most important applications of their insights will require the use of both approaches”
“Mathematical behavioural finance” leads to new problems in mathematics and finance.
Mathematical Behavioural Finance

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“misguided behaviors ... are systematic and predictable – making us predictably irrational” (Dan Ariely, Predictably Irrational, Ariely 2008)
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**Mathematical behavioural finance:** research is in its infancy, yet potential is unlimited – or so we believe.