A Benchmark Approach to Investing, Pricing and Hedging

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Pl. & Bruti-Liberati (2010), Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer

Baldeaux & Pl. (2013). Functionals of Multidimensional Diffusions with Applications to Finance. Springer
1 Best Performing Portfolio as Benchmark

- with focus on long term
- if possible pathwise
- should prefer more for less
- diversified portfolio
- robust, almost model independent construction
- sustainable (strictly positive long only)
- criterion independent from denomination
- criterion independent from sequence of drawdown events

$\Rightarrow$ maximize logarithm of portfolio
EWI141 and MCI
logarithms of EWI114 and MCI

log(EWI114)

log(MCI)
Long term growth of EWI114 and MCI
Key idea:

Make the “best” performing portfolio $S^*_t$ numéraire or benchmark
Portfolio:

\[ S^δ_t = \sum_{j=0}^{d} \delta_t^j S_t^j \]

\( S_t^j \) - \( j \)th constituent (e.g. cum dividend stocks)
Long Term Growth:

\[
\lim_{t \to \infty} \frac{1}{t} \log \left( \frac{S_t^\delta}{S_0^\delta} \right) \leq \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{S_t^{\delta^*}}{S_0^{\delta^*}} \right)
\]

a.s. pathwise

\( S^{\delta^*} \) is the \textit{numéraire portfolio} (NP)

Long (1990)
Main Assumption:

There exists NP $S_t^{\delta^*}$, which means

$$\frac{S_t^\delta}{S_t^{\delta^*}} \geq E_t \left( \frac{S_s^\delta}{S_s^{\delta^*}} \right) \quad (*)$$

for all $0 \leq t \leq s < \infty$ and all nonnegative portfolios $S_t^\delta$,

$E_t(\cdot)$ - real world conditional expectation under information given at time $t$

$S_t^{\delta^*}$ - numéraire portfolio (NP), see Long (1990)
Benchmarking:

make NP numéraire and benchmark

\[ \hat{S}^\delta_t = \frac{S^\delta_t}{S^\delta*} \quad \text{— benchmarked portfolio} \]

\[ \hat{S}^j_t = \frac{S^j_t}{S^\delta*} \quad \text{— } j\text{th benchmarked security} \]
Existence of NP $\iff$ Supermartingale Property

$$\hat{S}_t^\delta \geq E_t(\hat{S}_s^\delta) \quad (*)$$

for all $0 \leq t \leq s < \infty$ and all nonnegative $S_t^\delta$.

- The key property of financial market!
- model independent
- makes also in the short term “best” performance of $S^{\delta*}$ precise
- existence of NP, extremely general, Karatzas and Kardaras (2007)
Outperforming Long Term Growth

- long term growth rate

\[ g^\delta = \limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{S_t^\delta}{S_0^\delta} \right) \]

strictly positive portfolio \( S^\delta \)

pathwise characteristic
Theorem  The NP $S^{\delta_*}$ achieves the maximum long term growth rate. For any strictly positive portfolio $S^\delta$

$$g^\delta \leq g^{\delta_*}.$$ 

NP outperforms long term growth pathwise!

- Pathwise outperformance asymptotically over time!
- Ideal long term investment if no constraints!
Proof: Consider strictly positive portfolio $S^\delta$, with $S^\delta_0 = S^\delta_{0}^* = x > 0$. By supermartingale property (\*) of $\hat{S}^\delta$ and Doob (1953), for any $k \in \{1, 2, \ldots\}$ and $\varepsilon \in (0, 1)$ one has

$$\exp\{\varepsilon k\} P \left( \sup_{k \leq t < \infty} \hat{S}^\delta_t > \exp\{\varepsilon k\} \right) \leq E_0 \left( \hat{S}^\delta_k \right) \leq \hat{S}^\delta_0 = 1.$$
For fixed $\varepsilon \in (0, 1)$

$$\sum_{k=1}^{\infty} P \left( \sup_{k \leq t < \infty} \ln (\hat{S}_t^\delta) > \varepsilon k \right) \leq \sum_{k=1}^{\infty} \exp\{-\varepsilon k\} < \infty.$$ 

By Lemma of Borel and Cantelli there exists $k_\varepsilon$ such that for all $k \geq k_\varepsilon$ and $t \geq k$

$$\ln (\hat{S}_t^\delta) \leq \varepsilon k \leq \varepsilon t.$$
Therefore, for all \( k > k_{\varepsilon} \)

\[
\sup_{t \geq k} \frac{1}{t} \ln \left( \frac{\hat{S}_t^\delta}{S_t^\delta} \right) \leq \varepsilon,
\]

which implies

\[
\limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{S_t^\delta}{S_0^\delta} \right) \leq \limsup_{t \to \infty} \frac{1}{t} \ln \left( \frac{S_t^{\delta_*}}{S_0^{\delta_*}} \right) + \varepsilon.
\]

Since the inequality holds for all \( \varepsilon \in (0, 1) \), theorem follows. \( \square \)
Best Performance of NP also in the Short Term

Outperforming Expected Benchmarked Returns

\((\ast) \Rightarrow \text{nonnegative } \hat{S}^\delta \text{ is supermartingale and } \hat{S}^\delta_* = 1\)

\(\Rightarrow\)

- expected benchmarked returns

\[E_t \left( \frac{\hat{S}^\delta_{t+h} - \hat{S}^\delta_t}{\hat{S}^\delta_t} \right) \leq E_t \left( \frac{\hat{S}^\delta_*_{t+h} - \hat{S}^\delta_*}{\hat{S}^\delta_*} \right) = 0\]

\(t > 0, \ h > 0, \ \hat{S}^\delta \text{ nonnegative}\)

\(\Rightarrow\) No strictly positive expected benchmarked returns possible!
Outperforming Expected Growth

- *expected growth*

\[ g_{t,h}^\delta = E_t \left( \ln \left( \frac{S_{t+h}^\delta}{S_t^\delta} \right) \right) \]

for \( t, h \geq 0 \).
invest in strictly positive portfolio $S^δ$ and small fraction $ε \in (0, \frac{1}{12})$ in some nonnegative portfolio $S^δ$, which yields perturbed portfolio $S^{δε}$

- derivative of expected growth in the direction of $S^δ$:

$$\frac{∂g_{t,h}^{δε}}{∂ε} \bigg|_{ε=0} = \lim_{ε\to0^+} \frac{1}{ε} \left(g_{t,h}^{δε} - g_{t,h}^{δ}\right)$$
Definition \( S^\delta \) is called growth optimal if

\[
\frac{\partial g_{t,h}^{\delta,\varepsilon}}{\partial \varepsilon} \bigg|_{\varepsilon=0} \leq 0
\]

for all \( t \) and \( h \geq 0 \).
- classical characterization: maximization of expected logarithmic utility from terminal wealth
  Kelly (1956), Latané (1959), Breiman (1960), Hakansson (1971a), Thorp (1972), Merton (1973a), Roll (1973) and Markowitz (1976) equivalent
Theorem

The NP is growth optimal if its expected growth is finite.

NP outperforms expected growth!
Proof: Pl. (2011)
For \( t \geq 0, h > 0 \), and a nonnegative portfolio \( S^\delta \) we introduce the portfolio ratio \( A_{t,h}^\delta = \frac{S_{t+h}^\delta}{S_t^\delta} \), which is set to 1 for \( S_t^\delta = 0 \). For \( \varepsilon \in (0, \frac{1}{2}) \); and a nonnegative portfolio \( S^\delta \), with \( S_t^\delta > 0 \), consider the perturbed portfolio \( S^\delta_{\varepsilon} \) with \( S_t^\delta = S_t^{\delta_0} \), yielding portfolio ratio \( A_{t,h}^{\delta_\varepsilon} = \varepsilon A_{t,h}^\delta + (1 - \varepsilon) A_{t,h}^{\delta_0} > 0 \). By the inequality \( \ln(x) \leq x - 1 \) for \( x \geq 0 \),

\[
G_{t,h}^{\delta_\varepsilon} = \frac{1}{\varepsilon} \ln \left( \frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_0}} \right) \leq \frac{1}{\varepsilon} \left( \frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_0}} - 1 \right) = \frac{A_{t,h}^\delta}{A_{t,h}^{\delta_0}} - 1
\]

and

\[
G_{t,h}^{\delta_\varepsilon} = -\frac{1}{\varepsilon} \ln \left( \frac{A_{t,h}^{\delta_0}}{A_{t,h}^{\delta_\varepsilon}} \right) \geq -\frac{1}{\varepsilon} \left( \frac{A_{t,h}^{\delta_0}}{A_{t,h}^{\delta_\varepsilon}} - 1 \right) = \frac{A_{t,h}^\delta - A_{t,h}^{\delta_0}}{A_{t,h}^{\delta_\varepsilon}}.
\]
Because of $A_{t,h}^{\delta\varepsilon} > 0$ one obtains for $A_{t,h}^{\delta} - A_{t,h}^{\delta*} \geq 0$

$$G_{t,h}^{\delta\varepsilon} \geq 0,$$

and for $A_{t,h}^{\delta} - A_{t,h}^{\delta*} < 0$ because of $\varepsilon \in (0, \frac{1}{2})$ and $A_{t,h}^{\delta} \geq 0$

$$G_{t,h}^{\delta\varepsilon} \geq -\frac{A_{t,h}^{\delta*}}{A_{t,h}^{\delta\varepsilon}} = -\frac{1}{1 - \varepsilon + \varepsilon \frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta*}}} \geq -\frac{1}{1 - \varepsilon} \geq -2.$$
Summarizing yields

\[-2 \leq G_{t,h}^{\delta_{e}} \leq \frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_{*}}} - 1,\]

where by (*)

\[E_{t}\left(\frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_{*}}}\right) \leq 1.\]
By the Dominated Convergence Theorem

\[
\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \lim_{\varepsilon \to 0^+} E_t \left( G_{t,h}^{\delta_\varepsilon} \right) = E_t \left( \lim_{\varepsilon \to 0^+} G_{t,h}^{\delta_\varepsilon} \right)
\]

\[
= E_t \left( \frac{\partial}{\partial \varepsilon} \ln \left( \frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_*}} \right) \bigg|_{\varepsilon=0} \right) = E_t \left( \frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}} \right) - 1.
\]

This proves that NP $S^{\delta_*}$ is growth optimal. $\blacksquare$
Systematic Outperformance

**Definition**  
A nonnegative portfolio $S^δ$ systematically outperforms a strictly positive portfolio $S^{\tilde{\delta}}$ if

(i) $S_{t_0}^δ = S_{t_0}^{\tilde{\delta}}$;

(ii) $P(S_t^δ \geq S_t^{\tilde{\delta}}) = 1$ for some $t > 0$ and

(iii) $P(S_t^δ > S_t^{\tilde{\delta}}) > 0$.

- also called relative arbitrage in Fernholz & Karatzas (2005)
Theorem \textit{The NP cannot be systematically outperformed by any nonnegative portfolio.}

- NP can systematically outperform other portfolios.
Proof: Consider nonnegative $S^\delta$ with $\hat{S}_t^\delta = 1$ where $\hat{S}_s^\delta \geq 1$ almost surely at $s \in [t, \infty)$. By the supermartingale property (*)

$$0 \geq E_t \left( \hat{S}_s^\delta - \hat{S}_t^\delta \right) = E_t \left( \hat{S}_s^\delta - 1 \right).$$

Since $\hat{S}_s^\delta \geq 1$ and $E_t(\hat{S}_s^\delta) \leq 1$, it can only follow $\hat{S}_s^\delta = 1$. 

$$\implies S_s^\delta = S_s^{\delta^*}.$$
• NP needs shortest expected time to reach a given wealth level
  Kardaras & Platen (2010)
Strong Arbitrage

- only market participants can exploit arbitrage with their total wealth
- limited liability

\[ S_{T_0}^\delta = 0 \]

\[ P\left( S_{T}^\delta > 0 \right) > 0. \]

Definition \ A nonnegative portfolio \( S^\delta \) is a strong arbitrage if \( S_{T_0}^\delta = 0 \) and for some \( T > 0 \)

- Loewenstein & Willard (2000) - same notion of arbitrage through economic arguments
- Pl. (2002)-mathematical motivation through supermartingale property (*)
Theorem \hspace{1cm} \textit{There is no strong arbitrage.}
Proof:

\((*) \implies\)

\(\hat{S}^{\delta}\) nonnegative supermartingale

\(\implies\)

\[0 = \hat{S}^{\delta}_0 \geq E\left(\hat{S}^{\delta}_T \mid \mathcal{A}_0\right) = E(\hat{S}^{\delta}_T)\]

\(\implies\)

\[P(S^{\delta}_T > 0) = P(\hat{S}^{\delta}_T > 0) = 0.\]
Classical notion of arbitrage is too restrictive

- Delbaen & Schachermayer (1998)

  \(\textit{no free lunches with vanishing risk} \triangleq \text{existence of risk neutral measure} \triangleq \text{APT}\)

- Loewenstein & Willard (2000)

  \(\textit{free snacks \& cheap thrills} \triangleq \text{some free lunch with vanishing risk}\)

- exploiting weak forms of classical arbitrage requires to allow negative total wealth

- pricing via hedging by avoiding “strong arbitrage” makes no sense since there is no “strong arbitrage” under (*)
How to approximate the NP in the real market?

- diversification leads asymptotically to the NP (Pl. 2005)
- confirm empirically by visualizing diversification effect
- simplest diversification:
  
  equal value weighting = naive diversification
Pl. & Rendek (2012)

- **investment universe:**
  
  market capitalization weighted indices generating the MSCI

- **Industry Classification Benchmark** (Datastream, Reuters ...):
  
  54 countries with country index
  
  10 industry indices
  
  19 supersector indices
  
  41 sector indices
  
  114 country subsector indices
  
  all market capitalization weighted
Equi-Weighted Index (EWId)

\[ S_{t_n}^{\delta_{EWId}} = S_{t_{n-1}}^{\delta_{EWId}} \left( 1 + \sum_{j=1}^{d} \sum_{k=1}^{\ell_{d,j}} \pi_{j,k}^{\delta_{EWId},t_{n-1}} \frac{S_{t_n}^{j,k} - S_{t_{n-1}}^{j,k}}{S_{t_{n-1}}^{j,k}} \right), \]

- Naive diversification over \( \ell_{d,j} \) countries
  
  and then over \( d = 10, 19, 41, 114 \) world industries

- equal value weighting uncertainties of different economic activities
The MCI and five equi-weighted indices: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).
The MCI and five equi-weighted indices in log-scale: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).
• all based on the same country industry subsectors of the equity market
• all resulting indices mostly driven by nondiversifiable uncertainty
• Empirically: **Better performance through better diversification**!
Illustrating Diversification Phenomenon:

- return of benchmarked constituent over small period \([t, t + h]\)

\[
R^j = \frac{\hat{S}^j_{t+h} - \hat{S}^j_t}{\hat{S}^j_t}
\]

\(j \in \{1, 2, \ldots \}\)
Assume for illustration purposes: $R^1$, $R^2$, \ldots independent, and

\begin{align*}
E_t(R^j) &= 0 \\
E_t\left((R^j)^2\right) &= \sigma^2 h
\end{align*}

$h > 0, j \in \{1, 2, \ldots\}$
• return of portfolio equals sum of weighted returns of constituents

• return of benchmarked EWIE\(\ell\) with \(\ell\) constituents:

\[
R_{\delta_{EWIE}} = \frac{\hat{S}_{t+h} - \hat{S}_t}{\hat{S}_{t_{EWIE}}} = \frac{1}{\ell} \sum_{j=1}^{\ell} R^j
\]
\[
E_t(R^\delta_{EW\ell}) = 0
\]
\[
E_t((R^\delta_{EW\ell})^2) = \frac{1}{\ell^2} \sum_{j=1}^{\ell} \sigma^2 h = \frac{1}{\ell} \sigma^2 h
\]

\[
\lim_{\ell \to \infty} E_t((R^\delta_{EW\ell})^2) = 0
\]

Strong Law of Large Numbers (Kolmogorov)

\[
\lim_{\ell \to 0} R^\delta_{EW\ell} = 0
\]

almost surely
for $\ell \gg 1$

$$\implies R_{\delta \text{EW} \ell} \approx 0$$

$$\implies \hat{S}_{t}^{\delta \text{EW} \ell} \approx 1$$

$$\implies S_{t}^{\delta \text{EW} \ell} = \hat{S}_{t}^{\delta \text{EW} \ell} S_{t}^{\delta^{*}} \approx S_{t}^{\delta^{*}}$$

a.s. for all $t$
Diversification yields approximation of the NP!

- diversification holds more generally
- model independent phenomenon
- arises naturally
- can be systematically exploited
Naive Diversification Theorem

Pl. & Rendek (2012)

- continuous market, Merton (1972)

- benchmarked constituents

\[
\frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_k \sigma_{t,k}^j dW_t^k
\]

\(W^1, W^2, \ldots\) - Brownian motions
fraction of wealth invested

\[ \pi_{\delta, t} = \frac{\delta^j_t \hat{S}^j_t}{\hat{S}^\delta_t} = \frac{\delta^j_t S^j_t}{S^\delta_t} \]
• benchmarked portfolio

\[
\frac{d\hat{S}_t^\delta}{\hat{S}_t^\delta} = \sum_j \pi_{\delta,t}^j \frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_j \pi_{\delta,t}^j \sum_k \sigma_t^{j,k} dW_t^k
\]
- return process of a benchmarked portfolio $\hat{S}_t^δ$

$$d\hat{Q}_t^δ = \frac{1}{\hat{S}_t^δ} d\hat{S}_t^δ$$

$t \geq 0$ with $\hat{Q}_0^δ = 0$
• Quadratic Variation of Return Process

\[ \langle \hat{Q}^{\delta_{\ell}} \rangle_t = \int_0^t \sum_k \left( \sum_j \pi^{j}_{\delta_{\ell},s} \sigma^{j,k}_{s} \right)^2 ds \]

• Tracking rate

\[ T_{\delta_{\ell}}(t) = \frac{d}{dt} \langle \hat{Q}^{\delta_{\ell}} \rangle_t = \sum_k \left( \sum_j \pi^{j}_{\delta_{\ell},t} \sigma^{j,k}_{t} \right)^2 \]
- benchmarked NP equals constant one
  return process of benchmarked NP equals zero

\[
\frac{d\hat{S}_{t}^{\delta*}}{\hat{S}_{t}^{\delta*}} = \sum_{k} \sum_{j} \pi_{\delta*,t}^{j} \sigma_{t}^{j,k} dW_{t}^{k} = 0
\]

\[\Rightarrow \text{tracking rate}\]

\[
T_{\delta*}(t) = \frac{d}{dt} \langle \hat{Q}^{\delta*} \rangle_{t} = 0
\]
• $\ell$th equi-weighted index (EWI$\ell$)

$$
\pi_{\delta_{EWI\ell},t}^j = \begin{cases} 
\frac{1}{\ell} & \text{for } j \in \{1, 2, \ldots, \ell\} \\
0 & \text{otherwise.}
\end{cases}
$$
Definition:

A sequence \((\hat{S}^{\delta_\ell})_{\ell \in \{1, 2, \ldots\}}\) of strictly positive benchmarked portfolios, with initial value equal to one, is called a sequence of benchmarked approximate NPs if for each \(\epsilon > 0\) and \(t \geq 0\) one has

\[
\lim_{\ell \to \infty} P \left( T_{\delta_\ell} (t) = \frac{d}{dt} \langle \hat{Q}^{\delta_\ell} \rangle_t > \epsilon \right) = 0.
\]
- benchmark captures general, systematic or non-diversifiable market uncertainty
- benchmarked primary security accounts capture specific or idiosyncratic uncertainty
- different types of economic activity yield naturally different specific uncertainties
- specific uncertainty can be diversified
Particular specific uncertainty drives in reality usually only returns of restricted number of benchmarked primary security accounts:
Definition:

A market is **well-securitized** if there exists $q > 0$ and square integrable $\sigma^2 = \{\sigma_t^2, t \geq 0\}$ with finite mean for all $\ell, k \in \{1, 2, \ldots\}$ and $t \geq 0$

$$\frac{1}{\ell} \left| \sum_{j=1}^{\ell} \sigma_{t}^{j,k} \right|^2 \leq \frac{1}{\ell q} \sigma_t^2$$

$P$-almost surely.

- $k$-th uncertainty affects not too many constituents.
Naive Diversification Theorem:

In a well-securitized market the sequence of benchmarked equi-weighted indices is a sequence of benchmarked approximate NPs.

