Multi-Period Risk Sharing Under Financial Fairness

The Concept of PEFF in Inter-temporal Risk-Sharing

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Introduction.

The Problem of Interest

- Discrete-time multi-period risk sharing system
- Agents gather to share financial risks
- A fund exists and enables inter-temporal transfer
- Key problem: determine how much money shall be paid out now and how much shall be put into the fund for future use

What is PEFF?

Efficiency Utility-wise, Fairness Value-wise

- PE stands for Pareto efficiency – utility-wise.
- FF stands for financial fairness – value-wise.

Motivation

PEFF from Reality

Systems like

- collective pension schemes that allow collective risk sharing
- reinsurance contracts where companies gather to reallocate their risk exposures

have properties of both a *multilateral risk sharing system* and a *financial contract*.

- PE is fundamental in multilateral risk sharing systems, and
- FF is important in designing financial contracts.

Model Framework.

Discrete-Time Multi-Period Risk-Sharing Systems

A Generalized Setting

- N agents gather to share risks: they pay into the system stochastic cash inflows and expect cash outflows after risk sharing.
- Each agent gets one and only one cash outflow.
- Agents make use of a *fund* for inter-temporal capital transfers.
- Cash outflows happen at time points $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$.

Discrete-Time Multi-Period Risk-Sharing Systems

Generic Structure

Try to determine the C's from system

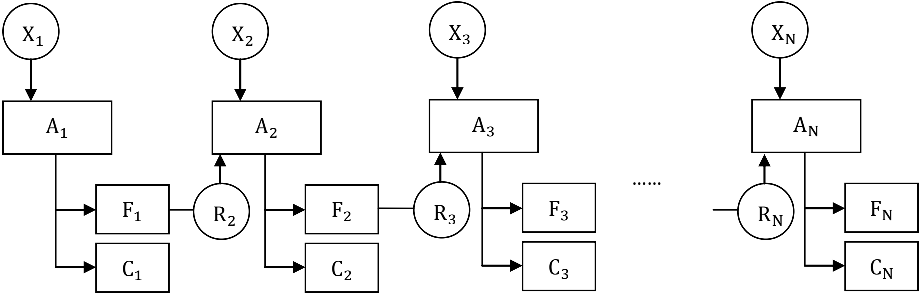
$$F_n + C_n = X_n + F_{n-1}R_n := A_n \quad n = 1, \cdots, N$$

where

- F_n: the fund size at time t_n. Can be positive or negative.
- C_n: the cash outflow paid out from the system at time t_n to agent n. Decision variable.
- X_n: aggregate risk to be shared which materializes from t_{n-1} to t_n.
- R_n: the gross return of the fund investment. Can also be stochastic.
- A_n: the total asset at time t_n.

Discrete-Time Multi-Period Risk-Sharing Systems

Illustration



Discrete-Time Multi-Period Risk-Sharing Systems

Financial Market

- We assume a *finite* probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})$.
  The measure $\mathbb{Q}$ can be chosen by the social planner or decided jointly by the agents.
- (X_n, R_n) is sequentially independent.
- The joint distribution of (X_n, R_n) is known under both $\mathbb{P}$ and $\mathbb{Q}$.

Discrete-Time Multi-Period Risk-Sharing Systems

Special Example

If we let

$$t_1 = t_2 = \cdots = t_N$$
$$X_2 = \cdots = X_N \equiv 0$$
$$R_2 = \cdots = R_N \equiv 1$$

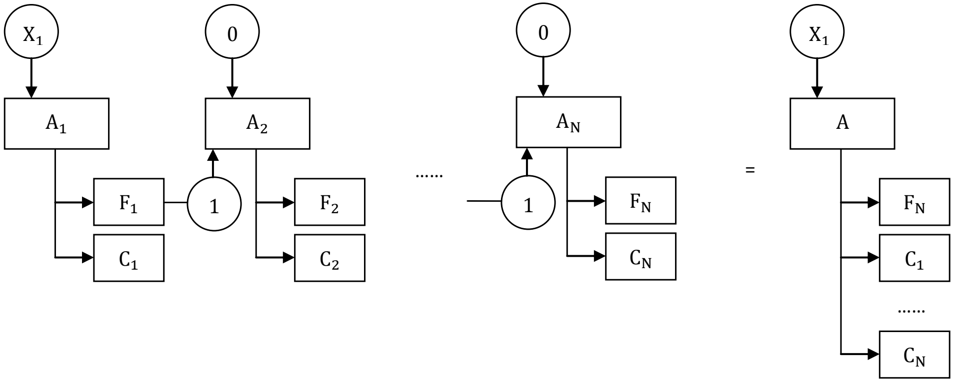
then the system degenerates to a single-period problem and the budget constraint becomes

$$\sum_{n=1}^{N} C_n + F_N = X_1$$

where X_1 represents the aggregate risk to be shared.

