Dual Formulation of the Optimal Consumption Problem with Multiplicative Habit Formation

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Outline

- Introduction
- Duality Mechanism
- Optimal Consumption Problem
- Main Duality Result
- Relevant Implications
- Conclusion

Introduction

- Habit formation
  - Individuals growing accustomed to a certain standard/level
  - Depending on an individual's past savings/consumption decisions
  - Subsistence level or standard of living
  - Affect utility levels $\rightarrow$ consumption behaviour
- Incorporate into consumption problems
  - Adjustment of conventional preference qualification
  - Exogenously or endogenously defined habit level
  - Different implications on life-cycle investment/consumption
- Additive or multiplicative specification
  - Economic relevance
  - Mathematical complexity
  - Presence in literature

- Additive or linear habits
  - Draw utility from difference between consumption and habit
  - Force individual to always consume above habit
  - Endogeneity encumbers subsistence interpretation
  - Mathematically easy (isomorphism)

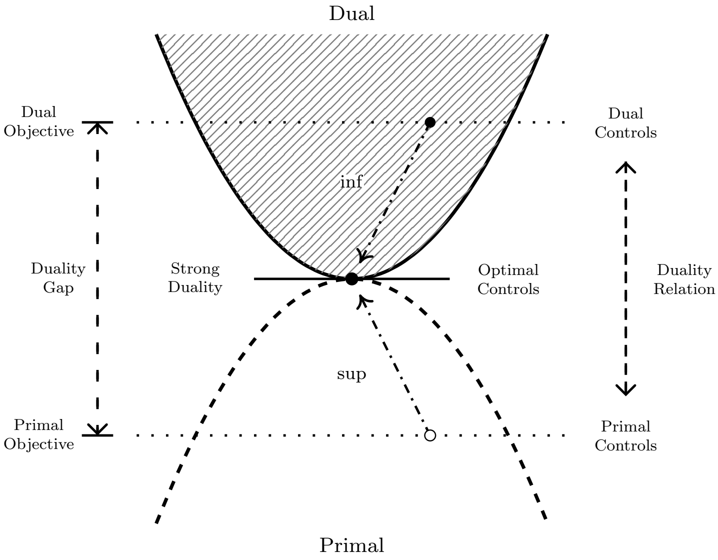
- Ratio or multiplicative habits
  - Draw utility from ratio of consumption to habit
  - Individual may consume below habit
  - Incentive to fix consumption near/above habit
  - Economically very relevant $\rightarrow$ mathematically troublesome

- In mathematical terms:
  - Additive: $U\left(c_t - h_t\right) \Rightarrow c_t > h_t$ must hold
  - Multiplicative: $U\left(c_t / h_t\right) \Rightarrow c_t / h_t > 0$ must hold

Introduction: what do we do?

- Analytical difficulties
  - Non-standard problem specification
  - Problem not strictly concave
  - Involves path-dependency
  - No closed-form solutions available
- Remedies?
  - Numerical methods: backward induction, grid search
  - Approximations: Taylor expansions, cf. van Bilsen et al. (2020)
  - Duality theory: **no** dual formulation known
- Our paper
  - Transforms non-concave problem into concave problem
  - Makes use of Fenchel Duality to derive dual formulation
  - Simultaneously proves that strong duality holds
  - Develops approximating/evaluation mechanism

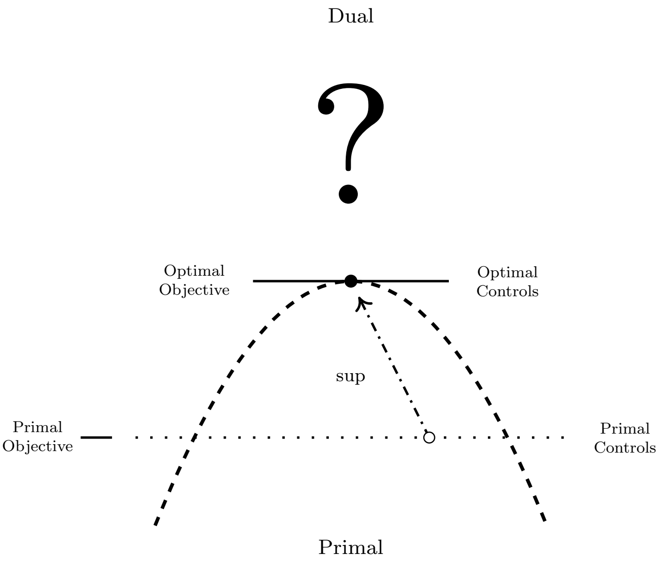
Duality Mechanism: how should we view this?