Pl. & Rendek (2012)
• return process of $\ell$th benchmarked EWI

$$\hat{Q}_{t}^{\delta_{EW_{I}}} = \sum_{j=1}^{\ell} \frac{1}{\ell} \sum_{k} \int_{0}^{t} \sigma_{s}^{j,k} dW_{s}^{k}$$
\begin{itemize}
  \item quadratic variation of return process

  \[
  \left\langle \hat{Q}^{\delta_{EWI\ell}} \right\rangle_t = \frac{1}{\ell} \int_0^t \sum_k \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^\ell \sigma_{s,k} \right|^2 ds
  \]

  \item tracking rate

  \[
  T_{\delta_{EWI\ell}}(t) = \frac{d}{dt} \left\langle \hat{Q}^{\delta_{EWI\ell}} \right\rangle_t = \frac{1}{\ell} \sum_k \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^\ell \sigma_{t,k} \right|^2
  \]
\end{itemize}
Lemma Assume that for all \( \varepsilon > 0 \) and \( t \geq 0 \)

\[
\lim_{\ell \to \infty} P \left( \frac{1}{\ell} \sum_{k} \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_{t,j,k} \right|^2 > \varepsilon \right) = 0,
\]
then the sequence of equi-weighted indices is a sequence of benchmarked approximate NPs.
Proof:
For $\varepsilon > 0$ and $t > 0$

\[
0 = \lim_{\ell \to \infty} P \left( \frac{1}{\ell} \sum_k \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_{t,k}^j > \varepsilon \right)^2 \geq \varepsilon \right).
\]

\[
= \lim_{\ell \to \infty} P \left( \frac{d}{dt} \langle \hat{Q}_{\delta_{EW \ell}} \rangle_t > \varepsilon \right)
\]

\[
= \lim_{\ell \to \infty} P (T_{\delta_{EW \ell}}(t) > \varepsilon)
\]

\[
\square
\]
Proof of Naive Diversification Theorem:

By Markov inequality in a well-securitized market

\[
\lim_{\ell \to \infty} P \left( \frac{1}{\ell} \sum_k \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^\ell \sigma_{t,j,k} \right|^2 > \epsilon \right)
\]

\[
\leq \lim_{\ell \to \infty} \frac{1}{\ell q} E \left( \sigma_t^2 \right) = 0,
\]

NDT follows from previous Lemma. \qed
Logarithms of MCI, EWI114 without transaction cost and EWI114_\xi with transaction costs of 5, 40, 80, 200 and 240 basis points.
EWI114$^m$ reallocated daily and every 2, 4, 8, 16 and 32 days.
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<th>0</th>
<th>5</th>
<th>40</th>
<th>80</th>
<th>200</th>
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<td>0.1834</td>
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<td>0.1135</td>
<td>0.1135</td>
<td>0.1135</td>
<td>0.1134</td>
<td>0.1134</td>
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<td>1.4046</td>
<td>1.2930</td>
<td>1.1654</td>
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<td>63166.73</td>
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<td>1.0591</td>
<td>0.9967</td>
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<td>Transaction cost</td>
<td>0</td>
<td>5</td>
<td>40</td>
<td>80</td>
<td>200</td>
<td>240</td>
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<tr>
<td>------------------</td>
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<tr>
<td>Reallocation terms</td>
<td>8</td>
<td></td>
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<tr>
<td>Final value</td>
<td>100505.37</td>
<td>97963.66</td>
<td>81881.57</td>
<td>66705.62</td>
<td>36055.10</td>
<td>29367.02</td>
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<tr>
<td>Annualised average return</td>
<td>0.1892</td>
<td>0.1885</td>
<td>0.1837</td>
<td>0.1783</td>
<td>0.1621</td>
<td>0.1566</td>
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<tr>
<td>Annualised volatility</td>
<td>0.1127</td>
<td>0.1127</td>
<td>0.1127</td>
<td>0.1127</td>
<td>0.1128</td>
<td>0.1128</td>
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<tr>
<td>Sharpe ratio</td>
<td>1.3531</td>
<td>1.3471</td>
<td>1.3051</td>
<td>1.2569</td>
<td>1.1119</td>
<td>1.0634</td>
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<td>16</td>
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<tr>
<td>Final value</td>
<td>98775.24</td>
<td>96892.29</td>
<td>84677.43</td>
<td>72588.14</td>
<td>45711.97</td>
<td>39177.91</td>
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<tr>
<td>Annualised average return</td>
<td>0.1887</td>
<td>0.1882</td>
<td>0.1847</td>
<td>0.1806</td>
<td>0.1684</td>
<td>0.1643</td>
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<tr>
<td>Annualised volatility</td>
<td>0.1130</td>
<td>0.1130</td>
<td>0.1130</td>
<td>0.1130</td>
<td>0.1131</td>
<td>0.1131</td>
</tr>
<tr>
<td>Sharpe ratio</td>
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<td>1.3418</td>
<td>1.3102</td>
<td>1.2740</td>
<td>1.1647</td>
<td>1.1281</td>
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<td>Reallocation terms</td>
<td>32</td>
<td></td>
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<tr>
<td>Final value</td>
<td>114592.50</td>
<td>112929.09</td>
<td>101939.59</td>
<td>90678.85</td>
<td>63804.84</td>
<td>56744.28</td>
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<tr>
<td>Annualised average return</td>
<td>0.1927</td>
<td>0.1923</td>
<td>0.1896</td>
<td>0.1865</td>
<td>0.1772</td>
<td>0.1741</td>
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<tr>
<td>Annualised volatility</td>
<td>0.1131</td>
<td>0.1131</td>
<td>0.1131</td>
<td>0.1131</td>
<td>0.1133</td>
<td>0.1133</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>1.3797</td>
<td>1.3763</td>
<td>1.3522</td>
<td>1.3245</td>
<td>1.2408</td>
<td>1.2127</td>
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</table>
Simulated GOP, EWI and MCI under the Black-Scholes model
Simulated benchmarked GOP, EWI and MCI under the Black-Scholes model
Example of a Multi-Asset BS Model

- \( j \)th primary security account

\[
dS^j_t = S^j_t \left[ \left( r + \sigma^2 \left( 1 + \frac{1}{\sqrt{d}} \right) \right) dt + \frac{\sigma}{\sqrt{d}} \sum_{k=1}^{d} dW^k_t + \sigma \ dW^j_t \right]
\]

\( t \in [0, T], \ j \in \{1, 2, \ldots, d\}, \ \sigma > 0 \)

- general market risk, specific market risk
• NP fractions

\[ \pi_{\delta^*,t}^j = \left( \sqrt{d} \left( 1 + \sqrt{d} \right) \right)^{-1} \]

\[ j \in \{1, 2, \ldots, d \} \]

\[ \pi_{\delta^*,t}^0 = \left( 1 + \sqrt{d} \right)^{-1} \]

• NP SDE

\[ dS_t^{\delta^*} = S_t^{\delta^*} \left( (r + \sigma^2) \, dt + \frac{\sigma}{\sqrt{d}} \, \sum_{k=1}^{d} dW_t^k \right) \]

• simulate over \( T = 32 \) years \( d = 50 \) risky primary security accounts

\[ \sigma = 0.15, \quad r = 0.05 \]
Benchmarked primary security account
BS - model

\[ d\hat{S}_t^j = d\left( \frac{S_t^j}{S_t^{\delta_t}} \right) = \hat{S}_t^j \sigma dW_t^j \]

martingale
Simulated primary security accounts.
Simulated GOP and savings account.
Equal Weighted Index

$S^{\delta_{\text{EWId}}} \text{ holds equal fractions}$

$$\pi^j_{\delta_{\text{EWId}},t} = \frac{1}{d + 1}$$

$j \in \{0, 1, \ldots, d\}$

• tracking rate for BS example:

$$T^d_{\delta_{\text{EWId}}} (t) = \sum_{k=1}^{d} \left( \frac{\sigma}{d + 1} \left( \frac{1}{\sqrt{d}} - 1 \right) \right)^2$$

$$= \ d \left( \frac{\sigma}{d + 1} \left( \frac{1}{\sqrt{d}} - 1 \right) \right)^2 \leq \frac{\sigma^2}{(d + 1)} \rightarrow 0$$
Simulated NP and EWI.
Simulated diversified accumulation index and NP.
Diversification in an MMM Setting

Pl. (2001), minimal market model (MMM)

- savings account
  \[
  S^0_{(d)}(t) = \exp\{r \ t\}
  \]

- discounted GOP drift
  \[
  \alpha^\delta\ast_t = \alpha_0 \ \exp\{\eta \ t\}
  \]

- \(j\)th benchmarked primary security account
  \[
  \hat{S}^j_{(d)}(t) = \frac{1}{Y_t^j \ \alpha^\delta\ast_t}
  \]
• square root process

\[ dY_t^j = \left(1 - \eta Y_t^j\right) dt + \sqrt{Y_t^j} \, dW_t^j \]

• NP

\[ S^{\delta_*}_{(d)}(t) = \frac{S^0_{(d)}(t)}{\hat{S}^0_{(d)}(t)} \]

• \( j \)th primary security account

\[ S^j_{(d)}(t) = \hat{S}^j_{(d)}(t) \, S^{\delta_*}_{(d)}(t) \]
Primary security accounts under the MMM
Benchmarked primary security accounts
NP and EWI
NP and market index
Diversified Portfolios

Fundamental phenomenon of **diversification** leads naturally to **NP** also for more general diversified portfolios than the EWI:

**Definition**  
A sequence of strictly positive portfolios \((S^d_\delta)d\in\{1,2,...\}\) is a sequence of diversified portfolios if

\[
|\pi^j_{d,t}| \leq \begin{cases} 
\frac{K_2}{d^{1/2}+K_1} & \text{for } j \in \{0, 1, \ldots, d\} \\
0 & \text{otherwise}
\end{cases}
\]

a.s. for \(t \in [0, T]\).

- still small fractions but more general than EWIs
• $j$th benchmarked primary security account

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$$

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_k s_t^{j,k} dW_t^k$$

• $s_t^{j,k}$  
  $(j, k)$th specific volatility  
  measures specific market risk of $S^j$  
  with respect to $W^k$ diversifiable uncertainty

• volatility of NP measures general market risk  
  $\triangleq$ nondiversifiable uncertainty
- $k$th total specific volatility

\[ \hat{s}_t^k = \sum_j |s_t^{j,k}| \]

**Definition**  
*A market is called regular if*

\[ E \left( (\hat{s}_t^k)^2 \right) \leq K_5 \]

*for all* $t \in [0, \infty)$, $k \in \{1, 2, \ldots \}$.
Tracking Rate

\[ T_\delta(t) = \frac{d}{dt} \langle \hat{Q}^\delta \rangle_t = \sum_k \left( \sum_{j=0}^d \pi_{\delta,t}^j s_{t,k}^j \right)^2 \]

\( T_\delta(t) \) - quantifies “distance” between \( S^\delta \) and \( S^{\delta_*} \)

\[ T_{\delta_*}(t) = 0 \]
Approximate NP

Definition  A strictly positive portfolio $S^{\delta_d}$ is an approximate NP if for all $t \in [0, T]$ and $\varepsilon > 0$

$$\lim_{d \to \infty} P (T_{\delta_d}(t) > \varepsilon) = 0.$$
Diversification Theorem  (Pl. 2005)

In a regular market

any diversified portfolio is an approximate NP.

- model independent
Proof:

\[ E(T_{\delta_d}(t)) \leq \sum_{k=1}^{d} E \left( \left( \sum_{j=0}^{d} |\pi_{\delta,t}^j| |s_{t}^{j,k}| \right)^2 \right) \]

\[ \leq \frac{(K_2)^2}{d(1+2K_1)} \sum_{k=1}^{d} E \left( (\hat{s}_t^k)^2 \right) \leq \frac{(K_2)^2 K_5}{d(1+2K_1)} d \to 0 \]

Markov inequality \[\Rightarrow\]

\[ \lim_{d \to \infty} P(T_{\delta_d}(t) > \varepsilon) \leq \lim_{d \to \infty} \frac{1}{\varepsilon} E(T_{\delta_d}(t)) = 0. \]
Summary on Diversification

- extremely robust approach to asset management
- does not rely on estimating some expected returns
- diversification can be refined
- diversified market indices are in reality similar in their dynamics
  ⇒ proxies for the NP
  e.g. MSCI, S&P 500, EWI114, EWI142 ...
Continuous Financial Market

- forces us to be more specific

$(\Omega, \mathcal{A}, \mathcal{F}, P)$ - filtered probability space

$\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ - filtration

$\mathcal{F}_t$ - information at time $t$

$E_t(\cdot) = E(\cdot | \mathcal{F}_t)$ - conditional expectation

- $d$ sources of continuous traded uncertainty

$W^1, W^2, \ldots, W^d$, $d \in \{1, 2, \ldots\}$

independent standard Brownian motions
Primary Security Accounts

\[ S_t^j \] - \( j \)th primary security account at time \( t \),

\( j \in \{0, 1, \ldots, d\} \)

**cum-dividend** share value or savings account value,

dividends or interest are reinvested

- vector process of primary security accounts

\[
S = \{ S_t = (S_t^0, \ldots, S_t^d)^\top, t \in [0, T] \}
\]
• assume **unique strong solution** of SDE

\[
dS^j_t = S^j_t \left( a^j_t \, dt + \sum_{k=1}^{d} b^{j,k}_t \, dW^k_t \right)
\]

\( t \in [0, \infty), \; S^j_0 > 0, \; j \in \{1, 2, \ldots, d\} \)

• predictable, integrable appreciation rate processes \( a^j \)

• predictable, square integrable volatility processes \( b^{j,k} \)
• savings account

\[ S^0_t = \exp \left\{ \int_0^t r_s \, ds \right\} \]

predictable, integrable short rate process

\[ r = \{ r_t, \ t \in [0, \infty) \} \]
- **Market price of risk**

\[ \theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top \]

**unique invariant** of the market

solution of relation

\[ b_t \theta_t = a_t - r_t \ 1 \]

where

\[ a_t = (a_t^1, a_t^2, \ldots, a_t^d)^\top \]

\[ 1 = (1, 1, \ldots, 1)^\top \]
Assumption:
Volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ is invertible

$\implies$

- market price of risk

$$\theta_t = b_t^{-1} (a_t - r_t 1)$$
can rewrite $j$th primary security account SDE

$$dS_t^j = S_t^j \left\{ r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} (\theta_t^k \, dt + dW_t^k) \right\}$$

$t \in [0, \infty), \ j \in \{1, 2, \ldots, d\}$
• strategy

\[ \delta = \{ \delta_t = (\delta^0_t, \ldots, \delta^d_t)^\top, \; t \in [0, T] \} \]

predictable and \( S \)-integrable,

\( \delta^j_t \) number of units of \( j \)th primary security account

• portfolio

\[ S^\delta_t = \sum_{j=0}^{d} \delta^j_t \cdot S^j_t \]
• $S^\delta$ self-financing $\iff$

$$dS_t^\delta = \sum_{j=0}^{d} \delta^j_t dS_t^j$$

• changes in the portfolio value only due to gains from trading,

• self-financing in each denomination
• $j$th fraction

\[ \pi^{j}_{\delta, t} = \delta^{j}_{t} \frac{S^{j}_{t}}{S^{\delta}_{t}} \]

$j \in \{0, 1, \ldots, d\}$

\[ \sum_{j=0}^{d} \pi^{j}_{\delta, t} = 1 \]
• portfolio SDE

\[ dS_t^\delta = S_t^\delta \left( r_t \, dt + \sum_{k=1}^{d} b_{\delta,t}^k \left( \theta_t^k \, dt + dW_t^k \right) \right) \]

with \( k \)th portfolio volatility

\[ b_{\delta,t}^k = \sum_{j=0}^{d} \pi_j^{\delta,t} b_{t}^{j,k} \]
• logarithm of strictly positive portfolio

\[ d \ln(S_t^\delta) = g_t^\delta \, dt + \sum_{k=1}^{d} b_{\delta,t}^k \, dW_t^k \]

with growth rate

\[ g_t^\delta = r_t + \sum_{k=1}^{d} b_{\delta,t}^k \left( \theta_t^k - \frac{1}{2} b_{\delta,t}^k \right) \]
maximize the growth rate

$$0 = \frac{\partial g_t^\delta}{\partial b_{\delta,t}^k} = \theta_t^k - b_{\delta^*,t}^k$$

for each $k \in \{1, 2, \ldots, d\}$

$$\theta_t^k = b_{\delta^*,t}^k = \sum_{\ell=1}^d \pi_{\delta^*,t}^\ell \ b_{t}^{\ell,k}$$

$$\theta_t^\top = \pi_{\delta^*,t}^\top \ b_t$$
Since $b_t$ invertible $\implies$

$$
\pi_{\delta^*,t} = (\pi^1_{\delta^*,t}, \cdots, \pi^d_{\delta^*,t})^\top
$$

$$
= (b_t^{-1})^\top \theta_t
$$

$\implies$  Kelly portfolio, growth optimal portfolio, GOP, log-optimal portfolio, NP

$$
dS_t^{\delta^*} = S_t^{\delta^*} \left( \left[ r_t + \sum_{k=1}^d (\theta_t^k)^2 \right] dt + \sum_{k=1}^d \theta_t^k dW_t^k \right)
$$

finite
⇒

- GOP

$$dS_{t}^{\delta * } = S_{t}^{\delta * } \left( r_{t} \ dt + \sum_{k=1}^{d} \theta_{t}^{k} \left( \theta_{t}^{k} \ dt + dW_{t}^{k} \right) \right)$$

- general portfolio

$$dS_{t}^{\delta } = S_{t}^{\delta } \left( r_{t} \ dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta, t}^{j} b_{t}^{j,k} \left( \theta_{t}^{k} \ dt + dW_{t}^{k} \right) \right)$$
Benchmarked Portfolio

\[ \hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta*}} \]

Itô formula

\[ d\hat{S}_t^\delta = -\hat{S}_t^\delta \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta,t}^j \left( \theta_t^k - b_t^{j,k} \right) dW_t^k \]

driftless \[\implies\] local martingale,

“zero expected return locally”

Local Efficient Market Hypothesis

similar to Fama (1965)
• nonnegative local martingale is supermartingale

\[ \implies \text{Kelly portfolio (GOP) is the NP} \]
• NP benchmark for investing
• numéraire for pricing
• captures general market risk in risk management
• central in benchmark approach
Benchmark Approach

- provides much wider framework for modeling than classical APT
- can capture more phenomena relevant in real world
- contains as special cases classical approaches
e.g. APT, actuarial approach, CAPM, ...
Capital Asset Pricing Model

- **risk premium** of security \( V(t) \)

\[
p_V(t) = \lim_{h \downarrow 0} \frac{1}{h} E_t \left( \frac{V(t + h) - V(t)}{V(t)} \right) - r_t
\]

excess expected return

- **NP**

\[
dS_t^{\delta_n} = S_t^{\delta_n} \left( (r_t + p S_t^{\delta_n}(t)) \ dt + \sum_{k=1}^{d} \theta_t^k \ dW_t^k \right)
\]

with volatility

\[
|\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2} = \sqrt{p S_t^{\delta_n}(t)}
\]
underlying security

\begin{equation}
    dS_t = S_t \left( r_t \ dt + \sum_{k=1}^{d} \sigma_t^k (\theta_t^k \ dt + dW_t^k) \right)
\end{equation}

risk premium

\begin{equation}
    p_S(t) = \sum_{k=1}^{d} \sigma_t^k \theta_t^k
    = \frac{d}{dt} \left[ \ln(S), \ln(S^{\delta*}) \right]_t
    = \lim_{h \downarrow 0} \frac{1}{h} E_t \left( \left( \frac{S_{t+h} - S_t}{S_t} \right) \left( \frac{S^{\delta*}_{t+h} - S^{\delta*}_t}{S^{\delta*}_t} \right) \right)
\end{equation}
portfolio

\[ dS_t^\delta = S_t^\delta \left( r_t \, dt + \pi_{\delta,t}^1 \sum_{k=1}^{d} \sigma_t^k (\theta_t^k \, dt + dW_t^k) \right) \]

risk premium

\[ p_{S^\delta}(t) = \pi_{\delta,t}^1 \sum_{k=1}^{d} \sigma_t^k \theta_t^k \]

\[ = \lim_{h \to 0} \frac{1}{h} E_t \left( \left( \frac{S_t^{\delta+h} - S_t^{\delta}}{S_t^{\delta}} \right) \left( \frac{S_t^{\delta*+h} - S_t^{\delta*}}{S_t^{\delta*}} \right) \right) \]

\[ = \frac{d}{dt} \left[ \ln(S^{\delta}), \ln(S^{\delta*}) \right]_t \]
Assumption: Market portfolio $S^{\delta_{MP}}$ is diversified and approximates NP

$$S^{\delta_{MP}}_t \approx S^{\delta_*}_t.$$
• systematic risk parameter beta

\[ \beta_{S^\delta}(t) = \frac{d}{dt} \left[ \ln(S^\delta), \ln(S_{MP}^\delta) \right]_t \]

\[ \Rightarrow \]

\[ \beta_{S^\delta}(t) \approx \frac{d}{dt} \left[ \ln(S^\delta), \ln(S_{\delta*}^\delta) \right]_t = \frac{p_{S^\delta}(t)}{p_{S_{\delta*}^\delta}(t)} \approx \frac{p_{S^\delta}(t)}{p_{S_{MP}^\delta}(t)} \]
• fundamental CAPM relation follows directly

\[ p_{S^\delta}(t) \approx \beta_{S^\delta}(t) p_{S^\delta_{MP}}(t) \]
• CAPM relationship still holds for $S^{\delta_{MP}}$ with SDE

$$dS^{\delta_{MP}}_t = S^{\delta_{MP}}_t \left( r_t \, dt + \pi_t \sum_{k=1}^{d} \theta^k_t \left( \theta^k_t \, dt + dW^k_t \right) \right)$$

combination of savings account and NP, two fund separation

•

$$\beta_{S^{\delta}}(t) = \frac{\frac{d}{dt} \left[ \ln(S^{\delta}), \ln(S^{\delta_{MP}}) \right]}{\frac{d}{dt} \left[ \ln(S^{\delta_{MP}}) \right]}$$

$$= \frac{\pi^1_{\delta,t} \sum_{k=1}^{d} \sigma^k_t \theta^k_t \pi_t}{(\pi_t)^2 \sum_{k=1}^{d} (\theta^k_t)^2}$$

$$= \frac{\pi^1_{\delta,t} \sum_{k=1}^{d} \sigma^k_t \theta^k_t}{\pi_t \sum_{k=1}^{d} (\theta^k_t)^2} \frac{p_{S^{\delta}}(t)}{p_{S^{\delta_{MP}}}(t)}$$
no utility function involved,
no principal agent,
no equilibrium assumption

follows simply by Itô calculus and BA

⇒

- capital asset pricing model (CAPM)

