UiO : Department of Mathematics
University of Oslo

Lecture IV

Stochastic volatility in energy forward price models

Fred Espen Benth

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Overview

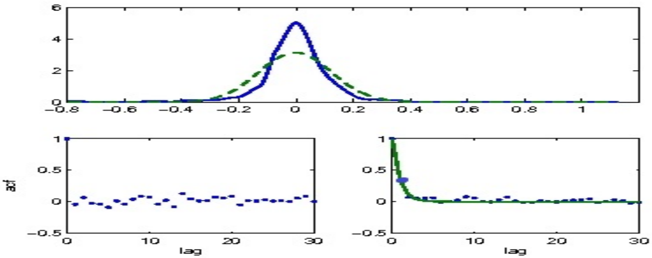
In collaboration with Barbara Rüdiger (Wuppertal) and André Süss (Zürich).

Overview

1. **Motivation and Background**
2. Operator-valued BNS SV model
3. Analysis of the OU model with BNS SV
4. Forward price with SV

Evidence for stochastic volatility?

- UK NBP gas spot prices
  - Residuals after de-seasonalization and regression
  - Non-Gaussian density (NIG), squared residuals correlated
- BNS SV model calibrates well



The BNS SV spot model

- Spot price of energy $S(t) = \Lambda(t)\exp(X(t))$

$$dX(t) = -\alpha X(t)\,dt + \sigma(t)\,dB(t)$$

- B is $\mathbb{R}$-valued Brownian motion, $\alpha > 0$
- $\sigma(t) := \sqrt{Y(t)}$

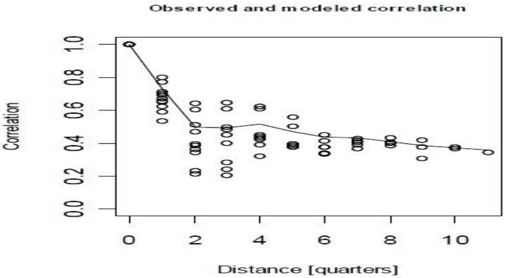
$$dY(t) = -\lambda Y(t)\,dt + dL(t)$$

- $L(t)$ is a Lévy process with increasing paths (subordinator), $\lambda > 0$.

- Implied forward dynamics from BNS SV spot model

$$\frac{dF(t,T)}{F(t-,T)} = e^{-\alpha(T-t)}\sigma(t)\,dB(t) + \int_0^\infty (e^{z\exp(-\lambda(T-t))} - 1)\,\widetilde{N}(dz,dt)$$

- But recall: indications of infinite dimensional noise
  - Spatial correlation between forwards with different maturities
  - Quarterly power forwards at NordPool (Andersen et al. (2010))



- GOAL: Define forward price dynamics with stochastic volatility
- Risk-neutral HJMM-dynamics for the forward price $f(t,x)$, $t,x \geq 0$,

$$df(t,x) = \partial_x f(t,x)\,dt + \sigma(t)dW(t,x)$$

- W Hilbert space valued Brownian motion, σ some "nice" operator-valued stochastic process

- Model should account for
  - Non-Gaussian spatial noise
  - Maturity dependent "BNS-type" stochastic volatility

Definition of stochastic model

- $\mathcal{A}$ unbounded operator (densely defined) on H, a separable Hilbert space
- $\mathcal{A}$ generates a C_0-semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$
- Ornstein-Uhlenbeck dynamics

$$dX(t) = \mathcal{A}X(t)\,dt + \sigma(t)\,dW(t)$$

- W H-valued Wiener process with covariance operator $\mathcal{Q}$

- σ predictable process with values in $L(H)$, the linear operators on H,

$$\mathbb{E}\left[\int_0^t \|\sigma(s)\mathcal{Q}^{1/2}\|_{\mathcal{H}}^2\,ds\right] < \infty$$

- $\mathcal{H} = L_{HS}(H)$, the space of Hilbert-Schmidt operators on H

$$\Psi \in \mathcal{H} \Leftrightarrow \|\Psi\|_{\mathcal{H}}^2 := \sum_{n=1}^{\infty} |\Psi e_n|_H^2 < \infty$$

- $\{e_n\}_{n\in\mathbb{N}}$ ONB in H
- Our focus: define $\sigma(t) = \mathcal{Y}^{1/2}(t)$, for some $\mathcal{Y}$