Duality Mechanism: duality explained

- Dual formulation as shadow problem
  - "Shadow": alternative to solving the primal problem
  - Typically easier to solve than primal problem
  - Conventional wisdom: allocation of resources vs. pricing of resources
  - Finance: allocation of assets vs. market prices of risk
- Optimal controls "sandwiched" between primal and dual
  - Minimising the dual $\Leftrightarrow$ maximising the primal
  - Dual renders upper bound on primal
  - Difference is called the duality gap
- Why is this so useful? Mere theoretical implications?
  - Provides alternative view on economic meaning
  - Facilitates solution techniques (Brennan and Xia (2002))
  - Applications: martingale method, super-replication, approximate methods, pricing of non-traded risk, shadow price (frictions), etc.

Duality Mechanism: situation for multiplicative habits



Optimal Consumption Problem

Primal problem

The optimal consumption problem is given by:

$$\sup_{\{c_t,\pi_t\}_{t\in[0,T]}\in\mathcal{A}_{X_0}} \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{\left(\frac{c_t}{h_t}\right)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$$

$$\text{s.t.}\quad \mathrm{d}X_t = X_t\left[\left(r_t+\pi_t^\top\sigma_t\lambda_t\right)\mathrm{d}t+\pi_t^\top\sigma_t\mathrm{d}W_t\right]-c_t\mathrm{d}t,$$

$$h_t = \exp\left\{\beta\int_0^t e^{-\alpha(t-s)}\log c_s\mathrm{d}s\right\}\ \forall\ t\in[0,T],\ X_0\in\mathbb{R}_+. \tag{1}$$

Optimal Consumption problem: difficulties

- Dependency of h_t on past consumption choices, $\{c_s\}_{s\in[0,t]}$
  - Complicated value function $\mathbb{E}\big[\int_0^T e^{-\delta t} U(c_t/h_t)\,\mathrm{d}t\big]$
  - Non-concave and cumbersome path-dependency
  - Elimination?

- Isomorphism Schroder and Skiadas (2002)
  - Re-define problem in terms of $\widehat{c}_t = \frac{c_t}{h_t}$
  - Re-situation of path-dependency
  - Relegated to static budget constraint: $\mathbb{E}\big[\int_0^T M_t \widehat{c}_t h_t \mathrm{d}t\big] \leq X_0$
  - Not very helpful

- Standard solution techniques fail to solve problem (1)
  - Carries over to applications of standard duality methods
  - $\rightarrow$ dual formulation not available (yet)

Main Duality Result: recap

- Standard duality applications
  - Make use of Legendre-Fenchel transformation
  - $V(x) = \sup_{z \in \mathbb{R}_+} \left(e^{-\delta t} U(z) - xz \right)$
  - Martingale method $\rightarrow$ derived from this transform
  - Useful inequality: $e^{-\delta t} U(x) \leq V(z) + xz$
- Legendre-Fenchel transformation **not** helpful
  - $\mathbb{E}\left[\int_0^T e^{-\delta t} U(c_t/h_t)\, \mathrm{d}t \right] \leq \mathbb{E}\left[\int_0^T V(Z_t)\, \mathrm{d}t \right] + \mathbb{E}\left[\int_0^T (c_t/h_t) Z_t \mathrm{d}t \right]$
  - Impossible to infer something about $\mathbb{E}\left[\int_0^T (c_t/h_t) Z_t \mathrm{d}t \right]$
  - Process $\int_0^T (c_t/h_t) Z_t \mathrm{d}t$ is not a positive martingale
- **Fenchel Duality**
  - Alternative to Legendre duality
  - Involves path-dependent linear transformations of controls
  - Implies dual that differs from conventional ones

