Hedging and optimal portfolio choice under endogenous permanent market impact

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Overview

- Introduction
- The model
- Characterization of optimality
- Examples

Introduction

Motivation

The premise of financial engineering and financial mathematics:

1. an asset price process model is exogenously given.
2. trading strategies do not affect the price process.

However:

1. a lot of derivatives are issued, for which the hedging activities explain the underlying asset movements ;
2. in the 1990s, Salomon Brothers suffered a huge loss due to volatility shrink due to delta hedging strategies. (A demon of our own design, Bookstaber (2008)).
3. ex. recent high volatility of Japanese market was attributed to the delta hedging strategy for an enormous amount of call options bought by foreign investors. (Nikkei)
4. Market crashes as the so called Flash-crash on May 6, 2010 have been attributed to such feedback effects.

Literature

Frey-Stremme (1997), Cvitanic-Ma (1996), Frey (1998):
hedging under exogenously modeled feedback.

Market impact

1. exogenously given order-flow see Garman (1976), Amihud and Mendelson (1980), Ho and Stoll (1981), Ohara and Oldfield (1986). Exogenously modeled instantaneous or temporary market impact models see Cetin et al. (2014), Fukasawa (2014), Guéant (2014)
2. transient market impact see Schied and Gatheral and the references therein
3. permanent market impact see Almgren and Chriss (1999, 2000), Guéant (2014) and Guéant and Pu (2015) among others
4. empirical analysis, Almgren et al. (2005), Tóth et al. (2016)
5. Bank and Baum (2004): In their model liquidity costs can approximately be avoided
6. closest to our work is Bank and Kramkov (2013,2014) and Anthropelos et al. (2021)

Framework

- We assume zero risk-free rates.
- Let $T > 0$ be the end of an accounting period. Each agent evaluates her utility based on her wealth at T.
- Consider n securities whose value S^i at T is exogenously determined. Denote the value by $S = (S^1, \ldots, S^n)$ and regard it as an $\mathcal{F}_T$ measurable random variable defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ generated by a d-dimensional Brownian motion W.
- The price of these securities at T is trivially S, but the price at $t < T$ should be $\mathcal{F}_t$ measurable and will be endogenously determined by an utility-based mechanism.

Agents

There are two agents in our model: Large trader and Market.

- The Market quotes a price for each volume of the securities. We have a limit order book in mind. She can be risk-averse and so her quotes can be nonlinear in volume and depend on her inventory of this securities.
- The Large trader refers to the quotes and makes a decision. She cannot avoid affecting the quotes by her trading due to the inventory consideration of the Market, and seeks an optimal strategy under this endogenous market impact.
- This setting follows the works of Stoll (1978), Ho und Stoll (1981) und Bank und Kramkov (2013,2014). This setting was also applied in Anthropelos et al. (2021).

Under a Bertrand-type competition among liquidity suppliers, the Market (a representative liquidity supplier) gives a quote according to the utility indifference principle. (c.f. Bank-Kramkov)

The representative liquidity supplier

Valuations

- In the sequel, we assume that representative liquidity supplier evaluates a payoff, X, using an evaluation method $\Pi(X)$.
- For instance, if the probabilistic model is known and the liquidity supplier through trading with other liquidity suppliers can perfectly diversify her risks, $\Pi(X)$ might be given through $\Pi(X) = \mathbb{E}_Q[X]$.
- In reality, however, this is quite a restrictive requirement: in many situations the liquidity supplier might not be able to perfectly diversify her risks and may face uncertainty about the true probabilistic model.
- In the sequel we will assume that Π is given by a convex risk measure (modulo a change of sign), see Artzner et al. (1999), Föllmer and Schied (2002), Frittelli and Gianin (2002), etc.

Convex Measures of Risk

- A convex risk measure means that a random future reward, say X, is evaluated according to

$$\Pi_t(X) = \inf_{Q \in \mathcal{Q}} \{\mathbb{E}_Q[X|\mathcal{F}_t] + c_t(Q)\},$$

  where $\mathcal{Q} = \{Q|Q \sim P\}$ is the set of probabilistic models Q that share the same null sets with a base reference model P, with each Q attaching a different probability law to the future reward X and c_t is a penalty function specifying the plausibility of the model Q.
- Models Q that have 'low' plausibility are associated with a high penalty, while models that have 'high' plausibility yield a low penalty, with $c_t(Q) = \infty$ corresponding to the case in which the model Q is considered fully implausible.
- By taking the infimum over $\mathcal{Q}$ a conservative worst-case approach occurs, also typical in (deterministic) robust optimization.