Discrete-Time Multi-Period Risk-Sharing Systems

Special Example



Discrete-Time Multi-Period Risk-Sharing Systems

Utility

The utility of each agent comes from the cash flow he receives.

- Agent n adopts utility function u_n for C_n. The expected utility is used for welfare evaluation.
- We also assume that the fund adopts utility function u_p to evaluate F_N.

Connection to the Consumption-Savings Problem

The setting of the classical consumption-savings problem (CSP) assumes that

- the timeline is equi-spaced,
- the final fund size is fixed,
- all cash flows belong to a single agent, and
- the social planner tries to maximize

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} d^{n-1} u_n(C_n)\right] \tag{1}$$

where d is the subjective discount factor.

Pareto Efficiency and Financial Fairness.

Pareto Efficiency

Definition of Multi-Period Pareto Efficiency

A risk-sharing rule $\rho = (C_1, C_2, \cdots C_N)$ is called *Pareto efficient*, or *Pareto optimal*, if there does not exist another risk-sharing rule $(\tilde{C}_1, \tilde{C}_2, \cdots \tilde{C}_N)$ such that

$$\left(\mathbb{E}^{\mathbb{P}} u_1(\tilde{C}_1), \cdots, \mathbb{E}^{\mathbb{P}} u_N(\tilde{C}_N), \mathbb{E}^{\mathbb{P}} u_p(\tilde{F}_N)\right) \gneqq \left(\mathbb{E}^{\mathbb{P}} u_1(C_1), \cdots, \mathbb{E}^{\mathbb{P}} u_N(C_N), \mathbb{E}^{\mathbb{P}} u_p(F_N)\right).$$

Theorem 1

(Characterization of Pareto efficiency.) For a risk-sharing rule $\rho = (C_1, C_2, \cdots, C_N)$, the following statements are equivalent.

- *The risk-sharing rule is Pareto efficient.*
- *The risk-sharing rule maximizes*

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N)\right] \tag{2}$$

for some positive constants $\theta = (\theta_1, \cdots, \theta_N, \theta_p)$.

Theorem 1 (cont.)

- *The risk-sharing rule will satisfy the following which are hereafter called the inter-temporal balance equations (IBEs) for some positive constants* $(\theta_1, \cdots, \theta_N, \theta_p)$*:*

$$\theta_n u'_n(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} \left[u'_{n+1}(C_{n+1}) R_{n+1} \right] \quad \forall n = 1, \cdots N-1,$$
$$\theta_N u'_N(C_N) = \theta_p u'_p(F_N).$$

Financial Fairness

Value Profile

- The notion of financial fairness can be characterized by the *value profile*

$$v = (v_1, v_2, \cdots, v_N, v_p) := \mathbb{E}^{\mathbb{Q}} \rho$$
$$= \left(\mathbb{E}^{\mathbb{Q}} C_1, \mathbb{E}^{\mathbb{Q}} C_2, \cdots, \mathbb{E}^{\mathbb{Q}} C_N, \mathbb{E}^{\mathbb{Q}} F_N \right) \in \mathbb{R}^{N+1}.$$

  Denote $\mathcal{V}$ as the set of all possible value profiles: it has dimension N.
- The value profile helps to determine the θ's.

Connection to the Consumption-Savings Problem
Cont.

Recall that in the consumption-savings problem one maximizes

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} d^{n-1} u_n(C_n)\right];$$

here one maximizes

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N} \theta_n u_n(C_n) + \theta_p u_p(F_N)\right]$$

where we use the value profile to determine the weights endogenously.

Existence and Uniqueness of Solutions.