Main Duality Result: Fenchel Duality

Fenchel Duality

Let $f : X \to \mathbb{R} \cup \{\infty\}$ *and* $g : Y \to \mathbb{R} \cup \{\infty\}$ *be two continuous and convex functions. Additionally, introduce the bounded linear map* $A : X \to Y$*. Here,* X *and* Y *outline two Banach spaces. Define:*

$$\begin{aligned} p^* &= \inf_{x \in X} \{f(x) + g(Ax)\} \\ d^* &= \sup_{y^* \in Y} \{-f^*(A^* y^*) - g^*(-y^*)\}, \end{aligned} \tag{2}$$

where, $f^*(x) = \sup_{z \in X} \{\langle x, z \rangle - f(z)\}$, $g^*(y) = \sup_{z \in Y} \{\langle y, z \rangle - g(z)\}$, *for all* $x \in X^*$ *and* $y \in Y^*$*. Moreover,* A^* *is the adjoint of* A*. Strong duality, i.e.* $p^* = d^*$*, holds if* $A \operatorname{dom} f \cap \operatorname{cont} g \neq \phi$

Main Duality Result: identification I

Define the following function:

$$\mathcal{J}\left(X_0, \log c_t, \eta\right) = \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{e^{[1-\gamma](\log c_t - \log h_t)}}{1-\gamma})\mathrm{d}t\right] - \eta\mathbb{E}\left[\int_0^T e^{\log c_t} M_t \mathrm{d}t\right] + \eta X_0. \tag{3}$$

Then, the optimal consumption problem can be written as:

$$\inf_{\eta \in \mathbb{R}_+} \sup_{-\log c_t \in L^2(\Omega \times [0,T])} \mathcal{J}\left(X_0, -\log c_t, \eta\right). \tag{4}$$

Note here that:

$$\log h_t = \beta \int_0^t e^{-\alpha(t-s)} \log c_s \mathrm{d}s \tag{5}$$

Main Duality Result: identification II

Recall that: $d^* = \sup_{y^* \in Y} \{-f^*(A^* y^*) - g^*(-y^*)\}$. Therefore, we have $\sup_{-\log c_t \in L^2(\Omega \times [0,T])} \mathcal{J}(X_0, -\log c_t, \eta) = d^*$, under:

$$\begin{aligned} -f^*(A^* y^*) &= \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{e^{-[1-\gamma]A^*(-\log c_t)}}{1-\gamma} \mathrm{d}t\right] \\ -g^*(-y^*) &= -\eta \mathbb{E}\left[\int_0^T e^{\log c_t} M_t \mathrm{d}t\right] + \eta X_0, \end{aligned} \tag{6}$$

where the linear map A^* is given by:

$$A^*(-\log c_t) = -\log c_t + \beta \int_0^t e^{-\alpha(t-s)} \log c_s \mathrm{d}s, \tag{7}$$

such that:

$$y^* = -\log c_t \quad \text{and} \quad Y = L^2(\Omega \times [0, T]) \tag{8}$$

Main Duality Result: identification III

Now, recall that: $p^* = \inf_{x \in X} \{f(x) + g(Ax)\}$. To be able to apply Fenchel duality, let $V(x) = x - x \log x$ and define:

$$
\begin{aligned}
f(x) &= \mathbb{E}\left[\int_0^T e^{-\delta t} \frac{1}{1-\gamma} V\left(e^{\delta t}\psi_t\right) \mathrm{d}t\right] \\
g(Ax) &= -\mathbb{E}\left[\int_0^T \eta M_t V\left(\frac{A\psi_t}{\eta M_t}\right) \mathrm{d}t\right] + \eta X_0,
\end{aligned}
\tag{9}
$$

where the bounded linear map (and adjoint of A^*) A reads:

$$
A\psi_t = \psi_t - \beta \mathbb{E}\left[\int_t^T e^{-\alpha(s-t)} \psi_s \mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \tag{10}
$$

such that:

$$
x_t = \psi_t \quad \text{and} \quad X = L^2(\Omega \times [0, T]). \tag{11}
$$

