Multi-Period Risk Sharing Under Financial Fairness
The Concept of PEFF in Inter-temporal Risk-Sharing

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Introduction.
The Problem of Interest

- Discrete-time multi-period risk sharing system
- Agents gather to share financial risks
- A fund exists and enables inter-temporal transfer
- Key problem: determine how much money shall be paid out now and how much shall be put into the fund for future use
What is PEFF?
Efficiency Utility-wise, Fairness Value-wise

- PE stands for Pareto efficiency – utility-wise.
- FF stands for financial fairness – value-wise.
Motivation
PEFF from Reality

Systems like
- collective pension schemes that allow collective risk sharing
- reinsurance contracts where companies gather to reallocate their risk exposures

have properties of both a multilateral risk sharing system and a financial contract.

- PE is fundamental in multilateral risk sharing systems, and
- FF is important in designing financial contracts.
Model Framework.
Discrete-Time Multi-Period Risk-Sharing Systems
A Generalized Setting

- $N$ agents gather to share risks: they pay into the system stochastic cash inflows and expect cash outflows after risk sharing.

- Each agent gets one and only one cash outflow.

- Agents make use of a *fund* for inter-temporal capital transfers.

- Cash outflows happen at time points $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$. 
Discrete-Time Multi-Period Risk-Sharing Systems

Generic Structure

Try to determine the C’s from system

\[ F_n + C_n = X_n + F_{n-1}R_n := A_n \quad n = 1, \ldots, N \]

where

- \( F_n \): the fund size at time \( t_n \). Can be positive or negative.
- \( C_n \): the cash outflow paid out from the system at time \( t_n \) to agent \( n \). Decision variable.
- \( X_n \): aggregate risk to be shared which materializes from \( t_{n-1} \) to \( t_n \).
- \( R_n \): the gross return of the fund investment. Can also be stochastic.
- \( A_n \): the total asset at time \( t_n \).
Discrete-Time Multi-Period Risk-Sharing Systems

Illustration
We assume a *finite* probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\).
The measure \(\mathbb{Q}\) can be chosen by the social planner or decided jointly by the agents.

\((X_n, R_n)\) is sequentially independent.

The joint distribution of \((X_n, R_n)\) is known under both \(\mathbb{P}\) and \(\mathbb{Q}\).
Discrete-Time Multi-Period Risk-Sharing Systems

Special Example

If we let

\[ t_1 = t_2 = \cdots = t_N \]
\[ X_2 = \cdots = X_N \equiv 0 \]
\[ R_2 = \cdots = R_N \equiv 1 \]

then the system degenerates to a single-period problem and the budget constraint becomes

\[ \sum_{n=1}^{N} C_n + F_N = X_1 \]

where \( X_1 \) represents the aggregate risk to be shared.
Discrete-Time Multi-Period Risk-Sharing Systems

Special Example
The utility of each agent comes from the cash flow he receives.

- Agent $n$ adopts utility function $u_n$ for $C_n$. The expected utility is used for welfare evaluation.

- We also assume that the fund adopts utility function $u_p$ to evaluate $F_N$. 
Connection to the Consumption-Savings Problem

The setting of the classical consumption-savings problem (CSP) assumes that

- the timeline is equi-spaced,
- the final fund size is fixed,
- all cash flows belong to a single agent, and
- the social planner tries to maximize

\[ \mathbb{E}^p \left[ \sum_{n=1}^{N} d^{n-1} u_n(C_n) \right] \]  

where \( d \) is the subjective discount factor.
Pareto Efficiency and Financial Fairness.
A risk-sharing rule $\rho = (C_1, C_2, \ldots, C_N)$ is called **Pareto efficient**, or **Pareto optimal**, if there does not exist another risk-sharing rule $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_N)$ such that

\[
\left( \mathbb{E}^{p} u_1(\tilde{C}_1), \ldots, \mathbb{E}^{p} u_N(\tilde{C}_N), \mathbb{E}^{p} u_p(\tilde{F}_N) \right) \not\geq \\
\left( \mathbb{E}^{p} u_1(C_1), \ldots, \mathbb{E}^{p} u_N(C_N), \mathbb{E}^{p} u_p(F_N) \right).
\]
Pareto Efficiency
Characterization of Multi-Period Pareto Efficiency

Theorem 1

(Characterization of Pareto efficiency.) For a risk-sharing rule \( \rho = (C_1, C_2, \cdots, C_N) \), the following statements are equivalent.

- The risk-sharing rule is Pareto efficient.
- The risk-sharing rule maximizes

\[
\mathbb{E}^p \left[ \sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N) \right]
\]

for some positive constants \( \theta = (\theta_1, \cdots, \theta_N, \theta_p) \).
Theorem 1 (cont.)

- The risk-sharing rule will satisfy the following which are hereafter called the inter-temporal balance equations (IBEs) for some positive constants \((\theta_1, \cdots, \theta_N, \theta_p)\):

\[
\theta_n u'_n(C_n) = \theta_{n+1} E_n^p \left[ u'_{n+1}(C_{n+1}) R_{n+1} \right] \quad \forall n = 1, \cdots, N - 1,
\]

\[
\theta_N u'_N(C_N) = \theta_p u'_p(F_N).
\]
Financial Fairness

Value Profile

- The notion of financial fairness can be characterized by the *value profile*

\[ \nu = (\nu_1, \nu_2, \cdots, \nu_N, \nu_p) := \mathbb{E}^Q \rho \]

\[ = \left( \mathbb{E}^Q C_1, \mathbb{E}^Q C_2, \cdots, \mathbb{E}^Q C_N, \mathbb{E}^Q F_N \right) \in \mathbb{R}^{N+1}. \]

Denote \( \mathcal{V} \) as the set of all possible value profiles: it has dimension \( N \).