Overview

- Define $\mathcal{H}$-valued "variance" process $\mathcal{Y}$

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t)\,dt + d\mathcal{L}(t)$$

- $\mathbb{C} \in L(\mathcal{H})$, bounded linear operator on $\mathcal{H}$
  - Uniformly continuous C_0-semigroup

$$\mathbb{S}(t) = \exp(t\mathbb{C})\,, t \geq 0$$

- $t \mapsto \mathcal{L}(t)$ $\mathcal{H}$-valued square-integrable Lévy process
  - Covariance operator $\mathbb{Q}_{\mathcal{L}}$
  - Self-adjoint, positive definite trace class operator on $\mathcal{H}$

Analysis of $\mathcal{Y}(t)$

- Unique mild solution

$$\mathcal{Y}(t) = \mathbb{S}(t)\mathcal{Y}_0 + \int_0^t \mathbb{S}(t-s)\, d\mathcal{L}(s)$$

- Bound on norm of stochastic integral

$$\int_0^t \|\mathbb{S}(s)\mathbb{Q}_{\mathcal{L}}^{1/2}\|^2_{L_{HS}(\mathcal{H})}\, ds \leq \frac{\mathrm{Tr}(\mathbb{Q}_{\mathcal{L}})}{2\|\mathbb{C}\|_{op}}(e^{2t\|\mathbb{C}\|} - 1) < \infty$$

- $\mathcal{Y}$ affine process in $\mathcal{H}$: for $s \leq t$, $\mathcal{T} \in \mathcal{H}$

$$\ln \mathbb{E}\left[e^{\mathrm{i}\langle \mathcal{Y}(t),\mathcal{T}\rangle_{\mathcal{H}}} \,|\, \mathcal{F}_s\right] = \mathrm{i}\langle \mathcal{Y}(s), \mathbb{S}^*(t-s)\mathcal{T}\rangle_{\mathcal{H}} + \int_0^{t-s} \Psi_{\mathcal{L}}(\mathbb{S}^*(u)\mathcal{T})\,du$$

- $\Psi_{\mathcal{L}}$ characteristic exponent of $\mathcal{L}$
- Result follows by:
  - Independent increment property of $\mathcal{L}$
  - The Lévy-Kintchine formula for $\mathcal{L}$ (given by $\Psi_{\mathcal{L}}$)

Proposition

Suppose that $(\mathbb{C}\mathcal{T})^* = \mathbb{C}\mathcal{T}^*$ for any $\mathcal{T} \in \mathcal{H}$. If $\mathcal{L}$ and $\mathcal{Y}_0$ are self-adjoint, then $\mathcal{Y}$ is self-adjoint

Proof.

"Sketch": For any $f, g \in H$,

$$(\mathcal{Y}(t)f, g)_H = \int_0^t (f, \mathbb{C}\mathcal{Y}^*(s))_H \, ds + (f, \mathcal{L}(t)g)_H$$

Thus,

$$d\mathcal{Y}^*(t) = \mathbb{C}\mathcal{Y}^*(t) \, dt + d\mathcal{L}(t)$$

and $\mathcal{Y}^* = \mathcal{Y}$. ■

Proposition

Suppose that $\mathbb{C}$ preserves positive definiteness. If increments of $\mathcal{L}$ and $\mathcal{Y}_0$ are positive definite, then $\mathcal{Y}$ is positive definite

Proof.

"Sketch": $\mathbb{S}(t)\mathcal{Y}_0$ positive definite by assumptions on $\mathbb{C}$ and $\mathcal{Y}_0$. Same holds for

$$\sum_{m=1}^{M} \mathbb{S}(t - s_m)\Delta\mathcal{L}(s_m)$$

by assumptions on $\mathbb{C}$ and $\mathcal{L}$. Result follows from the mild solution of $\mathcal{Y}$ after passing to the limit. ∎

A closer look at $\mathcal{L}$

- Positive definiteness of the increments of $\mathcal{L}$ is equivalent to $\mathcal{L}$ having "non-decreasing" paths:
- $\mathcal{H}$-valued Lévy process $\mathcal{L}$ has *non-decreasing paths* if $t \mapsto (\mathcal{L}(t)f, f)_H$ is non-decreasing for all $f \in H$.

$$0 \leq ((\mathcal{L}(t) - \mathcal{L}(s))f, f)_H = (\mathcal{L}(t)f, f)_H - (\mathcal{L}(s)f, f)_H$$

- Claim: $L_f(t) := (\mathcal{L}(t)f, f)_H$ is an $\mathbb{R}$-valued Lévy process with non-decreasing paths, i.e. a subordinator