Sharpe (1964), Lintner (1965),
Mossin (1966) and Merton (1973b)
Examples for Beta:

- savings account
  \[ \beta_{S^0}(t) = 0 \]

- underlying security
  \[ \beta_S(t) = \sum_{k=1}^{d} \sigma_k t \theta_k t^2 \]

- NP
  \[ \beta_{S^0}(t) = 1 \]
• portfolio

$$\beta_{S^\delta}(t) = \frac{\pi_{1,\delta,t}^1 \sum_{k=1}^{d} \sigma_{t}^k \theta_{t}^k}{|\theta_t|^2}$$

• If $S^\delta$ involves only diversifiable uncertainty, which is not in NP, then it has zero beta

$$\beta_{S^\delta}(t) = 0$$
Portfolio Optimization

- Kelly portfolio (GOP, NP) best long term investment
  - Kelly (1956), Latané (1959), Breiman (1961), Hakansson (1971b), Thorp (1972)

- optimal portfolio separated into two funds
  - NP and savings account
  - Tobin (1958), Sharpe (1964)
• mean-variance efficient portfolio
  Markowitz (1959)

• intertemporal capital asset pricing model
  Merton (1973a)
Discounted Portfolio

- strictly positive portfolio $S^\delta \in \mathcal{V}^+$

- discounted value

$$\bar{S}^\delta_t = \frac{S^\delta_t}{S^0_t}$$

$\Rightarrow$

- SDE

$$d\bar{S}^\delta_t = \sum_{k=1}^{d} \psi^k_{\delta,t} (\theta^k_t dt + dW^k_t)$$

with $k$th diffusion coefficient

$$\psi^k_{\delta,t} = \sum_{j=1}^{d} \delta^j_t \bar{S}^j_t b^{j,k}_t$$
• discounted drift

\[ \alpha_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \theta_t^k \]

link to macro economy

• fluctuations

\[ \tilde{M}_t = \sum_{k=1}^{d} \int_{0}^{t} \psi_{\delta,s}^k \ dW_s^k \]

\[ \Rightarrow \]

\[ \tilde{S}_t^\delta = \tilde{S}_0^\delta + \int_{0}^{t} \alpha_s^\delta \ ds + \tilde{M}_t \]
• aggregate diffusion coefficient (deviation)

\[ \gamma_t^\delta = \sqrt{\sum_{k=1}^{d} (\psi_{\delta,t}^k)^2} \]

\[ b_t^\delta = \frac{\gamma_t^\delta}{\bar{S}_t^\delta} \]

⇒ aggregate volatility
Locally Optimal Portfolios

Definition

\( S^\delta \in \mathcal{V}^+ \) **locally optimal**, if for all \( t \in [0, \infty) \) and \( S^\delta \in \mathcal{V}^+ \) with

\[
\gamma_t^\delta = \gamma_t^\tilde{\delta}
\]

almost surely:

\[
\alpha_t^\delta \leq \alpha_t^\tilde{\delta}.
\]

generalization of mean-variance optimality

Markowitz (1952, 1959)
Sharpe Ratio

Sharpe (1964, 1966)

- risk premium

\[ p S^\delta(t) = \frac{\alpha^\delta_t}{S_t^\delta} \]

- aggregate volatility \( b^\delta_t \)

- Sharpe ratio

\[ s^\delta_t = \frac{p S^\delta(t)}{b_t^\delta} = \frac{\alpha^\delta_t}{\gamma_t^\delta} = \frac{\text{mean}}{\text{deviation}} \]
• total market price of risk

\[ |\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2} \]

Assumption:

\[ 0 < |\theta_t| < \infty \]

and

\[ \pi^0_{\delta_*,t} \neq 1 \]

almost surely.
• Portfolio Selection Theorem
  For any $S^\delta \in \mathcal{V}^+$ it follows Sharpe ratio
  \[ s^\delta_t \leq |\theta_t|, \]
  where equality when $S^\delta \text{ locally optimal}$
  \[ d\bar{S}^\delta_t = \bar{S}^\delta_t \frac{b^\delta_t}{|\theta_t|} \sum_{k=1}^d \theta_t^k (\theta_t^k \, dt + dW^k_t), \]
  \[ \Rightarrow \text{ fractions} \]
  \[ \pi_{\delta,t}^j = \frac{b^\delta_t}{|\theta_t|} \pi_{\delta^*,t}^j, \]

Markowitz (1959), Sharpe (1964), Merton (1973a)
Sharpe ratio maximization
Proof of Portfolio Selection Theorem

Lagrange multiplier $\lambda$

consider

$$\mathcal{L}(\psi^{1}_\delta, \ldots, \psi^{d}_\delta, \lambda) = \sum_{k=1}^{d} \psi^{k}_\delta \theta^{k} + \lambda \left( (\gamma^{\tilde{\delta}})^2 - \sum_{k=1}^{d} (\psi^{k}_\delta)^2 \right)$$

suppressing time dependence

first-order conditions

$$\frac{\partial \mathcal{L}(\psi^{1}_\delta, \ldots, \psi^{d}_\delta, \lambda)}{\partial \psi^{k}_\delta} = \theta^{k} - 2\lambda \psi^{k}_\delta = 0$$

for all $k \in \{1, 2, \ldots, d\}$ as well as
\[ \frac{\partial \mathcal{L}(\psi^1_\delta, \ldots, \psi^d_\delta, \lambda)}{\partial \lambda} = \left( \gamma^{\tilde{\delta}} \right)^2 - \sum_{k=1}^{d} (\psi^k_\delta)^2 = 0 \]

\[ \implies \text{locally optimal portfolio } S^{(\tilde{\delta})} \text{ must satisfy} \]

\[ \psi^k_\delta = \frac{\theta^k}{2 \lambda} \]

for all \( k \in \{1, 2, \ldots, d\} \).

Then

\[ \sum_{k=1}^{d} (\psi^k_\delta)^2 = \left( \gamma^{\tilde{\delta}} \right)^2 \]
\[
\left( \gamma \delta \right)^2 = \sum_{k=1}^{d} (\psi_k^\delta)^2 = \sum_{k=1}^{d} (\theta_k^k)^2 = \frac{\sum_{k=1}^{d} (\theta_k^k)^2}{4 \lambda^2}
\]

we have \( |\theta| = \sqrt{\sum_{k=1}^{d} (\theta_k^k)^2} > 0 \),

then

\[
\psi_k^\delta = \frac{\gamma \delta}{|\theta|} \theta_k^k
\]

for all \( k \in \{1, 2, \ldots, d\} \)
\[
\alpha_t^\tilde{\delta} = \gamma_t^\tilde{\delta} \frac{|\theta_t|^2}{|\theta_t|} = \gamma_t^\tilde{\delta} |\theta_t|
\]

\[
d\bar{S}_t^\tilde{\delta} = \gamma_t^\tilde{\delta} \sum_{k=1}^{d} \frac{\theta_t^k}{|\theta_t|} (\theta_t^k \, dt + dW_t^k)
\]

\[
\psi_{\delta,t}^k = \sum_{j=1}^{d} \bar{S}_t^j \bar{S}_t^j b_t^{j,k} = \bar{S}_t^{(\tilde{\delta})} \sum_{j=1}^{d} \pi_{\delta,t}^j b_t^{j,k} = \frac{\gamma_t^\tilde{\delta}}{|\theta_t|} \theta_t^k = \bar{S}_t^{(\tilde{\delta})} b_t^\tilde{\delta} \frac{\theta_t^k}{|\theta_t|}
\]

invertibility of volatility matrix

\[
\pi_{\delta,t}^j = \frac{b_t^\tilde{\delta}}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k b_t^{-1} j,k
\]
Two Fund Separation

- locally optimal portfolio

fraction of wealth in the NP

\[
\frac{b^\delta_t}{|\theta_t|} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta^*,t}}
\]

remainder in savings account

\[
\pi^0_{\delta,t} = 1 - \frac{b^\delta_t}{|\theta_t|} \left(1 - \pi^0_{\delta^*,t}\right)
\]

also known as fractional Kelly strategy:

Risk Aversion Coefficient

\[ J_t^\delta = \frac{1 - \pi_{\delta*,t}^0}{1 - \pi_{\delta,t}^0} = \frac{|\theta_t|}{b_t^\delta} \]

Pratt (1964), Arrow (1965)

- discounted locally optimal portfolio

\[ d\bar{S}_t^\delta = \bar{S}_t^\delta \frac{1}{J_t^\delta} |\theta_t| \left( |\theta_t| \, dt + dW_t \right), \]

where

\[ dW_t = \sum_{k=1}^{d} \frac{\theta_t^k}{|\theta_t|} \, dW_t^k \]

\[ \frac{1}{J_t^\delta} \] fraction in the NP
Capital Market Line

- expected rate of return

\[ a_t^\delta = r_t + p_{S_t^\delta}(t) \]

- for a locally optimal portfolio

\[ a_t^\delta = r_t + |\theta_t| b_t^\delta \]

- portfolio process at the capital market line

has fraction \( \frac{1}{J_t^\delta} = \frac{b_t^\delta}{|\theta_t|} \) in NP

fractional Kelly strategy
Markowitz Efficient Frontier

- aggregate volatility

\[ b_t^\delta = \frac{1 - \pi_{\delta,t}^0}{1 - \pi_{\delta^*,t}^0} |\theta_t| \]

Definition

*Markowitz efficient portfolio has expected rate of return on efficient frontier* if

\[ a_t^\delta = r_t + \sqrt{(b_t^\delta)^2} |\theta_t|. \]
Theorem Any locally optimal portfolio $S^\delta$ is Markowitz efficient portfolio.

$$p_{S^\delta}(t) = \frac{1 - \pi_{\delta,t}^0}{1 - \pi_{\delta^*,t}^0} |\theta_t|^2 = b^\delta_t |\theta_t|$$

Sharpe ratio maximization
risk attitude expressed via $J^\delta_t = \frac{|\theta_t|}{b^\delta_t}$
short term view
Efficient frontier
Efficient Growth Rates

- efficient growth rate frontier

  of locally optimal portfolio

\[ g_t^\delta = r_t + \sqrt{|b_t^\delta|^2 |\theta_t|} - \frac{1}{2} |b_t^\delta|^2 = r_t + \frac{|\theta_t|^2}{J_t^\delta} \left( 1 - \frac{1}{2 J_t^\delta} \right) \]

- maximum

\[ g_t^{\delta*} = r_t + \frac{1}{2} |\theta_t|^2 \]

- long term view
Efficient growth rates
Market Portfolio

\[ S_t^{\delta_{\text{MP}}} = \sum_{\ell=1}^{n} S_t^{\delta_{\ell}} \]

portfolio of tradable wealth of \( \ell \)th investor \( S_t^{\delta_{\ell}} \)

**Remark:**

If each investor forms a nonnegative, locally optimal portfolio with her or his total tradable wealth, then MP is locally optimal.
\[ d\bar{S}_{t}^{\delta_{MP}} = \sum_{\ell=1}^{n} d\bar{S}_{t}^{\delta_{\ell}} \]

\[ = \sum_{\ell=1}^{n} \frac{\left( \bar{S}_{t}^{\delta_{\ell}} - \delta_{\ell}^{0} \right)}{\left( 1 - \pi_{0}^{0,\delta_{*,t}} \right)} \sum_{k=1}^{d} \theta_{t}^{k} \left( \theta_{t}^{k} dt + dW_{t}^{k} \right) \]

\[ = \bar{S}_{t}^{\delta_{MP}} \frac{\left( 1 - \pi_{0}^{0,\delta_{MP},t} \right)}{\left( 1 - \pi_{0}^{0,\delta_{*,t}} \right)} \sum_{k=1}^{d} \theta_{t}^{k} \left( \theta_{t}^{k} dt + dW_{t}^{k} \right) \]

- CAPM relationship still holds when MP locally optimal
Practical Feasibility of Sample Based Markowitz Mean-Variance Approach:

- NP theoretically known for given model
- Markowitz (1952)
- Best & Grauer (1991)
- DeMiguel, Garlappi & Uppal (2009)

practical application
requires for 50 assets observation window of about 500 years

- not realistic to be applied
- theoretically can be reconciled under strong assumptions (MP locally optimal)
Maximum Drawdown Constrained Portfolios

\( \mathcal{X} \) - set of nonnegative continuous discounted portfolios

- **Running maximum**
  
  for \( X = \{X_t, t \geq 0\} \in \mathcal{X} \)
  
  \[ X^*_t = \sup_{u \in [0,t]} X_u \]

- **Drawdown**

  \[ X^*_t - X_t \]
• relative drawdown

$$\frac{X_t}{X_t^*}$$

• maximum relative drawdown

• express attitude towards risk by restricting to $X \in \alpha \mathcal{X}$, where

$$\frac{X_t}{X_t^*} \geq \alpha, \alpha \in [0, 1)$$

pathwise criterion
• **drawdown**

  for $X \in \alpha \mathcal{X}$

  $X_t \geq \alpha X_t^*$

  $\implies$

  $X_t^* - X_t \leq (1 - \alpha) X_t^*$
• maximum drawdown constrained portfolio
\( \alpha \in [0, 1) \), \( X \in \mathcal{X} \)

\[
\alpha X_t = \alpha (X_t^*)^{1-\alpha} + (1 - \alpha) X_t (X_t^*)^{-\alpha}
\]

\[
= 1 + \int_0^t (1 - \alpha)(X_s^*)^{-\alpha} dX_s \in [\alpha X_t^*, (X_t^*)^{1-\alpha}]
\]

Kardaras, Obloj & Pl. (2012)
\[ \text{SDE} \]

\[ \frac{d^\alpha X_t}{\alpha X_t} = \alpha \pi_t \frac{dX_t}{X_t} \]

fraction

\[ \alpha \pi_t = 1 - \frac{(1 - \alpha) \frac{X_t}{X_t^*}}{\alpha + (1 - \alpha) \frac{X_t}{X_t^*}} \]

model independent

depends only on \( \frac{X_t}{X_t^*} \) and \( \alpha \)

\( \alpha \bar{S}_t^{\delta^*} \) - locally optimal portfolio
logarithm of discounted S&P500 and its running maximum
Relative drawdown of discounted S&P500
Logarithm of maximum drawdown constrained portfolio, $\alpha = 0.6$
Relative drawdown for drawdown constrained portfolio, $\alpha = 0.6$
Fraction of wealth in S&P500 for $\alpha = 0.6$
Relative hedge error, $\alpha = 0.6$, monthly hedging
• Long term growth rate

For $\alpha \in [0, 1)$, $X \in \mathcal{X}$

\[
\lim_{t \to \infty} \left( \frac{\log(X_t)}{G_t} \right) \leq 1 - \alpha = \lim_{t \to \infty} \left( \frac{\log(\alpha S^\delta_t)}{G_t} \right)
\]

\[
\log(S^\delta_t) = G_t + \int_0^t |\theta_s| dW_s
\]

Kardaras, Obloj & Pl. (2012)
• asymptotic maximum long term growth rate

\[
\lim_{t \to \infty} \frac{1}{t} \log(\alpha S_t^\delta) = (1 - \alpha) \lim_{t \to \infty} \frac{1}{t} \log(S_t^\delta)
\]

\[\alpha g = (1 - \alpha)g\]

restricted drawdown \(\implies\) reduced maximum growth rate

long term view with short term attitude towards risk

realistic alternative to Markowitz mean-variance approach and utility maximization
Logarithm of drawdown constrained portfolios
Long term growth of drawdown constrained portfolios
Expected Utility Maximization

- utility function \( U(\cdot), U'(\cdot) > 0 \quad U''(\cdot) < 0 \)

- examples

  power utility

  \[
  U(x) = \frac{1}{\gamma} x^\gamma
  \]

  for \( \gamma \neq 0 \) and \( \gamma < 1 \)

  log-utility

  \[
  U(x) = \ln(x)
  \]
Examples for power utility (upper graph) and log-utility (lower graph)
• assume $\hat{S}^0$ is scalar Markov process

• maximize expected utility

$$v^\delta = \max_{\bar{S}^\delta \in \mathcal{V}^+_S} E \left( U (\bar{S}_T^\delta) \mid \mathcal{A}_0 \right)$$

$\mathcal{V}^+_S$ set of strictly positive, discounted, fair portfolios

$$\bar{S}_0^\delta = S_0 > 0$$
Theorem

Benchmarked, expected utility maximizing portfolio:

\[
\hat{S}_t^\delta = \hat{u}(t, \hat{S}_t^0) = E \left( U'^{-1} \left( \lambda \hat{S}_T^0 \right) \hat{S}_T^0 | \mathcal{A}_t \right),
\]

\[
\lambda \text{ s.t. } S_0 = \hat{S}_0^\delta S_0^{\delta^*} = \hat{u}(0, \hat{S}_0) S_0^{\delta^*}
\]

two fund separation with risk aversion coefficient

\[
J_t^\delta = \frac{1}{1 - \frac{\hat{S}_t^0}{\hat{u}(t, \hat{S}_t^0)} \frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}_t^0}}.
\]

\[\implies\] locally optimal portfolio
Proof:

\[ v^\delta = E \left( U(\bar{S}_T^\delta) \right) - \lambda \left( E(\hat{S}_T^\delta) - \hat{S}_0^\delta \right) \]

\[ = E \left( F(\bar{S}_T^\delta) \right) \]

\[ F(\bar{S}_T^\delta) = U(\bar{S}_T^\delta) - \lambda \left( \frac{\bar{S}_T^\delta}{\bar{S}_T^\delta*} - \hat{S}_0^\delta \right) \]

\[ F'(\bar{S}_T^\delta) = U'(\bar{S}_T^\delta) - \frac{\lambda}{\bar{S}_T^\delta*} = 0 \]
\[
\begin{align*}
U'(\bar{S}^\delta_T) &= \frac{\lambda}{\bar{S}_T^{\delta*}} \\
\bar{S}^\delta_T &= U'^{-1} \left( \frac{\lambda}{\bar{S}_T^{\delta*}} \right) = U'^{-1}(\lambda \hat{S}^0_T) \\
\hat{S}^\delta_T &= U'^{-1}(\lambda \hat{S}^0_T) \hat{S}_T^0
\end{align*}
\]
• log-utility function $U(x) = \ln(x)$

$$U'(x) = \frac{1}{x}$$

$$U'^{-1}(y) = \frac{1}{y}$$

$$U''(x) = -\frac{1}{x^2}$$

utility concave

$$\hat{u}(t, \hat{S}_t^0) = E \left( U'^{-1} \left( \lambda \hat{S}_T^0 \right) \hat{S}_T^0 \left| \mathcal{A}_t \right. \right) = E \left( \frac{1}{\lambda \hat{S}_T^0} \hat{S}_T^0 \left| \mathcal{A}_t \right. \right) = \frac{1}{\lambda}$$

Lagrange multiplier
\[ \lambda = \frac{S_0^{\delta^*}}{S_0} \]

risk aversion coefficient

\[ J_t^\delta = 1 \]

- expected log-utility

\[ v^\delta = E \left( \ln \left( \bar{S}_{T^*} \right) \bigg| \mathcal{A}_0 \right) \]

\[ = \ln(\lambda) + \ln(S_0) + \frac{1}{2} \int_0^T E \left( \left( \theta(s, \bar{S}_{s}^{\delta^*}) \right)^2 \bigg| \mathcal{A}_0 \right) ds \]

if local martingale part forms a martingale
• power utility

\[ U(x) = \frac{1}{\gamma} x^{\gamma} \]

for \( \gamma < 1, \gamma \neq 0 \)

\[ U'(x) = x^{\gamma - 1} \]

\[ U'^{-1}(y) = y^{\frac{1}{\gamma - 1}} \]

\[ U''(x) = (\gamma - 1) x^{\gamma - 2} \]

concave

\[ \hat{u}(t, \hat{S}_t^0) = E \left( \left( \frac{\lambda}{\bar{S}_T^{\delta_*}} \right)^{\frac{1}{\gamma - 1}} \frac{1}{\bar{S}_T^{\delta_*}} \bigg| \mathcal{A}_t \right) = \lambda^{\frac{1}{\gamma - 1}} E \left( \left( \bar{S}_T^{\delta_*} \right)^{\frac{\gamma}{1 - \gamma}} \bigg| \mathcal{A}_t \right) \]
\( \bar{S}^{\delta_*} \) geometric Brownian motion

\[
\hat{u}(t, \hat{S}_t^0) = \lambda^{\frac{1}{\gamma - 1}} (\bar{S}_t^{\delta_*})^{\frac{\gamma}{1 - \gamma}} \times E \left( \exp \left\{ \frac{\gamma}{1 - \gamma} \left( \frac{\theta^2}{2} (T - t) + \theta (W_T - W_t) \right) \right\} \bigg| \mathcal{A}_t \right)
\]

\[
= \lambda^{\frac{1}{\gamma - 1}} (\bar{S}_t^{\delta_*})^{\frac{\gamma}{1 - \gamma}} \exp \left\{ \frac{\theta^2}{2} \frac{\gamma}{(1 - \gamma)^2} (T - t) \right\}
\]
Lagrange multiplier

\[ \lambda = S_0^{\gamma^{-1}} \tilde{S}_{\delta^*} \exp \left\{ \frac{\theta^2}{2} \frac{\gamma}{1 - \gamma} T \right\} \]

\[ \hat{S}_t^0 = (\tilde{S}_{\delta^*})^{-1} \]

\[ \frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}_t^0} = \frac{\hat{u}(t, \hat{S}_t^0)}{\hat{S}_t^0} \frac{\gamma}{\gamma - 1} \]

risk aversion coefficient

\[ J_t^\delta = 1 - \gamma \]

expected utility

\[ \nu^\delta = E \left( \frac{1}{\gamma} \left( \tilde{S}_{\delta^*}^T \right)^\gamma \bigg| \mathcal{A}_0 \right) = \exp \left\{ -\frac{\theta^2}{2} \frac{\gamma}{1 - \gamma} T \right\} (S_0)^\gamma \]
Various Approaches to Asset Pricing

- Actuarial Pricing Approach (APA)
- Capital Asset Pricing Model (CAPM)
- Arbitrage Pricing Theory (APT)
- Utility Indifference Pricing
- Benchmark Approach (BA)
Classical

Law of One Price

“All replicating portfolios of a payoff have the same price!”