Examples

a) A first example is given by Π simply being a linear conditional expectation

$$\Pi_t(X) = \mathbb{E}^{\mathbf{Q}}[X|\mathcal{F}_t],$$

b) The more interesting example is an exponential utility :

$$\Pi_t(X) = -\frac{\log \mathbb{E}[\exp\{-\gamma X\}|\mathcal{F}_t]}{\gamma},$$

where $\gamma > 0$ is a parameter of risk-aversion. In this case c corresponds to the Kulback-Leibler divergence also called the relative entropy, see Hansen and Sargent (2008)

c) In the theory of no-arbitrage pricing, attempts have been made to narrow the no-arbitrage bounds by restricting the set of pricing kernels considered. One of these approaches is the good-deal bounds ansatz in Cochrane and Saá-Requejo (2000) and Björk and Slinko (2006) defined as

$$\Pi_t(X) = \inf_{Q \in \mathcal{M}} \mathbb{E}_Q[X|\mathcal{F}_t],$$

where $\mathcal{M}$ is the set of all pricing kernels excluding the ones which induce a too high Sharpe ratio.

- In all of our examples above this is actually an alternative representation of Π. Namely, each of these evaluation is connected, so a fixed function $g(t, z)$ which is convex in z with $g(t, 0) = 0$ such that for each suitable integrable X there exists a square-integrable predictable process $Z(X)$ such that

$$d\Pi_t(X) = g(t, Z_t(X))dt - Z_t(X)dW_t. \tag{1}$$

  (Recall that $\Pi_T(X) = X$.)

- Interpreting the right-hand-side of (1) as a non-linear conditional expectation sometimes $\Pi_t(X)$ is called a g-expectation.
- Alternatively $(\Pi(X), Z(X))$ is also called the solution of a backward stochastic differential equation (BSDE). Alternatively to (1), a BSDE is often written as

$$\Pi_t(X) = X - \int_t^T g(s, Z_s(X))ds + \int_t^T Z_s(X)dW_s.$$

- More general, One may see that any Π satisfying our assumptions is equivalent to Π being a g-expectation modulo a compactness assumption.

Examples

In our examples we obtain the following driver function g

a) in the case of Π being a conditional expectation $g(s,z) = \nu_s z$

b) in the case of Π corresponding to an exponential utility function $g(t,z) = \frac{|z|^2}{\gamma}$

c) in the case of Π corresponding to a lower good-deal bound g is the norm of a projection of z on a suitable set, and g is positively homogeneous in the sense that $g(t,\lambda z) = \lambda g(t,z)$ for $\lambda > 0$.

Conditions

We assume in the sequel that Π is given by a g-expectation with $g(t,z)$ being convex in z, $g(t,0)=0$, and that either g is Lipschitz in z or grows at most quadratically in z. Additionally, in the following we will always assume that one of the following assumptions holds:

(H1) Additive separability: suppose that g is of the form $g(t,z)=\sum_{i=1}^{d} g^i(t,z^i)$ and $S^i=s^i(F_T^i)$ with F^i being a Markov process driven by W^i satifying a standard SDE for $i=1,\dots,n$.

(H2) g is positively homogeneous meaning that $g(t,\lambda z)=\lambda g(t,z)$.

Pricing and trading

The utility indifference price

- We assume that the Market is initially endowed with a risky asset H_{M} while the investor is holding an initial endowment of H_L.

If the Market is holding $a = (a^1, \dots, a^n)$ units of the securities in question other than H_{M} as her inventory at time $t \in [0, T]$, then her utility is measured as $\Pi_t(H_{\mathrm{M}} + aS)$. According to the utility indifference principle, the Market quotes a selling price for y units of the securities by

$$\begin{aligned} P_t(a, y) :=& \inf\{p \in \mathbb{R}; \Pi_t(H_{\mathrm{M}} + aS - yS + p) \geq \Pi_t(H_{\mathrm{M}} + aS)\} \\ =& \Pi_t(H_{\mathrm{M}} + aS) - \Pi_t(H_{\mathrm{M}} + (a - y)S). \end{aligned}$$

Example: In the risk-neutral case of Example a) $P_t(a, y) = y\mathbb{E}^{\mathbf{Q}}[S|\mathcal{F}_t]$.

