Completeness, arbitrage and optimal portfolio strategy in an Itô-Markov additive market

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Market model

Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be complete probability space,
- $\mathbb{T} := [0, T]$, for fixed $0 < T < \infty$,
- $J := \{J(t) : t \in \mathbb{T}\}$ be the observable and continuous-time Markov chain with a finite,canonical state space $E := \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$,
- $[\lambda_{ij}]_{i,j=1}^N$ be the intensity matrix of the Markov chain J.

Risk-free asset

We describe the dynamic of the price process of risk-free asset B as follows:

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \qquad B(0) = 1.$$

Here r is the interest rate of B and it is modulated by Markov chain J

$$r(t) := \langle \mathbf{r}, J(t) \rangle = \sum_{i=1}^{N} r_i \langle \mathbf{e}_i, J(t) \rangle,$$

where $\mathbf{r} = (r_1, \ldots, r_N)' \in \mathbb{R}_+^N$ and $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^N$. The value $r_i > 0$ represents the value of the interest rate when the Markov chain is in the state space $\mathbf{e}_i$.

Itô-Markov additive process

We define the process X as follows:

$$X(t) = \overline{\overline{X}}(t) + \overline{X}(t),$$

where

$$\overline{\overline{X}}(t) := \sum_{i=1}^{N} \Psi_i(t)$$

for

$$\Psi_i(t) := \sum_{n \geqslant 1} U_n^{(i)} \mathbf{1}_{\{J(T_n)=\mathbf{e}_i,\ T_n \leqslant t\}}$$

and for the jump epochs $\{T_n\}$ of J. Here $U_n^{(i)}$ $(n \geqslant 1, 1 \leqslant i \leqslant N)$ are independent random variables such that for every fixed i, the random variables $U_n^{(i)}$ are identically distributed. We can express the process Ψ_i as follows:

$$\Psi_i(t) = \int_0^t \int_{\mathbb{R}} x\, \Pi_U^i(\mathrm{d}s, \mathrm{d}x)$$

for the point measure

$$\Pi_U^i([0,t], \mathrm{d}x) := \sum_{n \geqslant 1} \mathbf{1}_{\{U_n^{(i)} \in \mathrm{d}x\}} \mathbf{1}_{\{J(T_n)=\mathbf{e}_i,\ T_n \leqslant t\}}, \quad i = 1, \ldots, N.$$

An Itô-Lévy process has the following decomposition:

$$\overline{X}(t) := \overline{X}(0) + \int_0^t \mu_0(s)\mathrm{d}s + \int_0^t \sigma_0(s)\mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}} \gamma(s-,x)\bar{\Pi}(\mathrm{d}s,\mathrm{d}x),$$

where W denotes the standard Brownian motion independent of J and $\bar{\Pi}(\mathrm{d}t,\mathrm{d}x) := \Pi(\mathrm{d}t,\mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure which is independent of J and W. Furthermore,

$$\mu_0(t) := \langle \boldsymbol{\mu}_0, J(t)\rangle = \sum_{i=1}^N \mu_0^i \langle \mathbf{e}_i, J(t)\rangle, \qquad \sigma_0(t) := \langle \boldsymbol{\sigma}_0, J(t)\rangle = \sum_{i=1}^N \sigma_0^i \langle \mathbf{e}_i, J(t)\rangle,$$

$$\gamma(t,x) := \langle \boldsymbol{\gamma}(x), J(t)\rangle = \sum_{i=1}^N \gamma_i(x)\langle \mathbf{e}_i, J(t)\rangle$$

for some vectors $\boldsymbol{\mu}_0 := (\mu_0^1,\dots,\mu_0^N)' \in \mathbb{R}^N$, $\boldsymbol{\sigma}_0 := (\sigma_0^1,\dots,\sigma_0^N)' \in \mathbb{R}_+^N$ and the vector-valued measurable function $\boldsymbol{\gamma}(x) := \big(\gamma_1(x),\dots,\gamma_N(x)\big)$.
A bivariate process (J,X) with above decomposition is called **Itô-Markov additive process**.

We assume the evolution of the price process of the risky asset S_0 is governed by the Itô-Markov additive process as follows:

$$\begin{cases} \mathrm{d}S_0(t) = S_0(t-)\Big[\mu_0(t)\mathrm{d}t + \sigma_0(t)\mathrm{d}W(t) + \int_{\mathbb{R}} \gamma(t-,x)\bar{\Pi}(\mathrm{d}t,\mathrm{d}x) \\ \qquad\qquad + \sum_{i=1}^{N} \int_{\mathbb{R}} x\bar{\Pi}_U^i(\mathrm{d}t,\mathrm{d}x)\Big], \\ S_0(0) = s_0 > 0. \end{cases}$$

We interpret the coefficient μ_0 as the appreciation rate and σ_0 as the volatility of the risky asset for each $i = 1, \ldots, N$.