Debreu (1959), Sharpe (1964), Lintner (1965),
Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),
Cochrane (2001), . . .

will be, in general, violated under the benchmark approach.
Logarithms of savings bond, fair zero coupon bond and savings account
benchmarked value

\[ \hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}} \]

Supermartingale Property

For nonnegative \( S^\delta \)

\[ \hat{S}_t^\delta \geq E_t \left( \hat{S}_s^\delta \right) \]

\( 0 \leq t \leq s < \infty \)

\( \hat{S}^\delta \) supermartingale
• One needs consistent pricing concept

• All benchmarked nonnegative securities are supermartingales
  (no upward trend)
**Definition**  
Price is fair if, when benchmarked, forms martingale (no trend)

\[ \hat{S}_t^\delta = E_t \left( \hat{S}_s^\delta \right) \]

\[ 0 \leq t \leq s < \infty. \]
Lemma  The minimal nonnegative supermartingale that reaches a given benchmarked contingent claim is a martingale.

see Revuz & Yor (1999)
Benchmarked savings bond and benchmarked fair zero coupon bond
Benchmark Approach:

Law of the Minimal Price

Pl. (2008)

**Theorem**  
*If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal possible hedge
- economically correct price in a competitive market
• contingent claim

\[ H_T \]

\[ E_0 \left( \frac{H_T}{S_T^{\delta_{\ast}}} \right) < \infty \]

• \( S_t^{\delta_H} \) fair if

\[ \hat{S}_t^{\delta_H} = \frac{S_t^{\delta_H}}{S_t^{\delta_{\ast}}} = E_t \left( \frac{H_T}{S_T^{\delta_{\ast}}} \right) \]

• real world expectation
  best performing portfolio as numéraire

\[ \implies \text{Direct link with real world and economy!} \]
Law of the Minimal Price \[ \implies \]

Corollary

*Minimal price for replicable* \( H_T \) *is fair* and given by

**real world pricing formula**

\[
S_t^{\delta_H} = S_t^{\delta_*} \cdot E_t \left( \frac{H_T}{S_t^{\delta_*}} \right).
\]

- Du & Pl. (2013) benchmarked risk minimization for nonreplicable claims
most pricing concepts become unified and generalized by real world pricing

⇒

actuarial pricing
risk neutral pricing
pricing with stochastic discount factor
pricing with numeraire change
pricing with deflator
pricing with state pricing density
pricing with pricing kernel
utility indifference pricing
pricing with numeraire portfolio
classical risk minimization
benchmarked risk minimization
• Equivalent to risk neutral pricing are pricing via:

  * **stochastic discount factor**, Cochrane (2001);
  * **deflator**, Duffie (2001);
  * **pricing kernel**, Constantinides (1992);
  * **state price density**, Ingersoll (1987);
  * **numeraire portfolio** as in Long (1990)
Actuarial Pricing

When $H_T$ independent of $S_T^{\delta *}$

$\implies$ rigorous derivation of

- actuarial pricing formula

\[
S_t^{\delta H} = P(t, T) \, E_t(H_T)
\]

with zero coupon bond as discount factor

\[
P(t, T) = S_t^{\delta *} \, E_t \left( \frac{1}{S_T^{\delta *}} \right)
\]
Risk Neutral Pricing

- real world pricing formula \[ S_{0}^{\delta H} = E_{0} \left( \Lambda_{T} \frac{S_{0}^{0}}{S_{T}^{0}} H_{T} \right) \]

Radon-Nikodym derivative (density)

\[ \Lambda_{t} = \frac{\hat{S}_{t}^{0}}{\hat{S}_{0}^{0}} \] supermartingale

\[ 1 = \Lambda_{0} \geq E_{0}(\Lambda_{T}) \]

\[ S_{0}^{\delta H} \leq \frac{E_{0} \left( \Lambda_{T} \frac{S_{0}^{0}}{S_{T}^{0}} H_{T} \right)}{E_{0}(\Lambda_{T})} \]
similar for any numeraire (here $S^0$ savings account)
Discounted S&P500 total return index
Radon-Nikodym derivative and total mass of putative risk neutral measure
• special case when savings account is fair:

\[ \Lambda_T = \frac{dQ}{dP} \text{ forms martingale; } E_0(\Lambda_T) = 1; \]

equivalent risk neutral probability measure \( Q \) exists;

Bayes’ formula \[ \implies \text{ risk neutral pricing formula} \]

\[ S_{0H}^\delta = E_0^Q \left( \frac{S_0^0}{S_T^0} H_T \right) \]

Harrison & Kreps (1979), Ingersoll (1987),

• otherwise “risk neutral price” \( \geq \) real world price
\(\ln(\text{S&P500 accumulation index})\) and \(\ln(\text{savings account})\)
Risk Neutral Pricing under BS Model

\[ \theta_t = \theta, \ a_t = a, \ r_t = r, \ \sigma_t = \sigma \]

Drifted Wiener Process

- drifted Wiener process \( W_\theta \)

\[ W_\theta(t) = W_t + \theta t \]

- market price of risk

\[ \theta = \frac{a - r}{\sigma} \]
underlying security

\[ dS_t = (a - \sigma \theta) S_t \, dt + \sigma S_t (\theta \, dt + dW_t) \]

\[ = (a - \sigma \theta) S_t \, dt + \sigma S_t \, dW_\theta(t) \]

\[ = r \, S_t \, dt + \sigma \, S_t \, dW_\theta(t) \]

\[ S_t = S_0 \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma \, W_\theta(t) \right\} \]

\( W_\theta \) is not a Wiener process under the real world probability \( P \), however, will be under some \( P_\theta \)
Simulated primary security accounts.
• Radon-Nikodym derivative

\[ \Lambda_\theta(T) = \frac{dP_\theta}{dP} = \frac{\hat{S}_T^0}{\hat{S}_0^0} \]

• risk neutral measure  \( P_\theta \)

\[
P_\theta(A) = \int_\Omega 1_A \, dP_\theta(\omega) = \int_A \frac{dP_\theta(\omega)}{dP(\omega)} \, dP(\omega)
\]

\[
= \int_A \Lambda_\theta(T) \, dP(\omega)
\]

for all subsets  \( A \in \Omega \)
Definition\quad \tilde{P} \text{ is equivalent to } \hat{P} \text{ if both have the same sets of events that have probability zero.}

- equivalence of $P$ and $P_\theta$
  fundamental for risk neutral pricing

Key Risk Neutral assumption:

$\Lambda_\theta$ is $(A, P)$ martingale
• under BS model

Radon-Nikodym derivative

\[ \Lambda_\theta(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0} \]

is \((\mathcal{A}, P)\)-martingale,

\[ \Rightarrow \] there exists equivalent risk neutral probability measure \(P_\theta\)

for BS model
Benchmarked savings account under BS model, martingale.
Risk Neutral Measure Transformation

\[ \frac{dP_\theta}{dP} \bigg|_{\mathcal{A}_t} = \Lambda_\theta(t) = \exp \left\{ -\frac{1}{2} \theta^2 t - \theta W_t \right\} \]

- total risk neutral measure at \( t = 0 \):

\[ \frac{dP_\theta}{dP} \bigg|_{\mathcal{A}_0} = E(\Lambda_\theta(T) | \mathcal{A}_0) = \Lambda_\theta(0) = 1 \]

\[ \implies P_\theta \text{ is a probability measure} \]
Let us show that $W_\theta$ is Wiener process under $P_\theta$:

For fixed $\tilde{y} \in \mathbb{R}$,

$t \in [0, T]$ and $s \in [0, t]$

let $A$ be the event

$$A = \{ \omega \in \Omega : W_\theta(t, \omega) - W_\theta(s, \omega) < \tilde{y} \}$$

$$\implies$$

$$A = \{ \omega \in \Omega : W_t(\omega) - W_s(\omega) < \tilde{y} - \theta(t - s) \}$$
\( E_\theta \) denoting expectation under \( P_\theta \)  

\[
P_\theta(A) = E_\theta(1_A|\mathcal{A}_0) = E\left(\frac{dP_\theta}{dP}_{\mathcal{A}_T} 1_A | \mathcal{A}_0\right)
\]

\[
= E(\Lambda_\theta(T) 1_A|\mathcal{A}_0) = E\left(\Lambda_\theta(t) 1_A \frac{\Lambda_\theta(T)}{\Lambda_\theta(t)} | \mathcal{A}_0\right)
\]

\[
= E \left( \Lambda_\theta(t) 1_A E \left( \frac{\Lambda_\theta(T)}{\Lambda_\theta(t)} | \mathcal{A}_{t} \right) | \mathcal{A}_0 \right)
\]

\[
= E \left( \Lambda_\theta(t) 1_A \left( \Lambda_\theta(s) \frac{\Lambda_\theta(t)}{\Lambda_\theta(s)} 1_A \right) | \mathcal{A}_0 \right)
\]

\[
= E \left( \Lambda_\theta(s) | \mathcal{A}_0 \right) E \left( \frac{\Lambda_\theta(t)}{\Lambda_\theta(s)} 1_A | \mathcal{A}_0 \right)
\]

\[
= E \left( \frac{\Lambda_\theta(t)}{\Lambda_\theta(s)} 1_A | \mathcal{A}_0 \right)
\]
since \( y = W_t - W_s \) is Gaussian \( \sim \mathcal{N}(0, t - s) \) under \( P \) \( \Rightarrow \)

\[
P_\theta(A) = E \left( \frac{\Lambda_\theta(t)}{\Lambda_\theta(s)} 1_A \right)
\]

\[
= \int_{-\infty}^{\tilde{y} - \theta(t-s)} \exp \left\{ -\frac{\theta^2}{2} (t - s) - \theta y \right\} \frac{1}{\sqrt{2\pi (t - s)}} \exp \left\{ -\frac{y^2}{2 (t - s)} \right\} dy
\]

\[
= \int_{-\infty}^{\tilde{y} - \theta(t-s)} \frac{1}{\sqrt{2\pi (t - s)}} \exp \left\{ -\frac{(y + \theta (t - s))^2}{2 (t - s)} \right\} dy
\]

\[
= \int_{-\infty}^{\tilde{y}} \frac{1}{\sqrt{2\pi (t - s)}} \exp \left\{ -\frac{z^2}{2 (t - s)} \right\} dz
\]

for \( z = y + \theta(t - s) \)
Theorem (Girsanov) Under the above BS model the process

\[ W_\theta = \{ W_\theta(t) = W_t + \theta t, \ t \geq 0 \} \]

is a standard Wiener process in the filtered probability space \((\Omega, \mathcal{A}_T, \mathcal{A}, P_\theta)\).

- \( W_\theta \) is \((\mathcal{A}, P_\theta)\)-Wiener process

- simply a transformation of variable

  (not always possible, exploits here BS model)
Real World Pricing \implies Risk Neutral Pricing

\[ V(0, S_0) = S_0^{\delta^*} \mathbb{E} \left( \frac{H(S_T)}{S_T^{\delta^*}} \middle| \mathcal{A}_0 \right) \]

\[ = \mathbb{E} \left( \frac{\hat{\Delta}_T}{\hat{S}_T^0} \frac{H(S_T)}{S_T^0} \middle| \mathcal{A}_0 \right) \]

\[ = \mathbb{E} \left( \Lambda_\theta(T) \left( \frac{H(S_T)}{S_T^0} \right) \middle| \mathcal{A}_0 \right) \]

\[ \leq \frac{\mathbb{E} \left( \Lambda_\theta(T) \frac{H(S_T)}{S_T^0} \middle| \mathcal{A}_0 \right)}{\mathbb{E} \left( \Lambda_\theta(T) \middle| \mathcal{A}_0 \right)} \]
Risk Neutral Pricing if $\Lambda_\theta$ is $(\mathcal{A}, P)$-martingale

$$V(0, S_0) = E \left( \exp \left\{ -\frac{\theta^2}{2} T - \theta W_T \right\} \exp\{ -r T \} H(S_T) \mid \mathcal{A}_0 \right)$$

$$= \exp\{ -r T \} \int_{-\infty}^{\infty} H \left( S_0 \exp \left\{ (a - \frac{1}{2} \sigma^2) T + \sigma y \right\} \right)$$

$$\times \exp \left\{ -\frac{\theta^2}{2} T - \theta y \right\} \frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right) dy$$

$$= \exp\{ -r T \} \int_{-\infty}^{\infty} \left( H \left( S_0 \exp \left\{ (r - \frac{1}{2} \sigma^2) T + \sigma z \right\} \right) \right)$$

$$\times \frac{1}{\sqrt{T}} N' \left( \frac{z}{\sqrt{T}} \right) dz$$

$$= \exp\{ -r T \} E_\theta \left( H(S_T) \mid \mathcal{A}_0 \right) = E_\theta \left( \frac{H(S_T)}{S_0^T} \mid \mathcal{A}_0 \right)$$

$z = y + \theta T \quad \implies \quad E_\theta$ - expectation with respect to $P_\theta$
Risk Neutral SDEs under BS model

- discounted underlying security price

\[ d\bar{S}_t = \sigma \bar{S}_t \, dW_{\theta}(t) \]

Driftless under \( P_\theta, \sigma \bar{S} \in \mathcal{L}_T^2 \implies (\mathcal{A}, P_\theta)\text{-martingale} \]
• discounted option price

$$d\bar{V}(t, \bar{S}_t) = \frac{\partial \bar{V}(t, \bar{S}_t)}{\partial \bar{S}} \sigma \bar{S}_t dW_\theta(t)$$

driftless under $P_\theta$, $\frac{\partial \bar{V}}{\partial \bar{S}} \sigma \bar{S} \in \mathcal{L}_T^2 \implies$

$(\mathcal{A}, P_\theta)$-martingale

• discounted savings account $\bar{S}_0^t = 1 \implies (\mathcal{A}, P_\theta)$-martingale


- discounted portfolio

\[ d\bar{S}_t^\delta = d(\bar{S}_t^{\delta*} \hat{S}_t^\delta) \]

\[ = \bar{S}_t^{\delta} \pi_1^\delta(t) \sigma (\theta \, dt + dW_t) \]

\[ = \bar{S}_t^{\delta} \pi_1^\delta(t) \sigma \, dW_{\theta}(t) \]

\((\mathcal{A}, P_{\theta})\)-local martingale

- if \( \bar{S}^{\delta} \pi_1^\delta \delta = \delta^1 \bar{S} \sigma \in \mathcal{L}_T^2 \) \( \implies \bar{S}^\delta \) is an \((\mathcal{A}, P_{\theta})\)-martingale,

\( \implies \) risk neutral pricing formula holds

\[ \bar{S}_t^\delta = E_{\theta} (\bar{S}_T^\delta \mid \mathcal{A}_t) = E_{\theta} \left( \frac{S_T^\delta}{S_T^0} \mid \mathcal{A}_t \right) \]

- \( \bar{S}^{\delta*} \) is \((\mathcal{A}, P_{\theta})\)-martingale under BS model
• strict supermartingale portfolio under BS model

if $Z_t$ strict $(\mathcal{A}, P_\theta)$-supermartingale, independent of $\bar{S}_t^{\delta^*}$ and

$$\bar{S}_t^\delta = \bar{S}_t^{\delta^*} Z_t$$

$$\implies \bar{S}_t^\delta \text{ strict } (\mathcal{A}, P_\theta)\text{-supermartingale}$$

• risk neutral pricing formula does not hold for such $S_t^\delta$

$$\bar{S}_t^\delta > E_\theta (\bar{S}_T^\delta \mid \mathcal{A}_t) \text{ for } T > t$$

$$\implies \text{not all discounted portfolios are } (\mathcal{A}, P_\theta)\text{-martingales}$$

even when Radon-Nikodym derivative $\Lambda_\theta$ is $(\mathcal{A}, P)$-martingale
Change of Variables

- we performed a change of variables from $W_t$ to $W_\theta(t)$

  $W$  Wiener processes  under  $P$

  $W_\theta$  Wiener processes  under  $P_\theta$

- relies on existence of an

  equivalent risk neutral probability measure $P_\theta$
General Girsanov Transformation

- $m$-dimensional standard Wiener process
  \[ W = \{ W_t = (W^1_t, \ldots, W^m_t)\top, \ t \in [0, T]\} \]

  \((\Omega, \mathcal{A}_T, \mathcal{A}, P)\)

- $\mathcal{A}$-adapted, predictable $m$-dimensional stochastic process
  \[ \theta = \{ \theta_t = (\theta^1_t, \ldots, \theta^m_t)\top, \ t \in [0, T]\} \]

  \[ \int_0^T \sum_{i=1}^m (\theta^i_t)^2 \ dt < \infty \]
• assume Radon-Nikodym derivative

\[ \Lambda_\theta(t) = 1 - \sum_{i=1}^{m} \int_0^t \Lambda_\theta(s) \theta_s^i dW_s^i \]

\[ = \exp \left\{ - \int_0^t \theta_s^\top dW_s - \frac{1}{2} \int_0^t \theta_s^\top \theta_s ds \right\} < \infty \]

forms \((\mathcal{A}, P)\)-martingale
\( \Lambda_{\theta} (\mathcal{A}, P) \)-martingale

\[ E(\Lambda_{\theta}(t) \mid \mathcal{A}_s) = \Lambda_{\theta}(s) \]

for \( t \in [0, T] \) and \( s \in [0, t] \)

\[ E(\Lambda_{\theta}(t) \mid \mathcal{A}_0) = \Lambda_{\theta}(0) = 1 \]
• define a candidate risk neutral measure $P_{\theta}$ with

$$\frac{dP_{\theta}}{dP} \bigg|_{\mathcal{A}_T} = \Lambda_{\theta}(T)$$

by setting

$$P_{\theta}(A) = E(\Lambda_{\theta}(T) 1_A)$$

$$= E_\theta(1_A)$$

• since $\Lambda_{\theta}$ is an $(\mathcal{A}, P)$-martingale

$$P_{\theta}(\Omega) = E(\Lambda_{\theta}(T)) = E(\Lambda_{\theta}(T) \mid \mathcal{A}_0) = \Lambda_{\theta}(0) = 1$$

$\implies P_{\theta}$ is a probability measure
Girsanov Theorem

**Theorem** (Girsanov)  
If a given strictly positive Radon-Nikodym derivative process \( \Lambda_\theta \) is an \((\mathcal{A}, P)\)-martingale, then the \( m \)-dimensional process \( W_\theta = \{W_\theta(t), \ t \in [0, T]\} \), given by

\[
W_\theta(t) = W_t + \int_0^t \theta_s \, ds
\]

for all \( t \in [0, T] \), is an \( m \)-dimensional standard Wiener process on the filtered probability space \((\Omega, \mathcal{A}_T, \mathcal{A}, P_\theta)\).

- \( W \) is \((\mathcal{A}, P)\)-Wiener process
- \( W_\theta \) is \((\mathcal{A}, P_\theta)\)-Wiener process
Novikov Condition

When is the strictly positive \((A, P)\)-local martingale \(\Lambda_\theta\) an \((A, P)\)-martingale?