Nonlinear stochastic integral

Let us consider trading strategies $\mathcal{S}$.

- Denote by $\mathcal{S}_0$ the set of the simple, predictable, left-continuous processes Y with $Y_0 = 0$. (The set of the simple trading strategies.)
- The price for the y units of the securities at time t is $P_t(-Y_t, y)$. (The Market holds $-Y_t$ units due to the preceding trades.)

The profit and loss at time T associated with $Y \in \mathcal{S}_0$ is given by

$$\mathcal{I}_T(Y) := Y_T S - \sum_{0 \le t < T} P_t(-Y_t, \Delta Y_t).$$

Using definition (1) let $\Pi^y = \Pi(H_{\mathrm{M}} - yS)$ and $\mathcal{Z}^y = Z(H_{\mathrm{M}} - yS)$ for $y \in \mathbb{R}$. We set $\mathcal{Z}(\omega, t, y) = \mathcal{Z}_t^y(\omega)$.

Proposition

If (trading strategies) $\theta^n \in \Theta_0$, and $\theta^n \to \theta$ in $L^2(\mathrm{d}\mathbf{P} \times \mathrm{d}t)$ as $n \to \infty$, we have

$$L^2\text{-}\lim_{n\to\infty} \mathcal{I}_T(\theta^n) = \mathcal{I}_T(\theta) := H_{\mathrm{M}} - \Pi_0(H_{\mathrm{M}}) - \int_0^T g(t, \mathcal{Z}_t^\theta)\mathrm{d}t + \int_0^T \mathcal{Z}_t^\theta \mathrm{d}W_t$$

where $\mathcal{Z}_t^\theta(\omega) := \mathcal{Z}(\omega, t, \theta_t(\omega))$.

Motivated by the above proposition we define the set of admissible strategies as

$$\Theta := \left\{\theta : \Omega \times [0, T] \to \mathbb{R} \text{ predictable with } \mathbb{E}\left[\int_0^T |\mathcal{Z}_t^\theta|^2 \mathrm{d}t\right] < \infty\right\}.$$

$\theta_t(\omega)$ gives how much securities the trader at time t holds in the scenario ω.

Results on optimal investment

A forward-backward SDE system and utility maximization

We study the following utility maximization problem for the Large Trader

$$\max_{(\theta_t), t \in [0,T]} \mathbb{E}[U(X_T^\theta + H_{\mathrm{L}})] \tag{2}$$

where θ_t is the number of risky asset held at time t, $X_0^\theta = x_0$ is the initial capital and

$$X_T^\theta = x_0 + \mathcal{I}_T^\theta,$$

is the terminal wealth of the large investor arising from the securties corresponding to the strategy θ. Here we assume that the utility function U is a strictly increasing, concave and three times differential function satisfying Inada's condition $\lim_{x\to-\infty} U'(x) = \infty$ and $\lim_{x\to+\infty} U'(x) = 0$.

Define $\mathrm{Im}(\mathcal{Z}) = (\mathrm{Im}(\mathcal{Z}_t(\omega)))_{t\in[0,T],\omega\in\Omega}$

$$\mathrm{Im}(\mathcal{Z}_t(\omega)) = \left\{\mathcal{Z}_t^\theta(\omega)\middle|\theta \in \Theta\right\}.$$

To simplify notation we in this case define the extension of the function g beyond the image space of $\mathcal{Z}$ as

$$\bar{g}(t,\omega,z) = \begin{cases} g(t,\omega,z) & \text{if } z \in \mathrm{Im}(\mathcal{Z}_t(\omega)) \\ \infty & \text{else,} \end{cases}, \tag{3}$$

A forward-backward SDE system and utility maximization

We are now able to characterize the optimal strategy in terms of a fully-coupled forward-backward system. For that we need the following additional proposition.

Proposition

Under assumption (H1) there exists a unique $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{n+2})$-measurable function, say
$\mathcal{H}(t, X, \zeta, M) = (\mathcal{H}^1(t, X, \zeta, M^1), \ldots, \mathcal{H}^n(t, X, \zeta, M^n)) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$,
such that

$$0 \in -U'(X+\zeta)\nabla \bar{g}^i(t, \mathcal{H}^i(t, X, \zeta, M^i)) + U''(X+\zeta)\Big(\mathcal{H}^i(t, X, \zeta, M^i) + M^i\Big), i = 1, \ldots, n, \quad (4)$$

with ∇ being the gradient with respect to the second component. Equality (4) should hold for each $(\omega, t, X, \zeta, M) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$.