Markovian jump securities

We define a marked point process Φ_j by

$$\Phi_j(t) := \Phi([0,t] \times \mathbf{e}_j) = \sum_{n \geqslant 1} \mathbf{1}_{\{J(T_n)=\mathbf{e}_j,\ T_n \leqslant t\}}, \quad j = 1, \ldots, N.$$

Let ϕ_j be the compensator of Φ_j, thus the process

$$\overline{\Phi}_j(t) := \Phi_j(t) - \phi_j(t), \quad j = 1, \ldots, N,$$

is a martingale and it is called the **jth Markovian jump martingale**.
The dynamics of prices of the Markovian jump securities S_j (for j = 1, . . . , N) is described as follows:

$$\begin{cases} \mathrm{d}S_j(t) = S_j(t-)\left[\mu_j(t)\mathrm{d}t + \sigma_j(t-)\mathrm{d}\overline{\Phi}_j(t)\right], \\ S_j(0) > 0, \end{cases}$$

where the appreciation rate μ_j and the volatility σ_j are determined by the Markov chain J as previously.

Markovian power-jump securities

We introduce the power-jump processes as follows:

$$X^{(k)}(t) := \sum_{0<s\leqslant t} (\Delta \overline{X}(s))^k, \qquad k \geqslant 2,$$

where $\Delta \overline{X}(s) = \overline{X}(s) - \overline{X}(s-)$. We set $X^{(1)}(t) = \overline{X}(t)$. We have

$$\mathbb{E}\left[X^{(k)}(t)\middle|\mathcal{J}_t\right] = \int_0^t \int_{\mathbb{R}} \gamma^k(s-,x)\nu(\mathrm{d}x)\mathrm{d}s < \infty,$$

$\mathbb{P} - a.e.$ for $k \geqslant 2$ and $\mathcal{J}_t := \sigma\{J(s) : s \leqslant t\}$. Hence the processes

$$\overline{X}^{(k)}(t) := X^{(k)}(t) - \int_0^t \int_{\mathbb{R}} \gamma^k(s-,x)\nu(\mathrm{d}x)\mathrm{d}s, \qquad k \geqslant 2,$$

are martingales.
The price process of Markovian kth-power-jump assets $S^{(k)}$ is described by:

$$\begin{cases} \mathrm{d}S^{(k)}(t) = S^{(k)}(t-)\left[\mu^{(k)}(t)\mathrm{d}t + \sigma^{(k)}(t-)\mathrm{d}\overline{X}^{(k)}(t)\right], \\ S^{(k)}(0) > 0, \end{cases}$$

where the coefficients are determined by the Markov chain J as previously.

Impulse regime switching securities

We define

$$\Psi_i^{(l)}(t) := \sum_{n\geqslant 1} \left(U_n^{(i)}\right)^l \mathbf{1}_{\{J(T_n)=\mathbf{e}_i,\ T_n\leqslant t\}}.$$

The compensated version of $\Psi_i^{(l)}$ is called an **impulse regime switching martingale**:

$$\overline{\Psi}_i^{(l)}(t) := \Psi_i^{(l)}(t) - \mathbb{E}\left(U_n^{(i)}\right)^l \phi_i(t).$$

We characterize the evolution of impulse regime switching securities $S_i^{(l)}$ as follows:

$$\begin{cases} \mathrm{d}S_i^{(l)}(t) = S_i^{(l)}(t-)\left[\mu_i^{(l)}(t)\mathrm{d}t + \sigma_i^{(l)}(t-)\mathrm{d}\overline{\Psi}_i^{(l)}(t)\right], \\ S_i^{(l)}(0) > 0, \end{cases}$$

where the coefficients are determined by the Markov chain J as previously.

$$
\begin{cases}
\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t, \\
\mathrm{d}S_0(t) = S_0(t-)\Big[\mu_0(t)\mathrm{d}t + \sigma_0(t)\mathrm{d}W(t) + \int_{\mathbb{R}} \gamma(t-,x)\bar{\Pi}(\mathrm{d}t,\mathrm{d}x) \\
\qquad\qquad + \sum_{i=1}^{N} \int_{\mathbb{R}} x\bar{\Pi}_U^i(\mathrm{d}t,\mathrm{d}x)\Big], \\
\mathrm{d}S_j(t) = S_j(t-)\Big[\mu_j(t)\mathrm{d}t + \sigma_j(t-)\mathrm{d}\overline{\Phi}_j(t)\Big], \\
\mathrm{d}S^{(k)}(t) = S^{(k)}(t-)\Big[\mu^{(k)}(t)\mathrm{d}t + \sigma^{(k)}(t-)\mathrm{d}\overline{X}^{(k)}(t)\Big], \\
\mathrm{d}S_i^{(l)}(t) = S_i^{(l)}(t-)\Big[\mu_i^{(l)}(t)\mathrm{d}t + \sigma_i^{(l)}(t-)\mathrm{d}\overline{\Psi}_i^{(l)}(t)\Big],
\end{cases}
$$

for $i, j = 1, \ldots, N$, $k \geqslant 2$ and $l \geqslant 1$.