- a sufficient condition is the Novikov condition

Novikov (1972)

\[
E \left( \exp \left\{ \frac{1}{2} \int_0^T |\theta_s^\top \theta_s| \, ds \right\} \right) < \infty
\]

(is satisfied for the BS model)

- for other sufficient conditions see Revuz & Yor (1999)
Bayes’s Theorem

Theorem (Bayes) Assume that $\Lambda_\theta$ is an $(\mathcal{A}, P)$-martingale, then for any given stopping time $\tau \in [0, T]$ and any $\mathcal{A}_\tau$-measurable random variable $Y$, satisfying the integrability condition

$$E_\theta(|Y|) < \infty,$$

one has the Bayes rule

$$E_\theta(Y \mid \mathcal{A}_s) = \frac{E(\Lambda_\theta(\tau) Y \mid \mathcal{A}_s)}{E(\Lambda_\theta(\tau) \mid \mathcal{A}_s)}$$

for $s \in [0, \tau]$. 

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Proof of Bayes’s Theorem (*)

For each $\mathcal{A}_\tau$-measurable random variable $Y$ and a set $A \in \mathcal{A}_s$ with $s \in [0, \tau]$, $\tau \leq T$

$$1_A E_\theta (Y|\mathcal{A}_s) = E_\theta (1_A Y|\mathcal{A}_s) = E (1_A Y \Lambda_\theta (T)|\mathcal{A}_s)$$

$$= E (1_A Y \Lambda_\theta (\tau)|\mathcal{A}_s) = E (1_A E (Y \Lambda_\theta (\tau)|\mathcal{A}_s)|\mathcal{A}_s)$$

$$= E \left( \Lambda_\theta (s) \left\{ \frac{1_A}{\Lambda_\theta (s)} E (Y \Lambda_\theta (\tau)|\mathcal{A}_s) \right\} \right| \mathcal{A}_s)$$

$$= E_\theta \left( \frac{1_A}{\Lambda_\theta (s)} E (Y \Lambda_\theta (\tau)|\mathcal{A}_s) \right| \mathcal{A}_s)$$

$$= E_\theta \left( 1_A \frac{E (\Lambda_\theta (\tau) Y|\mathcal{A}_s)}{E (\Lambda_\theta (\tau)|\mathcal{A}_s)} \right| \mathcal{A}_s)$$

$$= 1_A \frac{E (\Lambda_\theta (\tau) Y|\mathcal{A}_s)}{E (\Lambda_\theta (\tau)|\mathcal{A}_s)}.$$

$\square$
Change of Numeraire Technique

Geman, El Karoui & Rochet (1995)

Jamshidian (1997)

- **numeraire**

  \[ S^{\bar{\delta}} = \{ S^{\bar{\delta}}_t, t \in [0, T] \} \]

  normalizes all other portfolios

- may lead to simplifications in calculations

- relative price of a portfolio

\[ \frac{S^{\bar{\delta}}_t}{S^{\bar{\delta}}_0} \]
Self-financing under Numeraire Change

\[ S_t^\delta = \delta_t^0 S_t^0 + \delta_t^1 S_t \]

\[ dS_t^\delta = \delta_t^0 dS_t^0 + \delta_t^1 dS_t \]

\[ dS_t^{\bar{\delta}} = \bar{\delta}_t^0 dS_t^0 + \bar{\delta}_t^1 dS_t \]

Itô formula \( \implies \)
\[ d \left( \frac{S_t^\delta}{S_t^{\bar{\delta}}} \right) = \frac{1}{S_t^{\bar{\delta}}} dS_t^\delta + S_t^\delta d \left( \frac{1}{S_t^{\bar{\delta}}} \right) + d \left[ \frac{1}{S_t^{\bar{\delta}}}, S_t^\delta \right]_t \]

\[ = \delta_t^0 \left( \frac{1}{S_t^{\bar{\delta}}} dS_t^0 + S_t^0 d \left( \frac{1}{S_t^{\bar{\delta}}} \right) \right) \]

\[ + \delta_t^1 \left( \frac{1}{S_t^{\bar{\delta}}} dS_t + S_t d \left( \frac{1}{S_t^{\bar{\delta}}} \right) + d \left[ \frac{1}{S_t^{\bar{\delta}}}, S \right]_t \right) \]

\[ [X, Y]_t \approx \sum_{i=0}^{i_t} (X_{t_{i+1}} - X_{t_i}) (Y_{t_{i+1}} - Y_{t_i}) \text{ covariation} \]
Itô formula for

\[
\frac{S_t^0}{S_t^\delta} \quad \text{and} \quad \frac{S_t}{S_t^\delta}
\]

\[
\implies d \left( \frac{S_t^\delta}{S_t^\delta} \right) = \delta_t^0 d \left( \frac{S_t^0}{S_t^\delta} \right) + \delta_t^1 d \left( \frac{S_t}{S_t^\delta} \right)
\]

\[
\implies \quad \text{in denomination of another numeraire \textbf{always self-financing}}
\]
Numeraire Pairs

- numeraire pair \((S^{\delta}, P_{\theta\delta})\)

**real world** pricing formula:

\[
S_t = E \left( \frac{S^*_{\delta*}}{S^*_{\delta T}} S_T \mid \mathcal{A}_t \right)
\]

existence of GOP

\[\implies \text{ numeraire pair } (S^{\delta*}, P)\]
if \( \Lambda_\theta \) is \((\mathcal{A}, P)-\)martingale \( \implies \)

- risk neutral pricing formula:

\[
S_t = E_\theta \left( \frac{S_0^t}{S_0^T} S_T \bigg| \mathcal{A}_t \right)
\]

\( \implies \) numeraire pair \((S^0, P_\theta)\)
• for fair price process $S$

from real world pricing formula

$$\frac{S_t}{S_t^{\delta*}} = E \left( \frac{S_T}{S_T^{\delta*}} \middle| \mathcal{A}_t \right)$$

$\implies$ for any strictly positive numeraire $S^{\bar{\delta}}$

$$\frac{S_t}{S_t^{\bar{\delta}}} = E \left( \frac{S_t^{\delta*}}{S_t^{\bar{\delta}}} \frac{S_T^{\bar{\delta}}}{S_T^{\delta*}} \frac{S_T}{S_T^{\bar{\delta}}} \middle| \mathcal{A}_t \right)$$

$$= E \left( \frac{\Lambda_{\theta^{\bar{\delta}}}(T)}{\Lambda_{\theta^{\bar{\delta}}}(t)} \frac{S_T}{S_T^{\bar{\delta}}} \middle| \mathcal{A}_t \right)$$
with benchmarked normalized numeraire

\[ \Lambda_{\theta \bar{\delta}}(t) = \frac{\hat{S}_t^\delta}{\hat{S}_0^\delta} \]

\[ d\Lambda_{\theta \bar{\delta}}(t) = d \left( \frac{\hat{S}_t^\delta}{\hat{S}_0^\delta} \right) \]

\[ = \Lambda_{\theta \bar{\delta}}(t) \left( \pi_1^\delta(t) \sigma_t - \theta_t \right) dW_t \]

\((\mathcal{A}, P)\)-local martingale

\[ \Longrightarrow \text{ Radon-Nikodym derivative for candidate measure } P_{\theta \bar{\delta}} \]

\[ \frac{dP_{\theta \bar{\delta}}}{dP} = \Lambda_{\theta \bar{\delta}}(T) \]
• drifted Wiener process

\[ dW_{\theta\tilde{\delta}}(t) = dW_t + \theta\tilde{\delta}(t) \, dt, \]

where

\[ \theta\tilde{\delta}(t) = \theta_t - \pi_1^{\tilde{\delta}}(t) \sigma_t \]

• If \( \Lambda_{\theta\tilde{\delta}} \) is true \((\mathcal{A}, P)\)-martingale

\[ \implies P_{\theta\tilde{\delta}} \text{ equivalent to } P \text{ and probability measure} \]

allows to apply Girsanov transformation

\[ \implies W_{\theta\tilde{\delta}} \text{ standard Wiener process under } P_{\theta\tilde{\delta}} \]

\[ \implies \text{numeraire pair } (S^{\tilde{\delta}}, P_{\theta\tilde{\delta}}) \]
Some Examples

- GOP \( S^{\delta_*} \) as numeraire \( \implies \theta_{\delta_*}(t) = 0 \)

- savings account \( S^0 \) as numeraire
  \[
  \pi_{\delta}^{1}(t) = 0 \implies \theta_{\delta}(t) = \theta_{t}
  \]

- underlying security \( S \) as numeraire
  \[
  \pi_{\delta}^{1}(t) = 1 \implies \theta_{\delta}(t) = \theta_{t} - \sigma_{t}
  \]
• unfair portfolio \( S_t^\delta = S_t^{\delta*} Z_t \) as numeraire

with \( Z \) an \((\mathcal{A}, P)\)-strict supermartingale, independent of \( S^{\delta*} \)

\[ \Rightarrow \quad \hat{S}^\delta \text{ is not an } (\mathcal{A}, P)\text{-martingale} \]

Girsanov Theorem \textbf{can not} be applied!

\[ \Rightarrow \quad \text{If savings account is not fair,} \]

then the change of numeraire approach is not applicable!
Pricing Formula for General Numeraire $S^\delta$

$$V(0, S_0) = E \left( \frac{S_0^{\delta_*}}{S_T^{\delta_*}} H(S_T) \bigg| A_0 \right)$$

$$= E \left( \Lambda_{\theta \delta}(T) \frac{H(S_T)}{S_T^{\delta}} \bigg| A_0 \right)$$

the quantity

$$\frac{\Lambda_{\theta \delta}(T)}{S_T^{\delta}} = \frac{S_0^{\delta_*}}{S_T^{\delta_*}}$$

remains **numeraire invariant**
• if $\Lambda_{\theta \delta}$ forms an $(\mathcal{A}, P)$-martingale

Bayes’s Theorem $\implies$

$$E \left( \Lambda_{\theta \delta}(T) \frac{H(S_T)}{S_T^\delta} \mid \mathcal{A}_0 \right) = E_{\theta \delta} \left( \frac{H(S_T)}{S_T^\delta} \mid \mathcal{A}_0 \right)$$

$\implies$

**pricing formula with general numeraire**

$$V(0, S_0) = E_{\theta \delta} \left( \frac{H(S_T)}{S_T^\delta} \mid \mathcal{A}_0 \right)$$

• represents a **transformation of variables** in integration

• may provide **computational advantages**
Utility Indifference Pricing

discounted payoff \( \tilde{H} = \frac{H}{S_T^0} \) (not perfectly hedgable)

\[
v_{\tilde{\delta},V} = E \left( U \left( (S_0 - \epsilon V) \frac{S_{\tilde{\delta}}}{S_0} + \epsilon \tilde{H} \right) \bigg| \mathcal{A}_0 \right)
\]

- assume \( \tilde{S}^0 \) is scalar diffusion

Definition  Utility indifference price \( V \) s.t.

\[
\lim_{\epsilon \to 0} \frac{v_{\tilde{\delta},V} - v_{0,V}}{\epsilon} = 0
\]

e.g. Davis (1997)
• Taylor expansion

\[ v_{\epsilon, V} \approx E \left( U \left( \bar{S}\delta_T \right) + \epsilon U' \left( \bar{S}\delta_T \right) \left( \bar{H} - \frac{V \bar{S}\delta_T}{S_0} \right) \right| \mathcal{A}_0 \) \\
= v_{0, V} + \epsilon E \left( U' \left( \bar{S}\delta_T \right) \bar{H} \right| \mathcal{A}_0 \) - V \epsilon E \left( U' \left( \bar{S}\delta_T \right) \frac{\bar{S}\delta_T}{S_0} \right| \mathcal{A}_0 \)
utility indifference price

\[ V = \left( \frac{E \left( U' \left( \tilde{S}_{T}^{\delta} \right) \tilde{H} \right | \mathcal{A}_0) \right) \right) \left( \frac{E \left( U' \left( \tilde{S}_{T}^{\delta} \frac{\tilde{S}_{T}^{\delta}}{S_0} \right) \tilde{H} \right | \mathcal{A}_0) \right) \left( \frac{E \left( U' \left( \tilde{S}_{T}^{\delta} \frac{\tilde{S}_{T}^{\delta}}{S_0} \right) \tilde{H} \right | \mathcal{A}_0) \right) \right) \]

independent of \( U \) and independent of particular model

real world pricing formula

\[ V = S_{0}^{\delta} \ E \left( \frac{H \mathcal{A}_0}{S_{0}^{\delta} \mathcal{A}_0} \right) \]
Pricing via Hedging

- underlying security

\[ S = \{S_t, \ t \in [0, T]\} \]

\[ dS_t = a_t \ S_t \ dt + \sigma_t \ S_t \ dW_t \]

\[ t \in [0, T], \ S_0 > 0 \]
appropriate stochastic appreciation rate $a_t$

and volatility $\sigma_t$

Wiener process $W = \{W_t, t \in [0, T]\}$

• savings account

$$dS_t^0 = r_t S_t^0 \, dt$$

$t \in [0, T], S_0^0 = 1$

appropriate stochastic short rate
• replicate payoff

\[ f(S_T) \geq 0 \]

assuming

\[ E(f(S_T)) < \infty \]

• hedge portfolio

\[ V(t, S_t) = \delta_t^0 S_t^0 + \delta_t^1 S_t \]

\( \delta_t^0 \) units of savings account

\( \delta_t^1 \) units of underlying security
• hedging strategy

\[ \delta = \{ \delta_t = (\delta^0_t, \delta^1_t)^\top, \ t \in [0,T] \} \]

\[ \delta^0 = \{ \delta^0_t, \ t \in [0,T] \} \]

\[ \delta^1 = \{ \delta^1_t, \ t \in [0,T] \} \]

predictable processes
• replicating portfolio

\[ V(T, S_T) = f(S_T) \]

• SDE for hedge portfolio

\[
dV(t, S_t) = \delta^0_t dS^0_t + \delta^1_t dS_t \\
+ S^0_t d\delta^0_t + S_t d\delta^1_t + d[\delta^1, S]_t
\]
• self-financing

portfolio changes are caused by gains from trade

\[ dV(t, S_t) = \delta_0^t \, dS_0^t + \delta_1^t \, dS_t \]

⇒

self-financing condition:

\[ S_0^t \, d\delta_0^t + S_t \, d\delta_1^t + d[\delta_1^t, S]_t = 0 \]
• discounted underlying security

\[ \bar{S}_t = \frac{S_t}{S_t^0} \]

\( t \in [0, T] \)

Itô formula \( \Rightarrow \)

\[ d\bar{S}_t = (a - r) \bar{S}_t \, dt + \sigma_t \bar{S}_t \, dW_t \]

\( t \in [0, T], \bar{S}_0 = S_0 \)

• discounted value function

\( \bar{V} : [0, T] \times [0, \infty) \to [0, \infty) \)

\[ \bar{V}(t, \bar{S}_t) = \frac{V(t, S_t)}{S_t^0} \]
• **profit and loss (P&L)**

\[ C_t = \text{price} - \text{gains from trade} - \text{initial price} \]

• **discounted P&L**

\[
\bar{C}_t = \frac{C_t}{S^0_t} \\
= \bar{V}(t, \bar{S}_t) - I_{\delta^1, \bar{S}(t)} - \bar{V}(0, \bar{S}_0)
\]
• discounted gains from trade

\[ I_{\delta^1, \bar{S}}(t) = \int_{0}^{t} \delta_u^1 d\bar{S}_u \]

\[ = \int_{0}^{t} \delta_u^1 (a - r) \bar{S}_u du \]

\[ + \int_{0}^{t} \delta_u^1 \sigma \bar{S}_u dW_u \]

\[ I_{\delta^0, \bar{S}^0}(t) = \int_{0}^{t} \delta_u^0 d\bar{S}^0_u = 0 \]
• initial payment

\[ V(0, S_0) = \bar{V}(0, \bar{S}_0), \]

\[ \bar{C}_0 = 0 \]
• identify hedging strategy $\delta$

  for which discounted P&L zero

  $\bar{C}_t = 0$

  for all $t \in [0, T]$

  $\implies$ perfect hedge

  $V(t, S_t)$ - replicating portfolio
• increments of discounted P&L

\[ 0 = \bar{C}_t - \bar{C}_s = \bar{V}(t, \bar{S}_t) - \bar{V}(s, \bar{S}_s) - \int_s^t \delta_u^1 d\bar{S}_u \]

\[ = \int_s^t \left[ \frac{\partial \bar{V}(u, \bar{S}_u)}{\partial t} + \frac{1}{2} \sigma_u^2 \bar{S}_u^2 \frac{\partial^2 \bar{V}(u, \bar{S}_u)}{\partial \bar{S}^2} \right. \]

\[ + (a_u - r_u) \bar{S}_u \left( \frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) \]

\[ \left. + \int_s^t \sigma_u \bar{S}_u \left( \frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) dW_u \right\]
• hedge ratio

\[ \delta^1_u = \frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}} \]

• discounted PDE

\[
\frac{\partial \bar{V}(u, \bar{S})}{\partial t} + \frac{1}{2} \sigma^2_u \bar{S}^2 \frac{\partial^2 \bar{V}(u, \bar{S})}{\partial \bar{S}^2} = 0
\]

\[ u \in [0, T), \bar{S} \in (0, \infty) \]

with terminal condition

\[
\bar{V}(T, \bar{S}) = \frac{f(\bar{S} \ S^0_T)}{S^0_T} = \frac{f(S)}{S^0_T}
\]
PDE

\[
\frac{\partial V(u, S)}{\partial t} + r_u S \frac{\partial V(u, S)}{\partial S} + \frac{1}{2} \sigma_u^2 S^2 \frac{\partial^2 V(u, S)}{\partial S^2} - r_u V(u, S) = 0
\]

\(u \in [0, T), \ S \in (0, \infty)\)

with terminal condition

\[V(T, S) = f(S)\]

special boundary condition not specified

PDE may not provide minimal possible price
• units in savings account

\[ \delta^0_t = \frac{V(t, S_t) - \delta^1_t S_t}{S^0_t} = \bar{V}(t, \bar{S}_t) - \delta^1_t \bar{S}_t \]

• option price

\[ V(t, S_t) = \bar{V}(t, \bar{S}_t) S^0_t = \delta^0_t S^0_t + \delta^1_t S_t \]

\[ t \in [0, T] \]
strategy \[ \delta_t = (\delta_t^0, \delta_t^1)^\top \]
such that
\[ 0 = d\bar{C}_t = d\bar{V}(t, \bar{S}_t) - \delta_t^1 d\bar{S}_t \]

\[ d\bar{V}(t, \bar{S}_t) = \delta_t^0 d\bar{S}_t^0 + \delta_t^1 d\bar{S}_t \]

self-financing

P&L vanishes
\[ C_t = \bar{C}_t S_t^0 = 0 \]
for all $t \in [0, T]$

If $V(t, S_t)$ is not fair, then there may exist less expensive hedge portfolio!
Numeraire Invariance

\( \tilde{V} \) self-financing

\[
\begin{align*}
dV(t, S_t) &= d(\tilde{V}(t, \tilde{S}_t) S^0_t) \\
&= S^0_t d\tilde{V}(t, \tilde{S}_t) + \tilde{V}(t, \tilde{S}_t) dS^0_t + d[S^0, \tilde{V}(\cdot, \tilde{S}.)]_t \\
&= S^0_t \delta^1_t d\tilde{S}_t + (\delta^0_t + \delta^1_t \tilde{S}_t) dS^0_t + \delta^1_t d[S^0, \tilde{S}]_t \\
&= \delta^0_t dS^0_t + \delta^1_t (S^0_t d\tilde{S}_t + \tilde{S}_t dS^0_t + d[S^0, \tilde{S}]_t) \\
&= \delta^0_t dS^0_t + \delta^1_t d(S^0_t \tilde{S}_t) \\
&= \delta^0_t dS^0_t + \delta^1_t dS_t
\end{align*}
\]

\[\implies V \text{ also self-financing}\]
The Black-Scholes Formula

- solution $c_{T,K}(t, S)$ of PDE

  with call payoff function

  $$f(S) = (S - K)^+$$

- $a_t = a, \sigma_t = \sigma, r_t = r$

  Black & Scholes (1973)
  Merton (1973a)
Black-Scholes Formula

\[
c_{T,K}(t, S_t) = S_t N(d_1(t)) - K \frac{S_t^0}{S_T^0} N(d_2(t))
\]

with

\[
d_1(t) = \ln \left( \frac{S_t}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) (T - t) \frac{1}{\sigma \sqrt{T-t}}
\]

\[
d_2(t) = d_1(t) - \sigma \sqrt{T-t}
\]

\[t \in [0, T)\]

\[
\hat{c}_{T,K}(t, S_t) = \frac{c_{T,K}(t, S_t)}{S_t^{\delta^*}} \quad (P, \mathcal{A}) - \text{martingale}
\]

minimal possible price
• $N(\cdot)$ standard **Gaussian distribution function**

with density

$$N'(x) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\}$$
Black-Scholes European call option price.
Delta

- sensitivity with respect to the underlying $S_t$

\[ \Delta = \frac{\partial V(t, S_t)}{\partial S} = \delta^1_t \]

\[ \Delta = N(d_1(t)) \]
Delta as a function of $t$ and $S_t$. 