In the case of positive homogeneity we can define $\mathcal{H}$ similarly (and actually explicit, see below)

The next theorem gives necessary conditions for an optimal solution.

Theorem

If θ^* is an optimal strategy then it holds that

$$\mathcal{Z}_t^{\theta^*} = \mathcal{H}(t, X_t, \zeta_t, M_t),$$

where (X, ζ, M) is a triple of adapted processes which solves the (coupled) forward-BSDE(FBSDE)

$$\begin{cases} X_t &= x_0 - \sum_{i=1}^n \int_0^t g^i(s, \mathcal{H}^i(s, X_s, \zeta_s, M_s))\mathrm{d}s \\ & \quad + \sum_{i=1}^n \int_0^t \mathcal{H}^i(s, X_s, \zeta_s, M_s)\mathrm{d}W_s^i + \Pi_t(H_\mathrm{M}) - \Pi_0(H_\mathrm{M}), \\ \zeta_t &= H_L - \int_t^T M_s \mathrm{d}W_s + \frac{1}{2}\int_t^T \left\{ \sum_{i=n+1}^d \frac{U^{(3)}}{U''}(X_s + \zeta_s)| + M_s^i|^2 \right\} \mathrm{d}s \\ & \quad + \int_t^T \left\{ \sum_{i=1}^n \frac{1}{2} \frac{U^{(3)}}{U''}(X_s + \zeta_s)|\mathcal{H}^i(s, X_s, \zeta_s, M_s^i) - Z_s^i(H_\mathrm{M}) + M_s^i|^2 \right. \\ & \quad \left. + g^i(s, Z^i(H_\mathrm{M})) - g^i(s, \mathcal{H}^i(s, X_s, \zeta_s, M_s^i)) \right\} \mathrm{d}s. \end{cases} \tag{5}$$

Sufficient conditions for an optimum

Theorem

Let (X, ζ, M) be a triple of adapted processes which solves the FBSDE (5) and satisfies

$$\mathbb{E}[|U'(X_T)|^2] < \infty; \quad \mathbb{E}[U(X_T)] < \infty, \quad \mathbb{E}[\int_0^T |\mathcal{H}(t, X_t, \zeta_t, M_t)|^2 \mathrm{d}t] < \infty.$$

Assume that U'/U'' and U'''/U'' are bounded and Lipschitz continuous. Then the maximum is attained in an optimal strategy θ^* if and only if $\mathcal{H}(t, X_t, \zeta_t, M_t) \in Im(\mathcal{Z}_t)$. In this case

$$\mathcal{Z}_t^{\theta^*} = \mathcal{H}(t, X_t, \zeta_t, M_t)$$

is an optimal strategy.

Existence results & uniqueness for coupled FBSDEs

We note that since the coupled FBSDEs are quadratic (even with growth constants which are unbounded) existence results in the literature are not available. The following theorem heavily relies on the connections between the FBSDEs & the optimal control problem

Theorem

Suppose that one of the following conditions holds:

(i) *g grows at least quadratically meaning that there exists $K_1, K_2 > 0$ such that*

$$g(t, z) \geq -K_1 + K_2|z|^2.$$

(ii) *There exists an $x_0 \in \mathbb{R}$ such that $U(x) \leq K_1 - K_2|x|^2$ with $K_1, K_2 > 0$ and $U(x) = constant$ for all $x > x_0$*

(iii) *U is exponential meaning that $U(x) = a - be^{-x/\gamma}$ for $a \in \mathbf{R}$ and $b, \gamma > 0$*

Then there exists a solution to the coupled FBSDE (5). Furthermore, if g is strictly convex and U is strictly concave the solution of the coupled FBSDE is unique.