Existence and Uniqueness of PEFF Solution

The PEFF risk-sharing rule is the solution to the following equation systems:

1. budget constraints (BCs):

$$F_n + C_n = X_n + F_{n-1} R_n \quad n = 1, \cdots, N;$$

2. inter-temporal balance equations (IBEs):

$$\theta_n u_n'(C_n) = \theta_{n+1} \mathbb{E}_n^{\mathbb{P}} \left[u_{n+1}'(C_{n+1}) R_{n+1} \right] \quad \forall n = 1, \cdots N-1,$$
$$\theta_N u_N'(C_N) = \theta_p u_p'(F_N);$$

3. financial fairness constraints (FFs):

$$\mathbb{E}^{\mathbb{Q}} C_n = v_n \quad \forall n = 1, \cdots, N.$$

4. Measurability conditions: C_n is $\mathcal{F}_n$-measurable, and F_N is $\mathcal{F}_N$-measurable.

Existence and Uniqueness of PEFF Solution

Theorem 2

(The existence and uniqueness of the PEFF risk sharing rule.) For any given value profile vector $v \in \mathcal{V}$*, the PEFF risk-sharing rule exists and is unique. The corresponding* θ *is unique up to normalization.*

Existence and Uniqueness of PEFF Solution

An Implication

Denote

- $\mathcal{P}$: the set of all PE risk sharing rules; $\rho \in \mathcal{P}$.
- $\mathcal{U}$: the open simplex in $\mathbb{R}_{++}^{N+1}$ (i.e. $\forall x \in \mathcal{U}$, $\sum_{i=1}^{N+1} x_i = 1$); $\theta \in \mathcal{U}$.
- $\mathcal{V}$: the set of all possible value profiles; $v \in \mathcal{V}$.

The theorems then tell that there are one-to-one correspondences among

$$\mathcal{U} \leftrightarrow \mathcal{P} \leftrightarrow \mathcal{V}$$

Compared to the classical CSP, the weights are not given directly; instead, the value profile is given directly to determine the weights.

Finding the Solution.

The Form of the PEFF Solution

The solution is of the following form

$$C_n = f_n(A_n),$$

where the f_n's are strictly increasing functions determined by the utility functions, the value profile and the distributions of the risks.

How to Get the PEFF Solution?

- Explicit solutions only exist under special circumstances...
- For generic settings we need a numerical algorithm which is based on an iterative procedure on θ.

Explicit Solution

- Explicit solution exists when we assume exponential utility function

$$u_n(x) = 1 - e^{-\alpha_n x}, \quad u_p(x) = 1 - e^{-\alpha_p x}$$

 and that the R_n's are deterministic.
- The explicit solution is of the form

$$C_n = v_n + a_n(A_n - \mathbb{E}^{\mathbb{Q}} A_n),$$

 where the coefficients a_n's are determined recursively by

$$a_N = \frac{\alpha_p}{\alpha_p + \alpha_N},$$
$$a_n = \frac{a_{n+1}\alpha_{n+1}R_{n+1}}{\alpha_n + a_{n+1}\alpha_{n+1}R_{n+1}} \quad n = 1, \cdots, N-1.$$

Explicit Solution
Cont.

Suppose $R_n \equiv 1 + r$, $\alpha_n \equiv \alpha$ and $\alpha_p = k\alpha$.

- If N is sufficiently large, then we have that the coefficient $a_n \approx \frac{r}{1+r}$ and the explicit solution becomes

$$C_n \approx v_n + \frac{r}{1+r}(A_n - \mathbb{E}^{\mathbb{Q}} A_n).$$

- The last coefficient $a_N = \frac{k}{1+k}$. If $k = r$ then we shall have $a_n \equiv \frac{r}{1+r}$ for all n.

Conclusions.

PEFF Recap

- This paper establishes the existence and uniqueness of the PEFF risk-sharing rule in a generic model setting.
- The PEFF model works as a processing tool:
  - Input: the preferences of agents, distributions of risks, the value profile
  - Output: resulted PEFF risk sharing rule.
- A numerical algorithm is proposed for finding the PEFF solution.

Thanks!

Bedankt!

Numerical Procedure: Outline

1. Construct a mapping $\phi_1 : \mathcal{U} \to \mathcal{P}$ from the sets of equations BC and IBE.
2. Construct a mapping $\phi_2 : \mathcal{P} \to \mathcal{U}$ from the set of equations FF.
3. The composition $\phi := \phi_2 \circ \phi_1$ maps $\mathcal{U}$ into itself. Theorem 2 tells that a fixed point θ^* exists.
4. It can be shown that the sequence of iterates $\{\phi^{(n)}(\theta_0)\}$ will finally converge to θ^* for any given $\theta_0 \in \mathcal{U}$.