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Gamma

- sensitivity of the hedge ratio with respect to underlying

\[
\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V(t, S_t)}{\partial S^2}
\]

\[
\Gamma = N'(d_1(t)) \frac{1}{S_t \sigma \sqrt{T-t}}
\]
Gamma as a function of $t$ and $S_t$. 
Simulation of Delta Hedging

- simulate scenario along an equidistant time discretization

- price of call option

\[ c_{T,K}(t, S_t) = \delta^1_t S_t + \delta^0_t S^0_t \]

- hedging strategy

\[ \delta = \{ \delta_t = (\delta^0_t, \delta^1_t)^\top, \ t \in [0, T] \} \]
hedge ratio, units in the underlying

\[ \delta^1_t = N(d_1(t)) = N \left( \frac{\ln \left( \frac{S_t}{K} \right) + (r + \frac{1}{2} \sigma^2) (T - t)}{\sigma \sqrt{T - t}} \right) \]
• units in domestic savings account

\[
\delta^0_t = -\frac{K}{S^0_T} N(d_2(t))
\]

\[
= -\frac{K}{S^0_T} N \left( \frac{\ln \left( \frac{S_t}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) (T - t)}{\sigma \sqrt{T - t}} \right)
\]
• discounted P&L

\[ \bar{C}_t = \bar{V}(t, \bar{S}_t) - \int_0^t \delta_s^1 d\bar{S}_s - \bar{V}(0, \bar{S}_0) \]

\[ \bar{V}(t, \bar{S}_t) = \bar{V}(0, \bar{S}_0) + \int_0^t \delta_s^1 d\bar{S}_s \]

\[ \Rightarrow \text{theoretically under continuous hedging} \]

\[ \bar{C}_t = 0 \]

for all \( t \in [0, T] \)
In-the-Money Scenario

- security price ends up in-the-money $S_T > K$
  \[ \implies \text{hedge ratio converges to } \delta^1_T = 1 \]

- self-financing strategy $\delta_t = (\delta^0_t, \delta^1_t)^\top$

- intrinsic value
  \[ f(S_t) = (S_t - K)^+ \]

- discounted P&L $\bar{C}_t$ remains almost perfectly zero
Underlying security and hedge ratio for in-the-money call.
Price, intrinsic value and P&L for in-the-money call.
Out-of-the Money Scenario

Underlying security and hedge ratio for out-of-the-money call.
Price, intrinsic value and P&L for out-of-the-money call.
Hedging a European Call Option on the S&P500

Normalized S&P500 and hedge ratio for $K = 1.2$. 268
Quadratic variation of log-S&P500.
Call price on S&P500, intrinsic value and P&L.
Benchmarked Risk Minimization

Risk Minimization

- Föllmer and Sondermann (1986)
- Föllmer and Schweizer (1989)
- Schweizer (1991, 2001)
- Biagini et al. (1999)
- Biagini and Cretarola (2009)
- Du & Pl. (2012)
Financial Market

- $d$ primary security accounts
- $S_{t}^{i,j}$ - $j$th primary security account in $i$th security denomination
• locally riskless savings account

\[ S_{t}^{i,i} = S_{0}^{i,i} \exp \left\{ \int_{0}^{t} r_{s}^{i} ds \right\} \]

• portfolio

\[ S_{t}^{i,\delta} = \sum_{j=1}^{d} \delta_{t}^{j} S_{t}^{i,j} \]
self-financing portfolio

\[ S_t^{i,\delta} = S_0^{i,\delta} + \sum_{j=1}^{d} \int_0^t \delta_s^j dS_s^{i,j} \]
Numéraire Portfolio

- $S_t^{i,\delta*}$ numéraire portfolio

\[
\hat{S}_t^\delta = \frac{S_t^{i,\delta}}{S_t^{i,\delta*}} \geq E_t(\hat{S}_s^\delta)
\]

$0 \leq t \leq s < \infty$

- supermartingale property
• benchmarked $j$th primary security account

$$\hat{S}_t^j = \frac{S_t^{i,j}}{S_t^{i,\delta*}}$$

local martingale; supermartingale
• matrix valued optional covariation process

\[[\hat{S}] = \{[\hat{S}]_t = ([\hat{S}^i, \hat{S}^j]_t)_{i,j=1}^d, t \in [0, \infty)\}\]

• benchmarked primary security accounts

\[\hat{S} = \{\hat{S}_t = (\hat{S}_t^1, \ldots, \hat{S}_t^d)^\top, t \in [0, \infty)\}\]
**Definition** A dynamic trading strategy \( v \), initiated at time \( t = 0 \), \( v = \{v_t = (\eta_t, \vartheta_1^t, \ldots, \vartheta_d^t)^\top, t \in [0, \infty)\} \), where \( \vartheta = \{\vartheta_t = (\vartheta_1^t, \ldots, \vartheta_d^t)^\top, t \in [0, \infty)\} \) forms benchmarked self-financing part \( \vartheta_t^\top \hat{S}_t \) of benchmarked price process

\[
\hat{V}_t^v = \vartheta_t^\top \hat{S}_t + \eta_t
\]

\( \vartheta \) is predictable

\[
\int_0^t \vartheta_u^\top d[\hat{S}]_u \vartheta_u < \infty
\]
\( \eta = \{ \eta_t, t \in [0, \infty) \} \) adapted with \( \eta_0 = 0 \) monitors the benchmarked non-self-financing part of

\[
\hat{V}_t^\nu = \hat{V}_0^\nu + \int_0^t \vartheta_s^\top \, d\hat{S}_s + \eta_t
\]

\( \hat{V}_t^\nu \) forms a supermartingale.
In general, capital has to be added or removed to match desired price

\[ \hat{V}_t^v = \hat{S}_t^\delta = \sum_{j=1}^{d} \delta_t^j \hat{S}_t^j \]

with

\[ \delta_t^j = \vartheta_t^j + \eta_t \delta_{*,t}^j . \]
benchmarked contingent claim $\hat{H}_T$

dynamic trading strategy $\nu$ delivers $\hat{H}_T$ if

$$\hat{V}_T^\nu = \hat{S}_T^\delta = \hat{H}_T$$

$\implies$ replicable if self-financing
Proposition If for $\hat{H}_T$ a self-financing benchmarked portfolio $\hat{S}^{\delta_{\hat{H}_T}}$ exists, satisfying

$$\hat{S}^{\delta_{\hat{H}_T}}_t = E(\hat{H}_T | \mathcal{F}_t)$$

for all $t \in [0, T]$ $P$-a.s., then this provides least expensive hedge for $\hat{H}_T$. 
Real World Pricing

- least expensive replication by a self-financing benchmarked portfolio
  \[ \Rightarrow \]
  real world pricing formula

\[ \hat{S}_t^\delta \hat{H}_T = E_t(\hat{H}_T) \]

minimal possible price process
Definition  For a dynamic trading strategy $\nu$ which delivers

$$\hat{S}_t^\delta = \sum_{j=1}^{d} \delta_j^t \hat{S}_t^j$$

**benchmarked P&L**

$$\hat{C}_t^\delta = \hat{S}_t^\delta - \sum_{j=1}^{d} \int_0^t \vartheta_j^u d\hat{S}_u^j - \hat{S}_0^\delta$$
Corollary \( \hat{C}_t^\delta = \eta_t \) for \( t \in [0, \infty) \).

- usually fluctuating benchmarked P&L
- intrinsic risk
What criterion would be most natural?

- symmetric view with respect to all primary security accounts, including the domestic savings account
- pooling in large trading book
  \[\implies\] vanishing total hedge error
Proposition $\hat{H}_{T,l}$, $\hat{V}^{v_l}$ with $\hat{C}^{v_l}$ independent square integrable martingales with $E \left( \left( \frac{\hat{C}^{v_l}}{\hat{V}_0^{v_l}} \right)^2 \right) \leq K_t < \infty$ for $l \in \{1, 2, \ldots \}$, $t \in [0, T]$, $T \in [0, \infty)$. 

At initial time well diversified trading book holds equal fractions

- total benchmarked wealth $\hat{U}_t = \frac{\hat{U}_0}{m} \sum_{l=1}^m \frac{\hat{V}_t^{v_l}}{\hat{V}_0^{v_l}}$

- total benchmarked P&L

$$\hat{R}_m(t) = \frac{\hat{U}_0}{m} \sum_{l=1}^m \frac{\hat{C}^{v_l}_t}{\hat{V}_0^{v_l}} ,$$

$$\Rightarrow$$

$$\lim_{m \to \infty} \hat{R}_m(t) = 0$$

$P$-a.s.
**Definition**  A dynamic trading strategy \( \nu \), initiated at time zero, is called locally real-world mean-self-financing if its adapted process \( \eta_t = \hat{C}_t^\delta \) forms an local martingale.
• classical risk minimization
  Föllmer and Sondermann (1986)
  Föllmer and Schweizer (1989)
  Schweizer (1999)
  Föllmer-Schweizer decomposition assuming a risk neutral probability measure
  Delbaen et al. (1997) Pham et al. (1998)
- **local risk minimization**
  - Bouleau and Lamberton (1989)
  - Duffie and Richardson (1991)
  - Schweizer (1994)
  - Biagini et al. (1999)
  - Schweizer (2001)
• Benchmarked P&L is *orthogonal* if

\[ \eta_t \hat{S}_t \]

is vector local martingale.

- above product has no trend
- benchmarked hedge error orthogonal to any benchmarked traded wealth
• $\mathcal{V}_{\hat{H}_T}$ set of locally real-world mean-self-financing dynamic trading strategies $\nu$ initiated at time zero with

$$\hat{V}_t^{\nu} = \hat{S}_t^{\delta}, \quad t \geq 0$$

which deliver $\hat{H}_T$

with orthogonal benchmarked P&L
• market participants prefer more for less

⇒ **Benchmarked Risk Minimization**

For $\hat{H}_T$ strategy $\tilde{v} \in \mathcal{V}_{\hat{H}_T}$, *benchmark risk minimizing* (BRM) if for all $v \in \mathcal{V}_{\hat{H}_T}$ with $\hat{V}_t^v = \hat{S}_t^\delta$

price $\hat{S}_t^\delta$ is minimal

$$\hat{S}_t^\delta \leq \hat{S}_t^\delta$$

$P$-a.s. for all $t \in [0, T]$. 
Regular Benchmarked Contingent Claims

$\hat{H}_T$ is called regular if

$$\hat{H}_T = E_t(\hat{H}_T) + \sum_{j=1}^{d} \int_{t}^{T} \vartheta^j_{\hat{H}_T}(s)d\hat{S}^j_s + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t)$$

$\vartheta_{\hat{H}_T}$ - predictable

$\eta_{\hat{H}_T}$ - local martingale, orthogonal to $\hat{S}$
Theorem A regular $\hat{H}_T$ has a BRM strategy $\mathbf{v} = \{v_t = (\eta_t, \vartheta^1_t, \ldots, \vartheta^d_t)^\top, t \in [0, T]\} \in \mathcal{V}_{\hat{H}_T}$ with

$$\hat{S}^\delta_t = \hat{V}_t^\mathbf{v} = E_t(\hat{H}_T),$$

$$\hat{S}^\delta_T = \hat{H}_T,$$

and orthogonal benchmarked P&L: $\hat{C}^\delta_t = \eta_{\hat{H}_T}(t)$
\( \hat{C}_t^\delta - \text{local martingale} \)

\( \hat{C}_t^\delta \hat{S}_t - \text{vector local martingale} \)

orthogonal
hedgeable part:

\[ E_t(\hat{H}_T) - \hat{C}_{t}^{\delta} = E_0(\hat{H}_T) + \int_{0}^{t} \vartheta_{\hat{H}_T}^{\top}(s)d\hat{S}_s. \]

local martingale
\[ \hat{V}_t^\nu = E_t(\hat{H}_T) \text{ minimal} \]

- do not request square integrability
- no equivalent risk neutral probability measure required
- need only to check zero drift of \( \eta_{\Hat{H}_T} \)
- and \( \eta_{\Hat{H}_T}(t), \hat{S}_t \) orthogonality
Hedging a Regular Claim

- benchmarked contingent claim $\hat{H}_T$ driven by continuous local martingales $W^1, W^2, \ldots, W^d$ orthogonal to each other

- benchmarked primary security account $\hat{S}^j_t, j \in \{1, \ldots, d\}$

$$d\hat{S}^j_t = -\hat{S}^j_t \sum_{k=1}^{d-1} \theta^{j,k}_t dW^k_t$$

- $d \times d$ matrix $\Phi_t = [\Phi_t^{i,k}]_{i,k=1}^d$

$$\Phi_t^{i,k} = \begin{cases} 
\theta^{i,k}_t & \text{for } k \in \{1, \ldots, d-1\} \\
1 & \text{for } k = d
\end{cases}$$

for $t \in [0, T]$. 
**Proposition** assume $\Phi_t$ invertible and $\hat{V}_t = E(\hat{H}_T|\mathcal{F}_t)$ has representation

$$
\hat{V}_t = \hat{V}_0 + \sum_{k=1}^{d} \int_0^t x_s^k dW_s^k + \int_0^t x_s^d dW_s^d .
$$

Then $\hat{H}_T$ is a regular benchmarked contingent claim

$$
\hat{V}^v_{\hat{H}_T} = \hat{V}_t \text{ for all } t \in [0, T]
$$

$$
\vartheta_{\hat{H}_T}(t) = \text{diag} (\hat{S}_t)^{-1} (\Phi_t^\top)^{-1} \xi_t
$$

$$
\eta_{\hat{H}_T}(t) = \int_0^t x_s^d dW_s^d
$$

with $\xi_t = (x_t^1, \ldots, x_t^{d-1}, \hat{V}_0 + \sum_{k=1}^{d-1} \int_0^t x_s^k dW_s^k)^\top$. 
\[
\bar{S}^i,j_t = \frac{\hat{S}^j_t}{\hat{S}^i_t}
\]

\[
d\bar{S}^i,j_t = \bar{S}^i,j_t \sum_{k=1}^{d-1} (\theta^i,k_t - \theta^j,k_t) (\theta^i,k_t dt + dW^k_t)
\]

\[
b^i,j,k_t = (\theta^i,k_t - \theta^j,k_t)
\]

\[
b^d_t = [b^d,j,k_t]_{j,k=1}^{d-1,d-1}
\]
Proposition  The matrix $\Phi_t$ is invertible if and only if $b_t^d$ is invertible.

Proof: elementary transform of $b_t^d$

$$
\Phi_t \longleftrightarrow \begin{pmatrix}
& b_t^d & 0 \\
& \theta_t^{d,1} & \theta_t^{d,d-1} & 1
\end{pmatrix}
$$

$\Phi_t$ has full rank if and only if $b_t^d$ has full rank
Comparison with Quadratic Criterion

- \( \int_0^t \vartheta_{\hat{H}_T}^\top (s) \, d\hat{S}_s \) and \( \eta_{\hat{H}_T}(t) \) independent, square integrable martingales
- \( \hat{S}_t^1, \ldots, \hat{S}_t^d \) and \( \eta_{\hat{H}_T}(t) \) independent, square integrable martingales
- \( \eta_{\hat{H}_T} \) orthogonal to benchmarked primary security accounts
- \( \hat{H}_T \) square integrable
- \( \hat{V}_t^\nu = \hat{S}_t^\delta \) delivers \( \hat{H}_T \)
Quadratic Criterion

\[ E \left( (\hat{C}_T^\delta)^2 \right) = E \left( (\hat{H}_T - \int_0^T \vartheta_s^\top d\hat{S}_s - \hat{S}_0^\delta)^2 \right) \Rightarrow \min \]
\[ E \left( (\hat{C}_T^\delta)^2 \right) \]
\[ = E \left( \left( E(\hat{H}_T) - \hat{S}_0^\delta + \int_0^T (\vartheta_{\hat{H}_T}^\top(s) - \vartheta_s^\top) d\hat{S}_s + \eta_{\hat{H}_T}(T) \right)^2 \right) \]
\[ = E \left( E(\hat{H}_T) - \hat{S}_0^\delta \right)^2 \]
\[ + E \left( \int_0^T (\vartheta_{\hat{H}_T}^\top(s) - \vartheta_s^\top)^2 d[\hat{S}]_s \right) + E((\eta_{\hat{H}_T}(T))^2) \]
\[
\Rightarrow
\]

- \( \hat{S}_0^\delta = E(\hat{H}_T) \)
- \( \vartheta_t = \vartheta_{\hat{H}_T}(t), \ t \in [0, T] \)
- \( \eta_{\hat{H}_T}(T) = \hat{C}_T^\delta \)
- second moment of benchmarked P&L becomes minimal
Nonhedgeable Contingent Claim

- savings account $S^{1,1}_t = 1$
- numéraire portfolio $S^{1,\delta*}_t$
- $H_t = E_t(H_T), \; t \in [0, T]$
  - independent from $S^{1,\delta*}_t$
  - continuous
Benchmarked risk minimization

- real world pricing formula

\[
\hat{S}_t^{\delta H_T} = E_t(\hat{H}_T) = E_t(H_T)E_t(\hat{S}_t^1) = H_t\hat{P}(t, T)
\]

\[
\hat{P}(t, T) = E_t(\hat{S}_t^1)
\]

\[
d\hat{P}(t, T) = \frac{\partial \hat{P}(t, T)}{\partial \hat{S}_t^1} d\hat{S}_t^1
\]

- benchmarked NP \( \hat{S}_t^{\delta*} = \frac{S_t^{1,\delta*}}{S_t^{1,\delta*}} = 1 \)

- benchmarked P&L

\[
d\hat{C}_t^{\delta H_T} = d\hat{S}_t^{\delta H_T} - \vartheta_1^{\delta H_T}(t) d\hat{S}_t^1
\]

\[
= \left( H_t \frac{\partial \hat{P}(t, T)}{\partial \hat{S}_t^1} - \vartheta_1^{\delta H_T}(t) \right) d\hat{S}_t^1 + \hat{P}(t, T) dH_t
\]
• Assume orthogonal benchmarked primary security accounts

• product $\hat{C}^{\delta_{HT}} \hat{S}$ has zero drift if

\[ 0 = \frac{d[\hat{S}, \hat{C}^{\delta_{HT}}]}{dt} = \left( H_t \frac{\partial \hat{P}(t, T)}{\partial \hat{S}^1} - \vartheta_{H_T}^1(t) \right) \frac{d[\hat{S}^1]}{dt}. \]

\[ \implies \]

\[ \vartheta_{H_T}^1(t) = H_t \frac{\partial \hat{P}(t, T)}{\partial \hat{S}^1} \]

Depends on $H_t$!
• Benchmarked P&L

\[ \eta_{\hat{H}_T}(t) = \hat{C}_t^\delta_{\hat{H}_T} = \int_0^t \hat{P}(s, T) dH_s \]

local martingale, orthogonal

• self-financing benchmarked hedgeable part

\[ \hat{S}_{t}^{\delta_{\hat{H}_T}} - \eta_{\hat{H}_T}(t) = \hat{S}_0^{\delta_{\hat{H}_T}} + \int_0^t \vartheta_{\hat{H}_T}^1(s) d\hat{S}_s^1 \]

• \( \vartheta_{\hat{H}_T}^1(t) \) units in \( \hat{S}_t^1 \)

• remaining wealth in NP

• minimizing \( \frac{d}{dt}[\hat{C}^{\delta_{\hat{H}_T}}]_t \)
• Regular benchmarked contingent claim

\[
\hat{H}_T = \frac{H_T}{S_T^{1,\delta^*}}
\]

\[
= \hat{S}_0^{\delta_{\hat{H}_T}} + \int_0^T H_t \frac{\partial \hat{P}(t, T)}{\partial \hat{S}_t^1} d\hat{S}_t^1 + \int_0^T \hat{P}(t, T) dH_t
\]

\[
\vartheta_{\hat{H}_T}(t) = H_t \frac{\partial \hat{P}(t, T)}{\partial \hat{S}_t^1}
\]

\[
\eta_{\hat{H}_T}(t) = \int_0^t \hat{P}(s, T) dH_s
\]
Evolving Information

- consider special case when $\hat{S}^1$ true martingale
- classical risk minimization can be applied
- real world pricing formula yields

$$\bar{S}_t^{\delta_{\mathcal{H}T}} = \frac{\check{S}_t^{\delta_{\mathcal{H}T}}}{\hat{S}_t^1} = E_t \left( \frac{\hat{S}_T^1 \mathcal{H}_T}{\hat{S}_t^1 S_{T_t}^{1,1}} \right) = E_t \left( \frac{\Lambda_T \mathcal{H}_T}{\Lambda_t S_{T_t}^{1,1}} \right) = E^Q_t \left( \frac{\mathcal{H}_T}{S_{T_t}^{1,1}} \right)$$

- equivalent minimal martingale measure $Q$

$$\Lambda_T = \left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} = \frac{\hat{S}_T^1}{\hat{S}_t^1}$$

- same initial price as benchmarked risk minimization

$$\check{S}_0^{\delta_{\mathcal{H}T}} = H_0 = \check{S}_0^{\delta_{\mathcal{H}T}} (\check{S}_0^1)^{-1}$$
• classical risk minimization invests total initial value in the savings account

\[ \vartheta_{CRM}^1(t) = H_0 \]