- General theory: for any $\mathcal{T} \in \mathcal{H}$,

$$t \mapsto \langle \mathcal{L}(t), \mathcal{T} \rangle_{\mathcal{H}}$$

  is an $\mathbb{R}$-valued Lévy process
- Let $\mathcal{T} = f \otimes f$, $f \in H$: since $(f \otimes f)(g) = (f, g)_H f$,

$$\langle \mathcal{L}(t), f \otimes f \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} (\mathcal{L}(t)(f, e_n)_H e_n, f)_H = (\mathcal{L}(t) f, f)_H$$

- $L_f(t)$ is a subordinator when $\mathcal{L}$ has "non-decreasing paths".
  - Recall: the univariate BNS SV model is driven by a subordinator Lévy process to ensure positive variance

- "Variance" in continuous martingale part in Levy-Kintchine formula of $L_f(t)$:

$$\langle \mathbb{Q}_{\mathcal{L}}^0(f \otimes f), f \otimes f \rangle_{\mathcal{H}} = 0$$

- $\mathbb{Q}_{\mathcal{L}}^0$ covariance of continuous martingale part of $\mathcal{L}$
- $f \otimes f \in \ker(\mathbb{Q}_{\mathcal{L}}^0)$:

$$0 = \langle \mathbb{Q}_{\mathcal{L}}^0(f \otimes f), f \otimes f \rangle_{\mathcal{H}} = \|(\mathbb{Q}_{\mathcal{L}}^0)^{1/2}(f \otimes f)\|_{\mathcal{H}}^2$$

$$\Downarrow$$

$$\mathbb{Q}_{\mathcal{L}}^0(f \otimes f) = 0$$

- All symmetric $\mathcal{T} \in \mathcal{H}$

$$\mathcal{T} = \sum_{k,l \in \mathbb{N}} \gamma_{kl} e_k \otimes e_l$$

- By polarization

$$\mathcal{T} = \sum_{k \in \mathbb{N}} \gamma_{kk} e_k \otimes e_k + 2 \sum_{k \in \mathbb{N}, l<k} \gamma_{kl}((e_k + e_l) \otimes (e_k + e_l) - e_k \otimes e_k - e_l \otimes e_l)$$

- Hence, symmetric $\mathcal{T} \in \ker(\mathbb{Q}_{\mathcal{L}}^0)$
- Cannot conclude $\mathbb{Q}_{\mathcal{L}}^0 = 0$
  - $\mathcal{L}$ may have a continuous martingale part

Example: compound Poisson process

- Suppose $\{\mathcal{X}_i\}_{i\in\mathbb{N}}$ iid square-integrable $\mathcal{H}$-valued random variables

$$\mathcal{L}(t) = \sum_{i=1}^{N(t)} \mathcal{X}_i$$

- N is an $\mathbb{R}$-valued Poisson process with intensity $\lambda > 0$

- $L_f(t)$ $\mathbb{R}$-valued compound Poisson process

$$L_f(t) := \langle \mathcal{L}(t), f \otimes f \rangle_{\mathcal{H}} = \sum_{i=1}^{N(t)} (\mathcal{X}_i f, f)_H$$

- $\mathcal{L}$ self-adjoint and positive definite if and only if $\mathcal{X}_i$ are self-adjoint and positive definite
  - Latter: $(\mathcal{X}_i f, f)_H$ is distributed on $\mathbb{R}_+$, i.e., L_f has positive jumps

- Let $\{Z_i\}_{i\in\mathbb{N}}$ be iid H-valued Gaussian random variables,

$$\mathcal{X}_i := Z_i^{\otimes 2}$$

- $\mathcal{X}_i$ becomes self-adjoint, positive definite,

$$L_f(t) = \sum_{i=1}^{N(t)} (Z_i^{\otimes 2} f, f)_H = \sum_{i=1}^{N(t)} (Z_i, f)_H^2$$

- $(Z_i, f)_H$ is $\mathbb{R}$-valued centered Gaussian with variance $|\mathcal{Q}_Z^{1/2} f|_H^2$
  - $L_f(t)$ has positive jumps being Gamma distributed
  - Shape parameter 1/2, scale $2|\mathcal{Q}_Z^{1/2} f|_H^2$

Examples of $\mathbb{C}$:

- Two specific cases of $\mathbb{C}$: For $\mathcal{C} \in L(H)$.

$$\mathbb{C}_1 : \mathcal{H} \to \mathcal{H}, \qquad \mathcal{T} \mapsto \mathcal{C}\mathcal{T}\mathcal{C}^*$$

$$\mathbb{C}_2 : \mathcal{H} \to \mathcal{H}, \qquad \mathcal{T} \mapsto \mathcal{C}\mathcal{T} + \mathcal{T}\mathcal{C}^*$$