Main Duality Result: identification IV

In the sense of Fenchel Duality, it can be shown that we have $d^* = p^*$ for the following primal optimisation problem:

$$d^* = \sup_{-\log c_t \in L^2(\Omega\times[0,T])} \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{e^{[1-\gamma](\log c_t - \log h_t)}}{1-\gamma})\mathrm{d}t\right] - \eta\mathbb{E}\left[\int_0^T e^{\log c_t} M_t \mathrm{d}t\right] + \eta X_0, \tag{12}$$

and the corresponding dual problem:

$$p^* = \inf_{\psi_t \in L^2(\Omega\times[0,T])} \mathbb{E}\left[\int_0^T \left\{ e^{-\delta t}\frac{1}{1-\gamma} V\left(e^{\delta t}\psi_t\right) - \eta M_t V\left(\frac{\psi_t - \beta\mathbb{E}\left[\int_t^T e^{-\alpha(s-t)}\psi_s \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]}{\eta M_t}\right)\right\}\mathrm{d}t\right] + \eta X_0. \tag{13}$$

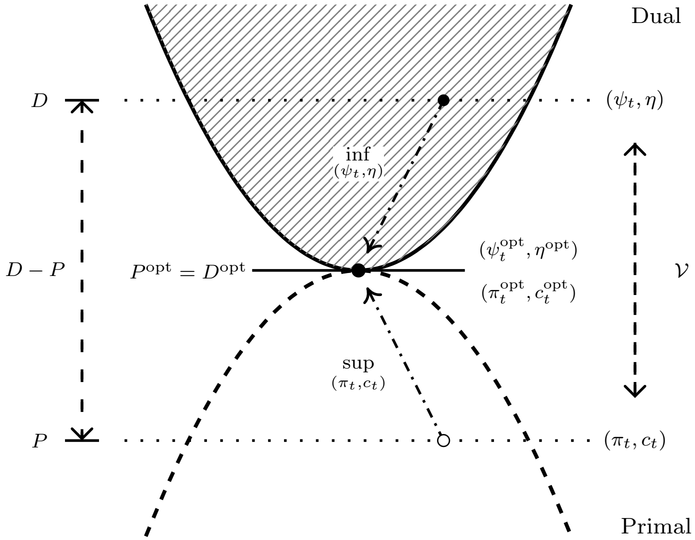
Main Duality Result

Dual problem

Define $V(x) = x - x \log x$. Then, the dual formulation of the optimal consumption problem in (1), satisfying strong duality, reads:

$$\inf_{\psi_t \in L^2(\Omega \times [0,T]), \eta \in \mathbb{R}_+} \mathbb{E}\left[\int_0^T \left\{ e^{-\delta t} \frac{1}{1-\gamma} V\left(e^{\delta t}\psi_t\right) - \eta M_t V\left(\frac{\psi_t - \beta \mathbb{E}\left[\int_t^T e^{-\alpha(s-t)}\psi_s \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]}{\eta M_t}\right)\right\} \mathrm{d}t\right] + \eta X_0. \tag{14}$$

Main Duality Result: we found it!