Theorem

Suppose that $V(t,x) = \text{esssup}_{\theta\in\tilde{\Theta},\theta_s,s\in[t,T]}\mathbb{E}\Big[U(x+\mathcal{I}_{t,T}(\theta))\Big|\mathcal{F}_t\Big]$ *is a regular family of semi-martinagles. Then* V *is a solution of the backward stochastic PDE*

$$V(t,x) = V(0,x) + \int_0^t \alpha(s,x)\mathrm{d}W_s + \int_0^t \mathcal{L}^V(s,x)\mathrm{d}s, \tag{6}$$

where the operator $\mathcal{L}$ *is defined by*

$$\mathcal{L}^V(t,x) := \text{esssup}_{\mathcal{Z}\in\text{Image}(\mathcal{Z}_t)}\Big(-g(t,\mathcal{Z})V_x(t,x) + \frac{1}{2}|\mathcal{Z}|^2 V_{xx}(t,x) + \mathcal{Z}\alpha_x(t,x)\Big),$$

with the essential supremum being attained in a progressively measurable function $v(t,x)$*. A strategy* $\theta^* \in \Theta$ *with* $V(t, X_t^{\theta^*})$ *being suitable integrable is optimal if and only if* $\mathcal{Z}_t^{\theta^*} = \upsilon(t, X_t^{\theta^*})$*, Finally, given the solution of the BSPDE the solution of the FBSDE can be defined in terms of the derivative of* V.

Corollary

Suppose that $g(t,z)$ is convex with quadratic growth and satisfies $g_z(t,0) = 0$ or in case of positive homogeneity $0 \in \nabla g(t,0)$. Then the optimal terminal wealth is given by $X_T^* = x_0$. This means that it is optimal for the Large Trader to invest nothing, i.e., $\mathcal{H}(t, X_t^*, \zeta_t^*) = 0$. In addition, the triple $(X_t^* = x_0, \zeta_t^* = 0, M_t^* = 0)$ is a solution of the FBSDE system (5).

Definition

A market is complete if for any H at T, a perfect replication is possible, in the sense that there exists $(a, \theta) \in \mathbb{R} \times \Theta$ such that

$$H = a + \mathcal{I}(\theta).$$

Examples

Explicit examples

We assume that the market is complete and the Market Maker evaluates the market risks in terms of an exponential utility indifferent principle under an equivalent measure $\mathbf{Q} \sim \mathbf{P}$. More precisely,

$$\Pi_t(X) = \frac{1}{\gamma} \log\Big(\mathbb{E}_{\mathbf{Q}}[e^{-\gamma X}|\mathcal{F}_t]\Big),$$

where the constant $\gamma > 0$ is the risk aversion. Assume that

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \exp\Big\{-\frac{1}{2}\int_0^T |\eta_t|^2 \mathrm{d}t - \int_0^T \eta_t \mathrm{d}W_t\Big\} := \xi_T,$$

where η is a deterministic and bounded process.
Define f as the inverse of the decreasing function $U'(x)e^{-\gamma x}\gamma^{-1}$.

Proposition

Assume that for any $\lambda > 0$, $\mathbb{E}[e^{\gamma f(\lambda \xi_T)} \xi_T] < \infty$. *The optimal terminal wealth of Problem* (2) *is then given by* $X_T^* := f(\lambda \xi_T)$, *where* λ *is determined such that the budget constraint is met. The optimal strategy can be characterized as the strategy* θ^* *that perfectly replicates the terminal optimal wealth* X_T^*, *i.e.,*

$$X_t^* = x_0 - \frac{\gamma}{2} \int_0^t |\mathcal{Z}_s^{\theta^*}|^2 \mathrm{d}s + \int_0^t \mathcal{Z}_s^{\theta^*} \mathrm{d}W_s^{\mathbf{Q}}, \quad t \in [0, T],$$

where $\mathcal{Z}_t^{\theta^*} := \frac{1}{\gamma}\left(\frac{\beta_t^*}{R_t^*} + \eta_t\right)$ *with* β_t^* *being the progressively measurable process resulting from the martingale representation* $R_t^* := \mathbb{E}[U'(X_T^*)|\mathcal{F}_t] = U'(X_T^*) - \int_t^T \beta_s^* \mathrm{d}W_s$. *Furthermore, define*

$$\zeta_t^* := I(R_t^*) - \frac{1}{\gamma} \log\left(\frac{R_t^*}{\gamma \lambda \xi_t}\right), \quad M_t^* := \frac{\beta_t^*}{U''(X_t^* + \zeta_t^*)} - \frac{1}{\gamma}\left(\frac{\beta_t^*}{R_t^*} + \eta_t\right).$$

Then the triple $(X_t^*, \zeta_t^*, M_t^*)$ *solves the FBSDE* (5) *and the optimality condition* (4) *holds.*