- $\mathbb{C}_2$ extension of the matrix-operator BNS SV model in Barndorff-Nielsen and Stelzer (2007)
- $\mathbb{C}_1\mathcal{T}$ is self-adjoint positive definite whenever $\mathcal{T} \in \mathcal{H}$ is
  - ...while $\mathbb{S}_2(t)\mathcal{T}$ is self-afdjoint, positive definite

The BNS SV model

- Assume $\mathcal{Y}$ satisfies:

1. $\mathcal{Y}_0$ is self-adjoint positive definite
2. $(\mathbb{C}\mathcal{T})^* = \mathbb{C}\mathcal{T}^*$
3. $\mathbb{C}\mathcal{T}$ positive definite whenever $\mathcal{T}$ is
4. $\mathcal{L}$ has "non-decreasing" paths

- Define BNS SV model

$$\sigma(t) := \mathcal{Y}^{1/2}(t)$$

Overview

- Recall our OU dynamics for $X(t)$

$$dX(t) = \mathcal{A}X(t)\,dt + \mathcal{Y}^{1/2}(t)\,dW(t)$$

- Mild solution

$$X(t) = \mathcal{S}(t)X_0 + \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dW(s)$$

- Well-defined stochastic integrals?

$$\mathbb{E}\left[\int_0^t \|\mathcal{Y}^{1/2}(s)\mathcal{Q}^{1/2}\|_{\mathcal{H}}^2\,ds\right] = \mathbb{E}\left[\int_0^t \mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{Y}(s)\mathcal{Q}^{1/2})\,ds\right] < \infty?$$

- It holds,

$$\mathbb{E}\left[\mathrm{Tr}(\mathcal{Q}^{1/2}\mathcal{Y}(t)\mathcal{Q}^{1/2})\right] = \mathrm{Tr}(\mathcal{Q}^{1/2}\mathbb{S}(t)\mathcal{Y}_0\mathcal{Q}^{1/2}) + \mathrm{Tr}\left(\mathcal{Q}^{1/2}\int_0^t \mathbb{S}(s)\,ds\mathbb{E}[\mathcal{L}(t)]\mathcal{Q}^{1/2}\right)$$

- $\int_0^t \mathbb{S}(s)\,ds$ is the Bochner integral and $\mathbb{E}[\mathcal{L}(t)]$ operator-valued expected value
- "Proof" goes by playing around with the Levy-Kintchine formula of $\mathcal{L}$ and definition of the trace

Characteristic function of X

- Characteristic function known under a strong commutativity hypothesis:
  - Assume there exists self-adjoint positive definite $\mathcal{D} \in L(H)$;

$$\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s) = \mathcal{D}^{1/2}\mathcal{Y}(s)\mathcal{D}^{1/2}$$

- Condition holds if $\mathcal{Q}$ commutes with $\mathcal{Y}(s)$
  - Choose $\mathcal{D} := \mathcal{Q}$
  - Strong conditions on $\mathcal{Y}$: $\mathcal{Q}$ commutes with $\mathcal{L}$, $\mathcal{Y}_0$ and $\mathbb{C}$
- Denote cumulant of $X(t)$ by $\Psi_X(t,f)$, $f \in H$

$$\mathbb{E}\left[\exp(\mathrm{i}(X(t),f)_H)\right] = \exp(\Psi_X(t,f))$$

Proposition

If $\mathcal{L}$ is independent of W, then

$$\begin{aligned}\Psi_X(t,f) &= \mathrm{i}(X_0, \mathcal{S}^*(t)f)_H \\ &- \frac{1}{2}\langle \mathcal{Y}_0, \int_0^t \mathbb{S}^*(s)((\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f)\otimes(\mathcal{D}^{1/2}\mathcal{S}^*(t-s)f))\,ds\rangle_{\mathcal{H}} \\ &+ \int_0^t \Psi_{\mathcal{L}}\left(-\frac{1}{2}\int_0^s \mathbb{S}^*(s-u)((\mathcal{D}^{1/2}\mathcal{S}^*(u)f)\otimes(\mathcal{D}^{1/2}\mathcal{S}^*(u)f))\,du\right)\,ds\end{aligned}$$

Proof.