Relevant Implications

- Strong duality result implies:
  - Semi-analytical expressions for optimal primal and dual processes
  - Discloses the interplay between primal and dual processes
  - Opens doors to applications involving duality

- Measure accuracy of approximations
  - Grid-search routine for "optimal" solution
  - Hambel et al. (2021): routines like Bick et al. (2013)'s and Kamma and Pelsser (2021)'s more accurate
  - Utilise strong duality to measure precision

- Strong duality $\Leftrightarrow$ weak duality
  - Duality gap $\triangleq$ dual (D) - primal (P)
  - Gap grows with inaccuracy of approximation

Relevant Implications: duality relation

Duality relation

The duality relations are given by:

$$c_t^* = \frac{\psi_t - \beta \mathbb{E}\left[\int_t^T e^{-\alpha(s-t)} \psi_s \mathrm{d}s \,\middle|\, \mathcal{F}_t \right]}{\eta M_t} \quad \text{and} \quad h_t^* = c_t^* \left(e^{\delta t} \psi_t \right)^{\frac{1}{\gamma - 1}} . \tag{15}$$

Suppose that ψ_t^{opt} *defines the optimal dual control, satisfying* $c_t^* = \widehat{c}_t^* h_t^*$. *Then, optimal consumption can be characterised as:*

$$c_t^{\text{opt}} = \left(e^{\delta t} \psi_t^{\text{opt}} \right)^{\frac{1}{1-\gamma}} \exp\left\{ \frac{\beta}{1-\gamma} \int_0^t e^{-[\alpha - \beta](t-s)} \left[\log\left(e^{\delta s} \psi_s^{\text{opt}} \right) \right] \mathrm{d}s \right\} . \tag{16}$$

Relevant Implications: duality relation explained

- Technical mechanism
  - Expressions for c_t and h_t not consistent
  - Recall: $h_t = e^{\beta \int_0^t e^{-\alpha(t-s)} \log c_s \mathrm{d}s}$
  - $\rightarrow$ Consumption does **not** imply expression for h_t
  - Dual determines ψ_t in a manner such that c_t and h_t are consistent

- Economic mechanism
  - Consumption is characterised as:
    $\eta M_t c_t = \psi_t - \beta \mathbb{E}\left[\int_t^T e^{-\alpha(s-t)} \psi_s \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]$
  - Note: h_t depends on past values; cond. expectation on future values
  - Consumption today affects via h_t consumption in the future
  - Dependency of c_t on ψ_t **and** $\{\psi_s\}_{s\in(t,T]}$ resembles this (smoothing)

- Special case: no habits ($\alpha = \beta = 0$)
  - $c_t = \frac{\psi_t^{\mathrm{opt}}}{\eta^{\mathrm{opt}} M_t}$, $h_t = 1$ and $\psi_t^{\mathrm{opt}} = (\eta^{\mathrm{opt}} M_t) \left[e^{\delta t} \eta^{\mathrm{opt}} M_t\right]^{-\frac{1}{\gamma}}$

Conclusion

- Non-standard specification of problem
  - Path-dependent and concave
  - Conventional Legendre duality fails
  - Cannot cope with non-linearity and path-dependency
- Derive dual formulation
  - Transform non-concave problem into concave problem
  - Make use of Fenchel Duality
  - Proof of strong duality
- Relevant implications:
  - One step closer to closed-form expressions
  - Interplay primal and dual controls
  - Simplified applications possible $\rightarrow$ martingale method
  - Numerically friendly evaluation of approximations

References

Bick, B., Kraft, H., and Munk, C. (2013). Solving constrained consumption–investment problems by simulation of artificial market strategies. *Management Science*, 59(2):485–503.

Brennan, M. J. and Xia, Y. (2002). Dynamic asset allocation under inflation. *The Journal of Finance*, 57(3):1201–1238.

Hambel, C., Kraft, H., and Munk, C. (2021). Solving life-cycle problems with biometric risk by artificial insurance markets. *Scandinavian Actuarial Journal*, pages 1–21.

Kamma, T. and Pelsser, A. (2021). Near-optimal asset allocation in financial markets with trading constraints. *European Journal of Operational Research, forthcoming*.

Schroder, M. and Skiadas, C. (2002). An isomorphism between asset pricing models with and without linear habit formation. *The Review of Financial Studies*, 15(4):1189–1221.

van Bilsen, S., Bovenberg, A. L., and Laeven, R. J. (2020). Consumption and portfolio choice under internal multiplicative habit formation. *Journal of Financial and Quantitative Analysis*, 55(7):2334–2371.

Questions?