• BRM strategy different

\[ \hat{P}(t, T) = \hat{S}_t^1, \quad \frac{\partial \hat{P}(t,T)}{\partial \hat{S}_t^1} = 1 \]

\[ \implies \vartheta_{HT}^1(t) = H_t = E_t(H_T) \]

• intuitive and practically appealing

• under classical risk minimization, evolving information ignored

• provides less fluctuations for benchmarked P&L

• easier diversified
Modeling the Dynamics of Diversified Indices

Pl. & Rendek (2012)

- conjecture for normalized aggregate wealth dynamics
time transformed square root process

- Naive Diversification Theorem $\Rightarrow$ equity index = proxy of numéraire portfolio

- empirical stylized facts $\Rightarrow$ falsify models
• ⇒ propose realistic one factor, two component long term index model

• realistic model outside classical theory ⇒ benchmark approach

• exact, almost exact simulation ⇒ verify empirical facts, effects of estimation techniques etc.
The Affine Nature of Diversified Wealth Dynamics

Object: normalized units of wealth

Total wealth: $Y^\Delta_{\tau_i}$

Time steps: $\tau_i = i \Delta$

Wealth unit value: $\sqrt{\Delta}$
**Economic activity:** during $[\tau_i, \tau_{i+1})$ ”projects”

- generate each independent wealth with variance increment $\nu^2 \Delta^{3/2}$;
- consume each $\eta \Delta$ fraction of wealth;
- generate together on average $\beta \sqrt{\Delta}$ new units.
Mean for increment of aggregate wealth: \((\beta - \eta Y_{\tau_i}^\Delta) \Delta\)
Variance for increment aggregate wealth: \( \Delta \triangleq \text{sum of variances} \)

\[ \implies \text{proportional to number of wealth units: } \frac{Y_{\tau_i}^\Delta}{\sqrt{\Delta}} \]

\[ \implies \text{proportional to aggregate wealth} \]

\[ \implies \text{variance of increment of aggregate wealth is } \nu^2 Y_{\tau_i}^\Delta \Delta \]
for $\Delta \to 0$

$$Y_{\tau_{i+1}}^{\Delta} - Y_{\tau_i}^{\Delta} = (\beta - \eta Y_{\tau_i}^{\Delta}) \Delta + \nu \sqrt{Y_{\tau_i}^{\Delta}} \Delta W_{\tau_i}$$

$$E(\Delta W_{\tau_i}) = 0, \quad E((\Delta W_{\tau_i})^2) = \Delta$$

conjectures drift and diffusion terms
Conjecture via Weak convergence
for parameters: $\beta = \eta = \nu = 1$

square root process:

$$dY_{\tau_t} = (1 - Y_{\tau_t}) \, d\tau_t + \sqrt{Y_{\tau_t}} \, dW_{\tau_t}$$
• Quadratic variation:

\[ [Y_{\tau.}]_t = \int_0^t Y_{\tau_s} d\tau_s = \int_0^t Y_{\tau_s} M_s ds \]

• Market activity:

\[ M_t = \frac{d\tau_t}{dt} \]

• integrate normalized index:

can we find \( M = \text{const.} \) s.t.

\[ M \int_0^t Y_{\tau_s} ds \approx [Y_{\tau.}]_t \ ? \]
Quadratic variation and integrated normalized S&P500 monthly data, calendar time, Shiller data

\[ M \approx 0.0178 \]

average long term fit
Market Activity: $M_t = \frac{d\tau_t}{dt}$ from model
Quadratic variation and integrated normalized S&P500 monthly data, $\tau$-time
Eight Stylized Empirical Facts

- falsify potential models, Popper (1959)
- TOTMKWD in 26 currency denominations

about 1000 years of daily data
(i) uncorrelated log-returns

Average autocorrelation function for log-returns
(ii) correlated absolute log-returns

Average autocorrelation function for absolute log-returns
(iii) Student-\( t \) distributed log-returns

Logarithm of empirical density of normalized log-returns with Student-\( t \) density
Results for log-returns of the EWI104s

Pl. & Rendek (2008)

<table>
<thead>
<tr>
<th></th>
<th>SGH</th>
<th>Student-(t)</th>
<th>NIG</th>
<th>Hyperbolic</th>
<th>VG</th>
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<td>0.97</td>
<td>0.96</td>
<td>0.96</td>
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<tr>
<td>(\bar{\alpha})</td>
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<td>0.97</td>
<td>0.72</td>
<td></td>
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<tr>
<td>(\lambda)</td>
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<td></td>
<td>1.49</td>
<td></td>
<td></td>
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<tr>
<td>(\nu)</td>
<td></td>
<td>4.33</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\ln(L^*))</td>
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<td>-285796.39</td>
<td>-286448.94</td>
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</tr>
<tr>
<td>(L_n)</td>
<td>0.0000004</td>
<td>1305.10</td>
<td>2711.38</td>
<td>3406.88</td>
<td></td>
</tr>
</tbody>
</table>

\(L_n = 0.0000004 < \chi^2_{0.001,1} \approx 0.000002\)
(iv) volatility clustering

Estimated volatility
(v) long term exponential growth

Logarithm of index with trend line
(vi) leverage effect

Logarithms of normalized index and its volatility
(vii) extreme volatility at major downturns

Logarithms of normalized index and its volatility
(viii) Integrated Normalized Index and its Quadratic Variation
monthly data, calendar time

\[ M \approx 0.0178 \]
average long term fit
\[ S_t = A_{\tau_t} (Y_{\tau_t})^q, \]
\[ A_{\tau_t} = A \exp\{a\tau_t\} \]

\( q > 0, \ a > 0, \ A > 0 \)

conjecture expects \( q = 1 \)
Normalized index: \((Y_{\tau_t})^q = \frac{S_t}{A_{\tau_t}}\)

\[
dY_{\tau} = \left( \frac{\delta}{4} - \frac{1}{2} \left( \frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} Y_{\tau} \right) d\tau + \sqrt{Y_{\tau}} dW(\tau)
\]

Long term mean: \(\lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau} (Y_s)^q ds = 1\)  P-a.s
Market activity time: \( d\tau_t = M_t \, dt \)

Inverse of market activity:

\[
d\left( \frac{1}{M_t} \right) = \left( \frac{\nu}{4} \gamma - \epsilon \frac{1}{M_t} \right) \, dt + \sqrt{\frac{\gamma}{M_t}} \, dW_t,
\]

where

\[
dW(\tau_t) = \sqrt{\frac{d\tau_t}{dt}} \, dW_t = \sqrt{M_t} \, dW_t
\]

◇ only one \( W_t \)
◇ two component model
Discounted index SDE:

\[ dS_t = S_t (\mu_t dt + \sigma_t dW_t) \]

Expected rate of return:

\[ \mu_t = \left( \frac{a}{M_t} - \frac{q}{2} \left( \frac{\Gamma \left( \frac{\delta}{2} \right) + q}{\Gamma \left( \frac{\delta}{2} \right)} \right)^{\frac{1}{q}} + \left( \frac{\delta}{4} q + \frac{1}{2} q(q - 1) \right) \frac{1}{M_t Y_{\tau_t}} \right) M_t \]

Volatility:

\[ \sigma_t = q \sqrt{\frac{M_t}{Y_{\tau_t}}} \]

Pl. & Rendek (2012c)
Benchmark Approach

\( \hat{B}_t \) – benchmark savings account

\[ d\hat{B}_t = \hat{B}_t \left( (-\mu_t + \sigma_t^2) \, dt - \sigma_t dW_t \right) \]

\( \sigma_t^2 \leq \mu_t \implies \hat{B}_t \) is an \((\mathcal{A}, P)\)-supermartingale

\( \implies \) no strong arbitrage; Pl. (2011)
Assumptions:

A1. $\delta = 2(q + 1)$  
A2. $\frac{q}{2} \left( \frac{\Gamma(2q+1)}{\Gamma(q+1)} \right)^{\frac{1}{q}} \leq \alpha$

$\implies \sigma_t^2 \leq \mu_t$
Fitting the model to TOTMKWD

Step 1: Normalization of Index

\[ A_{\tau_t} \approx A \exp\left\{ \frac{4a\epsilon}{\gamma(\nu-2)} t \right\} \Rightarrow A = 65.21, \quad \frac{4a\epsilon}{\gamma(\nu-2)} \approx 0.048 \]
Normalized TOTMKWD
Step 2: Power $q$:
Affine nature $\implies$ conjecture: $q = 1 = \frac{\delta}{2} - 1 \implies \delta = 4$

- student-t with about four degree of freedom is considered and cannot be falsified,

$\implies$ We set $q = 1$ and $\delta = 4$
Step 3: Observing Market Activity:

\[
\frac{d[\sqrt{Y}]_{\tau_t}}{d\tau_t} = \frac{1}{4} \frac{d\tau_t}{dt} = \frac{M_t}{4}
\]

\(\hat{Q}_{t_i} \approx \frac{[\sqrt{Y}]_{t_{i+1}} - [\sqrt{Y}]_{t_i}}{t_{i+1} - t_i}\)

\(\tilde{Q}_{t_{i+1}} = \alpha \sqrt{t_{i+1} - t_i} \hat{Q}_{t_i} + (1 - \alpha \sqrt{t_{i+1} - t_i}) \tilde{Q}_{t_i}, \alpha = 0.92\)

exponential smooth

robust
Market activity: $M_t \approx 4\tilde{Q}_t$

$M_0 = 0.0175$
Step 4: Parameter $\gamma$:

$\gamma = 265.12$
Step 5: Parameters $\nu$ and $\epsilon$ and
Step 6: Long Term Average Net Growth Rate $\alpha$:

$\nu \approx 4, \epsilon \approx 2.18 \Rightarrow \alpha = 2.55 \Rightarrow$ no strong arbitrage
Fitted model allows visualizing
calculated volatility

\[ \sigma_t \approx \sqrt{\frac{4\tilde{Q}_t}{Y_{\tau t}}} \], average volatility: 11.9%
Model applies also to proxies of numéraire portfolio
S&P500 and VIX

$A = 52.09, \epsilon = 2.15, \gamma = 172.3, a = 1.5$
Simulation Study

Step 1: Market activity:

\[
\frac{1}{M_{t_{i+1}}} = \frac{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}{4\epsilon} \left( \chi^2_{3,i} + \left( \sqrt{\frac{4\epsilon e^{-\epsilon(t_{i+1}-t_i)}}{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}} \frac{1}{M_{t_i}} + Z_i \right)^2 \right)
\]
Step 2: $\tau$-time:

$$\tau_{t_{i+1}} - \tau_{t_i} = \int_{t_i}^{t_{i+1}} M_s ds \approx M_{t_i}(t_{i+1} - t_i)$$

almost exact simulation
Step 3: Normalized index:

\[
Y_{\tau_{t_i+1}} = \frac{1 - e^{-\left(\tau_{t_{i+1}} - \tau_{t_i}\right)}}{4} \left(\chi_{3,i}^2 + \sqrt{\frac{4e^{-\left(\tau_{t_{i+1}} - \tau_{t_i}\right)}}{1 - e^{-\left(\tau_{t_{i+1}} - \tau_{t_i}\right)}} Y_{\tau_{t_i}} + Z_i}\right)^2
\]

almost exact simulation
Model recovers stylized empirical facts:

Model is difficult to falsify: Popper (1934)

1. Uncorrelated returns
2. Correlated absolute returns
3. Student-\(t\) distributed returns
<table>
<thead>
<tr>
<th>Simulation</th>
<th>Student-t</th>
<th>NIG</th>
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</table>
4. Volatility clustering
5. Long term exponential growth
6. Leverage effect and
7. Extreme volatility at major market downturns
Conclusions:

◊ equity index model: 3 initial parameters, 3 structural parameters and 1 Wiener process (nondiversifiable uncertainty)

◊ model recovers 7 stylized empirical facts

◊ allows long dated derivative pricing under benchmark approach

◊ leads outside classical theory

◊ if benchmarked savings account local martingale

\[ \implies \text{only 2 structural parameters} \]
Squared Bessel Processes (*)

Revuz & Yor (1999)

- squared Bessel process $\text{BESQ}_x^\delta$
  
  dimension $\delta \geq 0$
  
  \[
  dX_\varphi = \delta \, d\varphi + 2 \sqrt{|X_\varphi|} \, dW_\varphi
  \]
  
  $\varphi \in [0, \infty)$ with $X_0 = x \geq 0$

- scaling property
  
  \[
  Z_\varphi = \frac{1}{a} X_{a\varphi}
  \]
  
  $\text{BESQ}_{\frac{x}{a}}^\delta, \ a > 0$
- sum of squares

\[ X_\varphi = \sum_{k=1}^{\delta} (w^k + W^k_\varphi)^2 \]

\( W^k_\varphi \) Wiener process

\[ dX_\varphi = \delta \, d\varphi + 2 \sum_{k=1}^{\delta} \left( w^k + W^k_\varphi \right) dW^k_\varphi \]

\[ X_0 = \sum_{k=1}^{\delta} (w^k)^2 = x \]
setting

\[ dW_\varphi = |X_\varphi|^{-\frac{1}{2}} \sum_{k=1}^{\delta} \left( w^k + W^k_\varphi \right) dW^k_\varphi \]

quadratic variation

\[ [W]_\varphi = \int_0^{\varphi} \frac{1}{X_s} \sum_{k=1}^{\delta} \left( w^k + W^k_s \right)^2 ds = \varphi \]

Lévy’s theorem

\[ \implies W \text{ Wiener process in } \varphi \text{ time} \]
Squared Bessel process of dimension $\delta = 4$ in $\varphi$-time
• additivity property

Shiga & Watanabe (1973)

\[ X = \{ X_\varphi, \varphi \in [0, \infty) \} \quad \text{BESQ}_x \]

\[ Y = \{ Y_\varphi, \varphi \in [0, \infty) \} \quad \text{independent BESQ}_y \]

\[ X_\varphi + Y_\varphi \quad \text{forms BESQ}_{x+y}^{\delta+\delta'} \]

\[ x, y, \delta, \delta' \geq 0 \]
\begin{itemize}
  \item $\delta > 2$, strict positivity
    \[
    \text{with } X_0 = x > 0 \quad P \left( \inf_{0 \leq \varphi < \infty} X_\varphi > 0 \right) = 1
    \]
  \item $\delta = 2$, no strict positivity
    \[
    P \left( \inf_{0 \leq \varphi \leq \varphi'} X_\varphi > 0 \right) = 0
    \]
  \item $\delta \in [0, 2)$, no strict positivity
    \[
    X_0 = x > 0 \quad P \left( \inf_{0 \leq \varphi \leq \varphi'} X_\varphi = 0 \right) > 0
    \]
\end{itemize}
• transition density

\[ \delta > 0 \ , \ x > 0 \]

\[ p_\delta (\varrho, x; \varphi, y) = \frac{1}{2(\varphi - \varrho)} \left( \frac{y}{x} \right)^{\frac{\nu}{2}} \exp \left\{ - \frac{x + y}{2(\varphi - \varrho)} \right\} \ I_\nu \left( \frac{\sqrt{xy}}{\varphi - \varrho} \right) \]

\( I_\nu \) modified Bessel function of the first kind

index \( \nu = \frac{\delta}{2} - 1 \)
Transition density of squared Bessel process for $\delta = 4$
- \( Y = \frac{X}{\varphi} \) non-central chi-square distributed

  dimension \( \delta \)

  non-centrality parameter \( \ell = \frac{x}{\varphi} \)

\[
P \left( \frac{X}{\varphi} < u \right) = \sum_{k=0}^{\infty} \frac{\exp \left\{ -\frac{\ell}{2} \right\} \left( \frac{\ell}{2} \right)^k}{k!} \left( 1 - \frac{\Gamma \left( \frac{u}{2}; \frac{\delta+2k}{2} \right)}{\Gamma \left( \frac{\delta+2k}{2} \right)} \right)
\]

\( \Gamma(\cdot; \cdot) \) incomplete gamma function
• Moments for $\alpha > -\frac{\delta}{2}$, $\varphi \in (0, \infty)$ and $\delta > 2$

$$E \left( X_\varphi^\alpha \mid A_0 \right) = \begin{cases} (2 \varphi)^\alpha \exp \left\{ \frac{-X_0}{2 \varphi} \right\} \sum_{k=0}^{\infty} \frac{(X_0/2 \varphi)^k \Gamma(\alpha+k+\frac{\delta}{2})}{k! \Gamma(k+\frac{\delta}{2})} & \text{for } \alpha > -\frac{\delta}{2} \\ \infty & \text{for } \alpha \leq -\frac{\delta}{2} \end{cases}$$

$$\Rightarrow$$

$$E \left( X_\varphi \mid A_0 \right) = X_0 + \delta \varphi$$
by monotonicity of the gamma function

\[
E \left( X_\varphi^\alpha \mid \mathcal{A}_0 \right) \leq (2\varphi)^\alpha \exp\left\{ -\frac{X_0}{2\varphi} \right\} \left( \frac{\Gamma(\alpha + \frac{\delta}{2})}{\Gamma(\frac{\delta}{2})} + \exp\left\{ \frac{X_0}{2\varphi} \right\} \right)
\]

\[
< \infty
\]

for \( \delta = 4 \)

\[
E \left( X_\varphi^{-1} \mid \mathcal{A}_0 \right) = X_0^{-1} \left( 1 - \exp\left\{ -\frac{X_0}{2\varphi} \right\} \right)
\]
A Strict Local Martingale

\[ Z_\varphi = X_\varphi^{-1} \text{ inverse of a squared Bessel process } X_\varphi \text{ of dimension 4} \]

\[ dZ_\varphi = -2Z_\varphi^{3/2}dW_\varphi \]

\[ \Rightarrow \text{ local martingale} \]

\[ E\left(Z_\varphi \mid \mathcal{A}_0\right) = E\left(X_\varphi^{-1} \mid \mathcal{A}_0\right) \]

\[ = Z_0 \left(1 - \exp\left\{ \frac{-1}{2Z_0\varphi} \right\} \right) < Z_0 \]

\[ \Rightarrow \text{ strict local martingale} \]

\[ \Rightarrow \text{ strict supermartingale} \]
Inverse of a squared Bessel process of dimension $\delta = 4$ in $\varphi$-time
Expectation of the inverse of the squared Bessel process for $\delta = 4$ in $\varphi$-time
Time Transformation

- $\varphi$-time

$$\varphi(t) = \varphi(0) + \frac{1}{4} \int_0^t \frac{c_u^2}{s_u} \, du$$

$t \in [0, \infty), \ s_0 > 0$

with

$$s_t = s_0 \exp \left\{ \int_0^t b_u \, du \right\}$$
\begin{itemize}
\item \textbf{φ-time}

for constant $b < 0$, $c \neq 0$

$$
φ(t) = φ(0) + \frac{c^2}{4 b s_0} (1 - \exp\{-b t\})
$$

\item \textbf{time}

$$
t(φ) = -\frac{1}{b} \ln \left(1 - \frac{4 b s_0}{c^2} (φ - φ(0))\right)
$$
\end{itemize}
Time $t(\varphi)$ against $\varphi$-time
Squared Bessel process in dependence on time $t$
Expectation of a squared Bessel process in dependence on time $t$
Expectation of the inverse of a squared Bessel process in dependence on time $t$
Inverse of squared Bessel process in dependence on time $t$
Square Root Process

\[ Y_t = s_t X_{\varphi(t)} \]

\[
dY_t = s_t dX_{\varphi(t)} + X_{\varphi(t)} ds_t
\]

\[
= s_t \delta d\varphi(t) + s_t 2 \sqrt{X_{\varphi(t)}} dW_{\varphi(t)} + X_{\varphi(t)} s_t b_t dt
\]

\[
= \left( \frac{\delta}{4} c_t^2 + b_t Y_t \right) dt + c_t \sqrt{Y_t} \sqrt{\frac{4s_t}{c_t^2}} dW_{\varphi(t)}
\]
\[ dU_t = \sqrt{\frac{4s_t}{c_t^2}} \, dW_{\varphi(t)} \]

has quadratic variation

\[ [U]_t = \int_0^t \frac{4s_z}{c_z^2} \, d\varphi(z) = t \]

\( U \) Wiener process

\[ \implies \text{SR process} \]

\[ dY_t = \left( \frac{\delta}{4} c_t^2 + b_t Y_t \right) \, dt + c_t \sqrt{Y_t} \, dU_t \]
Sample path of a square root process in dependence on time $t$
• moment

\[ E \left( Y_t^\alpha \mid A_0 \right) = (2 \bar{\varphi}_t \bar{s}_t)^\alpha \exp \left\{ -\frac{Y_0}{2 \bar{\varphi}_t} \right\} \sum_{k=0}^{\infty} \left( \frac{Y_0}{2 \bar{\varphi}_t} \right)^k \frac{\Gamma(\alpha + k + \frac{\delta}{2})}{k! \Gamma(k + \frac{\delta}{2})} \]

\[ \delta > 2, \; \alpha > -\frac{\delta}{2} \]

• moment estimate

\[ E \left( Y_t^\alpha \mid A_0 \right) \leq (2 \bar{\varphi}_t \bar{s}_t)^\alpha \exp \left\{ -\frac{Y_0}{2 \bar{\varphi}_t} \right\} \left( \frac{\Gamma(\alpha + \frac{\delta}{2})}{\Gamma(\frac{\delta}{2})} + \exp \left\{ \frac{Y_0}{2 \bar{\varphi}_t} \right\} \right) \]