Apply conditional Gaussianity of stochastic integral given $\mathcal{Y}$ together with Levy-Kintchine formula of $\mathcal{L}$. Next a Fubini theorem to resolve integration of $\mathcal{Y}$ ■

"Price returns" model

- Define "adjusted returns" by

$$R(t,\Delta t) = X(t+\Delta t) - \mathcal{S}(\Delta t)X(t)$$

- $R(t,\Delta t)$ given $\mathcal{Y}$ is a mean-zero H-valued Gaussian random variable with covariance operator ($\mathcal{L}$ independent of W)

$$\mathcal{Q}_{R|\mathcal{Y}} := \int_t^{t+\Delta t} \mathcal{S}(t+\Delta t - s)\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s)\mathcal{S}^*(t+\Delta t - s)\,ds$$

- Adjusted returns conditionally independent, Gaussian "variance-mixture" model

Overview

- Choose $H = H_w$, the (Filipovic) space of real-valued absolutely continuous functions on $\mathbb{R}_+$

$$|f|_w^2 = f^2(0) + \int_0^\infty w(x)|f'(x)|^2\,dx < \infty$$

- w increasing function, $w(0) = 1$, $\int_0^\infty w^{-1}(x)\,dx < \infty$
- H_w separable Hilbert space with $\delta_x(f) = f(x)$ a continuous linear functional
- Let $\mathcal{A} = \partial/\partial x$ with C_0-semigroup $\mathcal{S}(t)(g) = g(\cdot + t)$

- Define the forward price at time t and maturity $x \geq 0$

$$f(t,x) := \delta_x(X(t)),$$

- Note

$$\delta_x \mathcal{S}(t) g = g(t+x) = \delta_{t+x} g$$

- Forward price:

$$f(t,x) = X_0(t+x) + \delta_x \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\, dW(s)$$

- Stochastic integral has zero mean

$$\lim_{x\to\infty}\mathbb{E}[f(t,x)] = \lim_{x\to\infty} X_0(t+x) = X_0(\infty)$$

- "Long end" of market is constant. Moreover, $f(t,x)$ "stationary in mean" as

$$\lim_{t\to\infty}\mathbb{E}[f(t,x)] = X_0(\infty)$$

- Model tends towards a flat (in mean) forward curve

- Result from Lecture II (B. Krühner (2014)):

$$\delta_x \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dW(s) = \int_0^t \sigma_x(t,s)\,dB_x(s)$$

- B_x univariate Brownian motion, σ_x stochastic volatility process

$$\sigma_x^2(t,s) = \delta_{x+t-s}(\mathcal{Y}^{1/2}(s)\mathcal{Q}\mathcal{Y}^{1/2}(s))\delta_{x+t-s}^*(1)$$

- $t \mapsto f(t,x)$ a Brownian-driven Volterra process
  - Barndorff-Nielsen, B, Veraart (2013). Volterra processes for energy spot price modelling
  - Spot price: $f(t,0)$

- Note in H_w: $\delta_x^*(1) = h_x(\cdot)$,

$$h_x(y) = 1 + \int_0^{y \wedge x} w^{-1}(z)\,dz \qquad y \geq 0$$

- If $\mathcal{Y}(s)$ and $\mathcal{Q}$ commutes

$$\sigma_x^2(t,s) = \mathcal{Y}_{\mathcal{Q}}(s)(h_{x+t-s})(x+t-s) = \langle \mathcal{Y}_{\mathcal{Q}}(s), h_{x+t-s} \otimes h_{x+t-s} \rangle_{\mathcal{H}}$$

- Here $\mathcal{Y}_{\mathcal{Q}}(t) = \mathcal{Y}(t)\mathcal{Q}$, $\mathcal{L}_Q(t) = \mathcal{L}(t)\mathcal{Q}$ $\mathcal{H}$-valued Lévy process

$$d\mathcal{Y}_{\mathcal{Q}}(t) = \mathbb{C}\mathcal{Y}_{\mathcal{Q}}(t)\,dt + d\mathcal{L}_{\mathcal{Q}}(t)$$

$f(t,x)$ as a random field

- Global noise in time and space rather than B_x marginal Brownian motion?
- Using properties of H_w....

$$\delta_x \int_0^t \mathcal{S}(t-s)\mathcal{Y}^{1/2}(s)\,dW(s) = \int_0^t (\mathcal{Y}^{1/2}(s)h_{x+t-s})(0)\,dW(s,0)$$
$$+ \int_0^t \int_0^\infty w(y)(\mathcal{Y}^{1/2}(s)h_{x+t-s})'(y)W(ds,dy)$$

- $W(t,0)$ real-valued Brownian motion with variance $|\mathcal{Q}^{1/2}1|_w^2$
- Second integral resembles an "Ambit Field"
  - Barndorff-Nielsen & Co

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UiO : Department of Mathematics
University of Oslo

Fred Espen Benth

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