Proposition

Assume $U(x) = -e^{-\gamma_A x}$*, for some constant* $\gamma_A > 0$*. The optimal wealth and optimal investment strategy are then given by*

$$X_t^* = x_0 - \frac{\gamma}{2}\int_0^t |\mathcal{Z}_s^{\theta^*}|^2 \mathrm{d}s + \int_0^t \mathcal{Z}_s^{\theta^*} \mathrm{d}W_s^{\mathbf{Q}}, \quad t \in [0, T],$$

with

$$\mathcal{Z}_t^{\theta^*} := \frac{\eta_t}{\gamma + \gamma_A}.$$

Furthermore, the triple $\left(X_t^*, \zeta_t^* = \frac{1}{2(\gamma+\gamma_A)}\int_t^T |\eta_s|^2 \mathrm{d}s, M_t^* = 0\right)$ *is a solution of the FBSDE system* (5)*.*

Conclusion

- We considered a continuous-time setting with permanent endogeneous price impact induced by a change in the inventory of the market maker
- We showed that trading in such a setting corresponds to non-linear stochastic integrals
- We gave necessary and sufficient conditions for an optimal strategy
- Existence and uniqueness results for the underlying FBSDEs were given
- Some explicit examples were given

Many thanks for your attention!

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Connection to Decision Theory

- Decision-making under ambiguity, with probabilities of events unknown to the decision-maker, has been extensively studied in economics since the seminal work of Ellsberg (1961).
- Popular approaches to decision-making under ambiguity are provided by the multiple priors preferences of Gilboa and Schmeidler (1989), also referred to as maxmin expected utility, and the significant generalization of variational preferences developed by Maccheroni, Marinacci and Rustichini (2006).
- With linear utility, variational preferences reduces to a convex risk measure.
- In macroeconomics such robustness critera were pioneered by Hansen and Sargent in a series of papers.
- Such preferences induce aversion to ambiguity. A version of multiple priors was also studied by Huber (1981) in robust statistics; see also the early Wald (1950).

Definition of $\mathcal{H}$ in case of (H2)

In case that g is positively homogeneous we define
$\mathcal{H}^i(t, X, \zeta, M) = \theta(t, X, \zeta, M) Z_t(-S)$ with

$$\theta(t, X, \zeta, M) = \frac{-Z_t(-S)M + \frac{U'}{U''}(X + \zeta) g(t, Z_t(-S))}{|Z_t(-S)|^2} \tag{7}$$

if the RHS of (7) is positive, and

$$\theta(t, X, \zeta, M) = -\frac{-Z_t(S)M + \frac{U'}{U''}(X + \zeta) g(t, Z_t(S))}{|Z_t(S)|^2}, \tag{8}$$

if the RHS of (8) is negative.

Definition (regular family of semimartingales)

The process $V(t,x), t \in [0,T]$ is a regular family of semimartingales if

(a) $V(t,x)$ is twice continuously differentiable with respect to x for any $t \in [0,T]$.

(b) For any $x \in \mathbb{R}$, $V(t,x), t \in [0,T]$ is a special semimartingale with progressively measurable finite variation part $A(t,x)$ which admits the representation $A(t,x) = \int_0^t b(s,x)\mathrm{d}s$, where $b(s,x)$ is progressively measurable, i.e.,

$$V(t,x) = V(0,x) - \int_0^t b(s,x)\mathrm{d}s + \int_0^t \alpha(s,x)\mathrm{d}W_s.$$

(c) For any $x \in \mathbb{R}$, the derivative process $V_x(t,x)$ is a special semimartingale with decomposition

$$V_x(t,x) = V_x(0,x) - \int_0^t b_x(s,x)\mathrm{d}s + \int_0^t \alpha_x(s,x)\mathrm{d}W_s,$$

where α_x denotes the derivative of α with respect to x.

Theorem

The triple $(X_t^{\theta^*}, \zeta_t, M_t)$ *defined by*

$$\zeta_t = I(V_x(t, X_t^{\theta^*})) - X_t^{\theta^*}, \quad M_t = \frac{\upsilon(t, X_t^{\theta^*}) V_{xx}(t, X_t^{\theta^*}) + \alpha_x(t, X_t^{\theta^*})}{U''(X_t^{\theta^*} + \zeta_t)} - \upsilon(t, X_t^{\theta^*}),$$

is a solution of the FBSDE (5).