- The value profile helps to determine the \( \theta \)'s.
Recall that in the consumption-savings problem one maximizes

$$\mathbb{E}^P \left[ \sum_{n=1}^{N} d^{n-1} u_n(C_n) \right] ;$$

here one maximizes

$$\mathbb{E}^P \left[ \sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N) \right]$$

where we use the value profile to determine the weights endogenously.
Existence and Uniqueness of Solutions.
Existence and Uniqueness of PEFF Solution

The PEFF risk-sharing rule is the solution to the following equation systems:

1. budget constraints (BCs):
   \[ F_n + C_n = X_n + F_{n-1}R_n \quad n = 1, \ldots, N; \]

2. inter-temporal balance equations (IBEs):
   \[
   \theta_n u_n'(C_n) = \theta_{n+1} \mathbb{E}^p_n \left[ u_{n+1}'(C_{n+1})R_{n+1} \right] \quad \forall n = 1, \ldots, N - 1, \\
   \theta_N u_N'(C_N) = \theta_p u_p'(F_N);
   \]

3. financial fairness constraints (FFs):
   \[ \mathbb{E}^\cap C_n = v_n \quad \forall n = 1, \ldots, N. \]

4. Measurability conditions: \( C_n \) is \( \mathcal{F}_n \)-measurable, and \( F_N \) is \( \mathcal{F}_N \)-measurable.
Existence and Uniqueness of PEFF Solution

Theorem 2
(The existence and uniqueness of the PEFF risk sharing rule.) For any given value profile vector $v \in V$, the PEFF risk-sharing rule exists and is unique. The corresponding $\theta$ is unique up to normalization.
Existence and Uniqueness of PEFF Solution

An Implication

Denote

- \( P \): the set of all PE risk sharing rules; \( \rho \in P \).
- \( U \): the open simplex in \( \mathbb{R}_{++}^{N+1} \) (i.e. \( \forall x \in U, \sum_{i=1}^{N+1} x_i = 1 \); \( \theta \in U \).
- \( V \): the set of all possible value profiles; \( v \in V \).

The theorems then tell that there are one-to-one correspondences among

\[
U \leftrightarrow P \leftrightarrow V
\]

Compared to the classical CSP, the weights are not given directly; instead, the value profile is given directly to determine the weights.
Finding the Solution.
The Form of the PEFF Solution

The solution is of the following form

\[ C_n = f_n(A_n), \]

where the \( f_n \)'s are strictly increasing functions determined by the utility functions, the value profile and the distributions of the risks.
How to Get the PEFF Solution?

- Explicit solutions only exist under special circumstances...

- For generic settings we need a numerical algorithm which is based on an iterative procedure on $\theta$. 
Explicit Solution

- Explicit solution exists when we assume exponential utility function

\[ u_n(x) = 1 - e^{-\alpha_n x}, \quad u_p(x) = 1 - e^{-\alpha_p x} \]

and that the \( R_n \)'s are deterministic.

- The explicit solution is of the form

\[ C_n = v_n + a_n (A_n - \mathbb{E}^Q A_n), \]

where the coefficients \( a_n \)'s are determined recursively by

\[
\begin{align*}
a_N &= \frac{\alpha_p}{\alpha_p + \alpha_N}, \\
a_n &= \frac{a_{n+1} \alpha_{n+1} R_{n+1}}{\alpha_n + a_{n+1} \alpha_{n+1} R_{n+1}} \quad n = 1, \cdots, N - 1.
\end{align*}
\]
Explicit Solution
Cont.

Suppose $R_n \equiv 1 + r$, $\alpha_n \equiv \alpha$ and $\alpha_p = k\alpha$.

- If $N$ is sufficiently large, then we have that the coefficient $a_n \approx \frac{r}{1+r}$ and the explicit solution becomes

  $$C_n \approx v_n + \frac{r}{1+r} (A_n - E^Q A_n).$$

- The last coefficient $a_N = \frac{k}{1+k}$. If $k = r$ then we shall have $a_n \equiv \frac{r}{1+r}$ for all $n$. 
Conclusions.
PEFF Recap

- This paper establishes the existence and uniqueness of the PEFF risk-sharing rule in a generic model setting.
- The PEFF model works as a processing tool:
  - Input: the preferences of agents, distributions of risks, the value profile
  - Output: resulted PEFF risk sharing rule.
- A numerical algorithm is proposed for finding the PEFF solution.
Thanks! 
Bedankt!
Numerical Procedure: Outline

1. Construct a mapping $\phi_1 : U \to P$ from the sets of equations BC and IBE.
2. Construct a mapping $\phi_2 : P \to U$ from the set of equations FF.
3. The composition $\phi := \phi_2 \circ \phi_1$ maps $U$ into itself. Theorem 2 tells that a fixed point $\theta^*$ exists.
4. It can be shown that the sequence of iterates $\{\phi^{(n)}(\theta_0)\}$ will finally converge to $\theta^*$ for any given $\theta_0 \in U$. 