< \infty \quad \text{for} \; \alpha \in (-\frac{\delta}{2}, 0)

\[ \bar{s}_t = \frac{s_t}{s_0} = \exp \left\{ \int_0^t b_u \, du \right\} \]

\[ \bar{\varphi}_t = s_0 \left( \varphi(t) - \varphi(0) \right) = \frac{1}{4} \int_0^t \frac{c_u^2}{s_u} \, du \]
• first moment of SR process

\[
E \left( Y_t \mid \mathcal{A}_0 \right) = E \left( Y_0 \mid \mathcal{A}_0 \right) \exp \left\{ \int_0^t b_s \, ds \right\} + \int_0^t \frac{\delta}{4} c_s^2 \exp \left\{ \int_s^t b_z \, dz \right\} \, ds
\]

• case \( \delta = 4 \)

for \( c_t^2 = c^2 > 0 \) and \( b_t = b < 0 \)

\[
\lim_{t \to \infty} E \left( Y_t \mid \mathcal{A}_0 \right) = -\frac{c^2}{b}
\]

\[
\lim_{t \to \infty} E \left( Y_t^{-1} \mid \mathcal{A}_0 \right) = -2 \frac{b}{c^2}
\]
transition density

\[ p(s, Y_s; t, Y_t) = \left( \varphi(s), \frac{Y_s}{s_s}; \varphi(t), \frac{Y_t}{s_t} \right) \]

\[ p(0, x; t, y) = \frac{1}{2\bar{s}_t \bar{\varphi}_t} \left( \frac{y}{x \bar{s}_t} \right)^{\frac{\nu}{2}} \exp \left\{ -\frac{x + \frac{y}{s_t}}{2 \bar{\varphi}_t} \right\} I_{\nu} \left( \frac{\sqrt{x \frac{y}{s_t}}}{\bar{\varphi}_t} \right) \]

for \( 0 < t < \infty, \ x, y \in (0, \infty) \),

where \( \nu = \frac{\delta}{2} - 1, \bar{s}_t = \exp\{b t\} \) and \( \bar{\varphi}_t = \frac{c^2}{4b}(1 - \frac{1}{\bar{s}_t}) \)
• stationary density

is gamma density

\[
p_{Y_{\infty}}(y) = \left(\frac{-2b}{c^2}\right)^{\frac{\delta}{2}} y^{\frac{\delta}{2}-1} \frac{\Gamma\left(\frac{\delta}{2}\right)}{\exp\left\{\frac{2b}{c^2} y\right\}}
\]
Minimal Market Model

Pl. (2001), special case of Pl.-Rendek model

Volatility Parametrization

- **Diversification Theorem** $\implies$
  well diversified stock market index approximates NP
- **discounted NP**

\[
d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} |\theta_t| (|\theta_t| \, dt + dW_t),
\]

where

\[
dW_t = \frac{1}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k \, dW^k_t
\]
Discounted NP
Logarithm of discounted NP
Drift Parametrization

- discounted NP drift
  \[ \alpha_t = \bar{S}_t^{\delta_*} |\theta_t|^2 \]
  strictly positive, predictable
  \[ \implies \]
- volatility
  \[ |\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}} \]

leverage effect creates natural feedback
under reasonably independent drift
• discounted NP

\[ d\bar{S}_t^{\delta} = \alpha_t \, dt + \sqrt{\bar{S}_t^{\delta}} \alpha_t \, dW_t \]

drift has economic meaning

• transformed time

\[ \varphi_t = \frac{1}{4} \int_0^t \alpha_s \, ds \]
squared Bessel process of dimension four

\[ X_{\varphi_t} = \bar{S}^{\delta_*}_t \]

\[ dW(\varphi_t) = \sqrt{\frac{\alpha_t}{4}} dW_t \]

\[ dX_{\varphi} = 4 \, d\varphi + 2 \sqrt{X_{\varphi}} \, dW(\varphi) \]

Revuz & Yor (1999)

economically founded dynamics in \( \varphi \)-time

still no specific dynamics in \( t \)-time assumed
Time Transformed Bessel Process

\[ d\sqrt{\bar{S}_t^{\delta_*}} = \frac{3 \alpha_t}{8 \sqrt{\bar{S}_t^{\delta_*}}} \, dt + \frac{1}{2} \sqrt{\alpha_t} \, dW_t \]

- quadratic variation

\[ \left[ \sqrt{\bar{S}_t^{\delta_*}} \right]_{t} = \frac{1}{4} \int_{0}^{t} \alpha_s \, ds \]

- transformed time

\[ \varphi_t = \left[ \sqrt{\bar{S}_t^{\delta_*}} \right]_{t} \]

observable
Empirical quadratic variation of the square root of the discounted S&P 500
Stylized Minimal Market Model


- assume discounted NP drift as

\[ \alpha_t = \alpha \exp \{ \eta t \} \]

- initial parameter \( \alpha > 0 \)

- net growth rate \( \eta \)
• transformed time

\[
\varphi_t = \frac{\alpha}{4} \int_0^t \exp \{\eta z\} \, dz
\]

\[
= \frac{\alpha}{4\eta} (\exp \{\eta t\} - 1)
\]
Fitted and observed transformed time
normalized NP

\[ Y_t = \frac{\bar{S}_t^\delta}{\alpha_t} \]

\[ dY_t = (1 - \eta Y_t) \, dt + \sqrt{Y_t} \, dW_t \]

square root process of dimension four

\[ \implies \text{parsimonious model, long term viability} \]

confirmation of stylized Pl.-Rendek model

Pl. & Rendek (2012)
\[
\bar{S}_t^{\delta^*} = Y_t \alpha_t
\]

- discounted NP

\[
S_t^{\delta^*} = S_t^0 \tilde{S}_t^{\delta^*} = S_t^0 Y_t \alpha_t
\]

- NP

- scaling parameter \( \alpha = 0.043 \)
- net growth rate \( \eta = 0.0528 \)
- reference level \( \frac{1}{\eta} = 18.93939 \)
- speed of adjustment \( \eta \)
- half life time of major displacement \( \frac{\ln(2)}{\eta} \approx 13 \) years
• logarithm of discounted NP

\[ \ln(\bar{S}_t^{\delta^*}) = \ln(Y_t) + \ln(\alpha) + \eta t \]

• no need for extra volatility process in MMM

• realistic long term dynamics
Volatility of NP under the stylized MMM

\[ |\theta_t| = \frac{1}{\sqrt{Y_t}} \]

- squared volatility

\[ d|\theta_t|^2 = d\left(\frac{1}{Y_t}\right) = |\theta_t|^2 \eta \, dt - (|\theta_t|^2)\frac{3}{2} \, dW_t \]

3/2 volatility model

Volatility of the NP under the stylized MMM
Transition Density of Stylized MMM

- transition density of discounted NP $\bar{S}^{\delta*}$

$$p(s, x; t, y) = \frac{1}{2 (\varphi_t - \varphi_s)} \left( \frac{y}{x} \right)^{\frac{1}{2}} \exp \left\{ -\frac{x + y}{2 (\varphi_t - \varphi_s)} \right\}$$

$$\times I_1 \left( \frac{\sqrt{xy}}{\varphi_t - \varphi_s} \right)$$

$$\varphi_t = \frac{\alpha}{4 \eta} \left( \exp \{ \eta t \} - 1 \right)$$

$I_1 (\cdot)$ modified Bessel function of the first kind
• non-central chi-square distributed random variable:

\[
\frac{y}{\varphi_t - \varphi_s} = \frac{\bar{S}_t^{\delta_*}}{\varphi_t - \varphi_s}
\]

with \( \delta = 4 \) degrees of freedom and non-centrality parameter:

\[
\frac{x}{\varphi_t - \varphi_s} = \frac{\bar{S}_s^{\delta_*}}{\varphi_t - \varphi_s}
\]
Transition density of squared Bessel process for $\delta = 4$
Zero Coupon Bond under the stylized MMM

- zero coupon bond

\[ P(t, T) = S^*_t \ E_t \left( \frac{1}{S^*_T} \right) = P^*_T(t) \ E_t \left( \frac{\bar{S}^*_t}{\bar{S}^*_T} \right) \]

with savings bond

\[ P^*_T(t) = \exp \left\{ - \int_t^T r_s \ ds \right\} \]
\[ \delta = 4 \implies P(t, T) = P^*_T(t) \left(1 - \exp \left\{-\frac{\bar{S}_{t}^\delta}{2(\varphi_T - \varphi_t)}\right\}\right) < P^*_T(t) \]

for \( t \in [0, T) \), Pl. (2002)

with time transform

\[ \varphi_t = \frac{\alpha}{4\eta} \left(\exp\{\eta t\} - 1\right) \]

\[ P(0, T) = 0.00423 \]

\[ P^*_T(0) = 0.04947 \]

\[ \frac{P(0,T)}{P^*_T(0)} \approx 0.0855 \]
Zero coupon bond and savings bond
\[ \ln(\text{savings bond}) \]
\[ \ln(\text{hedge portfolio}) \]
\[ \ln(P(t,T)) \]

\text{In from Zero coupon bond, hedge portfolio and savings bond}
Forward Rates under the MMM

- **forward rate**

\[
 f(t, T) = -\frac{\partial}{\partial T} \ln(P(t, T)) \\
= r_T + n(t, T)
\]
market price of risk contribution

\[ n(t, T) = -\frac{\partial}{\partial T} \ln \left( 1 - \exp \left\{ -\frac{\bar{S}_{t}^{\delta^{*}}}{2(\varphi_{T} - \varphi_{t})} \right\} \right) \]

\[ = \frac{1}{\left( \exp \left\{ \frac{\bar{S}_{t}^{\delta^{*}}}{2(\varphi_{T} - \varphi_{t})} \right\} - 1 \right)} \cdot \frac{\bar{S}_{t}^{\delta^{*}}}{(\varphi_{T} - \varphi_{t})^{2}} \cdot \frac{\alpha_{T}}{8} \]

\[ \lim_{T \to \infty} n(t, T) = \eta \]

Pl. (2005), Miller & Pl. (2005)
Market price of risk contribution in dependence on $\eta$ and $T$
Free Snack from Savings Bond

- potential existence of weak form of arbitrage?

borrow amount $P(0, T)$ from savings account

$$S_t^\delta = P(t, T) - P(0, T) \exp\{r t\}$$

such that $S_0^\delta = 0$

$$S_T^\delta = 1 - P(0, T) \exp\{r T\} > 0$$

lower bond

$$S_t^\delta \geq -P(0, T) \exp\{r t\}$$
• since $S_t^\delta$ may become negative

not strong arbitrage

**Fundamental Theorem of Asset Pricing**

Delbaen & Schachermayer (1998) $\implies$ the MMM does not admit an equivalent risk neutral probability measure

$\implies$

there is a **free lunch with vanishing risk**

Delbaen & Schachermayer (2006)
Absence of an Equivalent Risk Neutral Probability Measure

- fair zero coupon bond has a lower price than the savings bond

\[ P(t, T) < P_T^*(t) = \exp \left\{ - \int_t^T r_s ds \right\} = \frac{S_0^t}{S_0^T} \]

- Radon-Nikodym derivative

\[ \Lambda_t = \frac{\hat{S}_t^0}{\hat{S}_0^0} = \frac{\bar{S}_0^{\delta_*}}{\bar{S}_t^{\delta_*}} \]

strict supermartingale under MMM
Radon-Nikodym derivative and total mass of candidate risk neutral measure
• hypothetical risk neutral measure

total mass:

\[ P_{\theta,T}(\Omega) = E(\Lambda_T) = 1 - \exp \left\{ - \frac{\bar{S}_{0}^{d\ast}}{2 \varphi_T} \right\} < \Lambda_0 = 1 \]

\( P_\theta \) is not a probability measure
Benchmarked zero coupon bond
Hedge Simulation

- **delta** units in the NP

\[
\delta_*(t) = \frac{\partial P(t, T)}{\partial S^*_{t}}
\]

\[
= \exp \left\{ - \int_0^T r_s \, ds \right\} \exp \left\{ - \frac{\bar{S}^*_{t}}{2(\varphi_T - \varphi_t)} \right\} \frac{1}{2(\varphi_T - \varphi_t)}
\]
Ratio in the NP
Benchmarked P&L for hedge portfolio
European Call Options under the MMM

real world pricing formula $\implies$

\[
c_{T,K}(t, S^\delta_t) = S^\delta_t E_t \left( \frac{(S^\delta_T - K)^+}{S^\delta_T} \right)
\]

\[
= E_t \left( \left( S^\delta_t - \frac{K S^\delta_t}{S^\delta_T} \right)^+ \right)
\]

\[
= S^\delta_t \left( 1 - \chi^2(d_1; 4, \ell_2) \right)
\]

\[
- K \exp\{-r(T-t)\} \left( 1 - \chi^2(d_1; 0, \ell_2) \right)
\]

$\chi^2(\cdot; \cdot, \cdot)$ non-central chi-square distribution
with
\[ d_1 = \frac{4 \eta K \exp\{-r(T - t)\}}{S_t^0 \alpha_t (\exp\{\eta(T - t)\} - 1)} \]

and
\[ \ell_2 = \frac{2 \eta S_t^{\delta*}}{S_t^0 \alpha_t (\exp\{\eta(T - t)\} - 1)} \]

Hulley, Miller & Pl. (2005)

Miller & Pl. (2008)
Implied volatility surface for the stylized MMM
• implied volatility

in BS-formula adjust short rate to

$$\hat{r} = -\frac{1}{T-t} \ln(P(t,T))$$

otherwise put and call implied volatilities do not match
European Put Options under the MMM

- fair put-call parity relation

\[ p_{T,K}(t, S_t^\delta) = c_{T,K}(t, S_t^\delta) - S_t^\delta + K P(t, T) \]

- European put option formula

\[ p_{T,K}(t, S_t^\delta) = -S_t^\delta \left( \chi^2(d_1; 4, \ell_2) \right) \]
\[ + K \exp\{-r(T-t)\} \left( \chi^2(d_1; 0, \ell_2) - \exp\{-\ell_2\} \right) \]

Hulley, Miller & Pl. (2005)
Risk neutral and fair put on index
Benchmarked “risk neutral” and fair put on index
• put-call parity **breaks down** if one uses the savings bond $P^*_T(t)$

$$p_{T,K}(t, \bar{S}_t^{\delta^*}) < c_{T,K}(t, \bar{S}_t^{\delta^*}) - S_t^{\delta^*} + K \exp\{-r(T-t)\}$$

for $t \in [0,T)$
Comparison to Hypothetical Risk Neutral Prices

- hypothetical risk neutral price $c_{T,K}^*(t, S_{t}^{\delta_*})$

  of a European call option on the GOP

- benchmarked hypothetical risk neutral call price

  $$\hat{c}_{T,K}^*(t, S_{t}^{\delta_*}) = \frac{c_{T,K}^*(t, S_{t}^{\delta_*})}{S_{t}^{\delta_*}}$$

  local martingale

  $\hat{c}_{T,K}^*(\cdot, \cdot)$ uniformly bounded
\[ \implies \hat{c}_{T,K}^* \text{ - martingale} \implies \]

\[ \hat{c}_{T,K}^*(t, S_t^{\delta^*}) = \hat{c}_{T,K}(t, S_t^{\delta^*}) \]

\[ \implies \]

\[ c_{T,K}^*(t, S_t^{\delta^*}) = c_{T,K}(t, S_t^{\delta^*}) \]
• hypothetical risk neutral put-call parity

\[ p_{T,K}^*(t, S_{t}^{\delta*}) = c_{T,K}^*(t, S_{t}^{\delta*}) - S_{t}^{\delta*} + K P_{T}^*(t) \]

since \( P_{T}^*(t) > P(t, T) \) \( \implies \)

\[ p_{T,K}(t, S_{t}^{\delta*}) < p_{T,K}^*(t, S_{t}^{\delta*}) \]

\( t \in [0, T) \)
Difference in Asymptotic Put Prices

- hypothetical risk neutral prices can become extreme if NP value tends towards zero

- asymptotic fair zero coupon bond

\[ \lim_{\bar{S}_t^\delta \to 0} P_T(t, T) \overset{\text{a.s.}}{=} 0 \]

for \( t \in [0, T) \)

- asymptotic fair European call

\[ \lim_{\bar{S}_t^\delta \to 0} c_{T,K}(t, S_t^\delta) \overset{\text{a.s.}}{=} \lim_{\bar{S}_t^\delta \to 0} S_t^\delta \hat{c}_{T,K}(t, S_t^\delta) = 0 \]
• asymptotic fair put

fair put-call parity \implies

\lim_{\tilde{S}_t^{\delta*} \to 0} p_{T,K}(t, S_t^{\delta*}) \overset{a.s.}{=} 0
asymptotic hypothetical risk neutral put

hypothetical risk neutral put-call parity

\[
\begin{align*}
\lim_{\bar{S}_{t}^{\delta^{*}} \to 0} p_{T,K}^{*}(t, S_{t}^{\delta^{*}}) &= \lim_{\bar{S}_{t}^{\delta^{*}} \to 0} \left( p_{T,K}(t, S_{t}^{\delta^{*}}) + K \frac{S_{t}^{0}}{S_{T}^{0}} \exp \left\{ -\frac{\bar{S}_{t}^{\delta^{*}}}{2(\phi_{T} - \phi_{t})} \right\} \right) \\
&\overset{\text{a.s.}}{=} K \frac{S_{t}^{0}}{S_{T}^{0}} > 0
\end{align*}
\]

dramatic differences can arise
“Risk neutral” and fair put on index
Pricing Annuities

Baldeaux and Pl. (2012)

Interest Indexed Payouts

\[
\bar{Q}(t, T) = \bar{S}_t^* \ E_t \left( \frac{B_T}{S^*_T} \right)
\]

\[
\bar{S}^*_t = \frac{S^*_t}{B_t}
\]

\[B_t\ - \ savings\ account\]
e.g.: Minimal Market Model (MMM), Pl. (2001)

\[ \bar{Q}(t, T) = 1 - \exp \left\{ - \frac{2 \eta \bar{S}_t}{\alpha (\exp\{\eta T\} - \exp\{\eta t\})} \right\} \]
Interest Indexed Life Annuity

\( \theta \)-entitlement level

\( \bar{T} \)-retirement date

\( T_0 < T_1 < T_2 < \cdots \)

\( \xi_x \) - remaining life time of individual aged \( x \) at time 0

\[
\bar{U}^{\theta}_{x, \bar{T}}(t) = \bar{S}_t^* \ E_t \left( \sum_{T_i \geq \bar{T}} \mathbb{I}\{\xi_x > T_i\} \ \frac{\theta B_{T_i}}{S_{T_i}^*} \right) \\
= \theta \sum_{T_i \geq \bar{T}} P_t (\xi_x > T_i) \ \bar{Q}(t, T_i)
\]
Threshold Life Table

\[ P(\xi_x > T) = \begin{cases} 
\exp \left\{ -\frac{b}{\ln(C)} \left( C^x + T - 1 \right) \right\} & \text{for } T \leq N - x \\
q \left( 1 + \frac{x + T - N}{\omega} \right)^{-\frac{1}{\lambda}} & \text{for } T > N - x 
\end{cases} \]
Figure 1: Mortality rates fitted to mortality data of the US population in 2007 with threshold at $N = 93$. 
Figure 2: Discounted fair interest indexed life annuity as function of time to retirement.
Figure 3: Discounted hedging portfolio and discounted fair interest indexed life annuity evolving over the years.
Figure 4: Fraction of the hedging portfolio invested in the benchmark portfolio.
Targeted Pension

\( \Gamma_l(t) \)-account of \( l \) th member in units of interest indexed life annuity

- total discounted value

\[
\sum_{l=1}^{M_{t_i}} \Gamma_l(t_i) \bar{U}_{x_l, T_l} (t_i) = \delta^0_{\text{total}}(t_i) + \delta^*_{\text{total}}(t_i) \bar{S}^*_{t_i}
\]

\( \rightarrow \) entitlement level

\[
\theta_{t_i} = \frac{\delta^0_{\text{total}}(t_i) + \delta^*_{\text{total}} \bar{S}^*_{t_i}}{\sum_{l=1}^{M_{t_i}} \Gamma_l(t_i) \sum_{T_k \geq T_l} P_{t_i}(\xi_{x_l} > T_k) \bar{Q}(t_i, T_k)}
\]
Figure 5: Entitlement level over the years.
Figure 6: Entitlement level using fast growing portfolio and daily hedging.
Variable Annuity

Guaranteed Minimum Death Benefits

Marquardt, Pl. & Jaschke (2008)

- payout to the policyholder

\[
\max(e^{g\tau} V_0, V_\tau)
\]

\(\tau\) time of death

\(g \geq 0\) is the guaranteed instantaneous growth rate

\(V_0\) is the initial account value

\(V_\tau\) is the unit value of the policyholder’s account at time of death \(\tau\)

embedded put option
Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for 
\[ \eta = 0.05, \alpha_0 = 0.05, r = 0.05, \xi = 0.01 \text{ and } Y_0 = 20. \]
References