Asymptotic completeness of the enlarged market

A market is said to be **complete** if each claim can be replicated by a strategy, that is, the claim can be represented as a stochastic integral with respect to the asset prices.
In the case of market models with an infinite number of assets, we define completeness in terms of approximate replication of claims.
A market is **asymptotically complete** in the sense that for every contingent claim A we can set up a sequence of finite self-financing portfolios whose final values converge to A.

Theorem 1. [Palmowski Z., Stettner L., S. A., 2019]
The enlarged Itô-Markov additive market is asymptotically complete.

We say that there is an **asymptotic arbitrage** opportunity if we have a sequence of strategies such that, for some real number $c > 0$, the value process V^n on a finite market satisfies:

- $V^n(t) \geqslant -c$ for each $0 < t \leqslant T$ and for each $n \in \mathbb{N}$,
- $V^n(0) = 0$ for each $n \in \mathbb{N}$,
- $\liminf_{n \to \infty} V^n(T) \geqslant 0$, $\mathbb{P}$-a.s,
- $\mathbb{P}\left(\liminf_{n \to \infty} V^n(T) > 0\right) > 0$.

Proposition 1. [Björk, T. and Näslund, B., 1998] If there exists a martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ then the market is asymptotic-arbitrage-free.

Let $\mathcal{L}^2(W)$ be the set of all predictable, $\{\mathcal{F}_t\}$-adapted processes ξ such that $\mathbb{E}\int_0^T \xi^2(s)\mathrm{d}s < \infty$ and $\xi \in \mathcal{L}^1(\phi_j)$ iff ξ is predictable, $\{\mathcal{F}_t\}$-adapted and satisfies $\mathbb{E}\int_0^T |\xi(s)|\lambda_j \mathrm{d}s < \infty$.

Proposition 2. [Boel, R. and Kohlmann, M., 1980]
Let $\psi_0 \in \mathcal{L}^2(W)$, $\psi_j \in \mathcal{L}^1(\phi_j)$ for all $j = 1, \dots, N$ and $\psi_j(s) > -1$. Then

$$\ell(t) := \exp\left[\int_0^t \psi_0(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t \psi_0^2(s)\mathrm{d}s - \sum_{j=1}^{N}\int_0^t \psi_j(s)\phi_j(\mathrm{d}s)\right]$$

$$\times \prod_{j=1}^{N} \prod_{\substack{J(t-)\neq J(t) \\ J(t)=\mathbf{e}_j}} (1+\psi_j(t))$$

is a non-negative local martingale. If additionally $\mathbb{E}\ell(t) = 1$ then it is a true martingale. Let $\mathbb{Q}$ be the probability measure defined by the Radon-Nikodym derivative

$$\ell(t) = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Theorem 2. [Boel, R. and Kohlmann, M., 1980] The process $\overline{X}$ under the new martingale measure $\mathbb{Q}$ has the form

$$\overline{X}^{\mathbb{Q}}(t) = \int_0^t \sigma_0(s)\mathrm{d}W^{\mathbb{Q}}(s) + \int_0^t \int_{\mathbb{R}} \gamma(s-, x)\bar{\Pi}(\mathrm{d}s, \mathrm{d}x),$$

where

$$W^{\mathbb{Q}}(t) = W(t) - \int_0^t \psi_0(s)\mathrm{d}s$$

is a standard $\mathbb{Q}$-Brownian motion.
Moreover,

$$\overline{\Phi}_j^{\mathbb{Q}}(t) = \Phi_j(t) - \int_0^t \Big(1 + \psi_j(s)\Big)\phi_j(\mathrm{d}s),$$

$$\phi_j^{\mathbb{Q}}(t) = \int_0^t \Big(1 + \psi_j(s)\Big)\phi_j(\mathrm{d}s).$$

and

$$\overline{\Psi}_j^{(l),\mathbb{Q}}(t) = \Psi_j^{(l)}(t) - \mathbb{E}\Big(U_n^{(i)}\Big)^l \phi_j^{\mathbb{Q}}(t).$$

Theorem 3. [Palmowski Z., Stettner L., S. A., 2019]
Assume that $\mu_j^j = r_j$ for all $j = 1, \ldots, N$ and

$$\begin{cases} \psi_0(t) = \dfrac{r(t-) - \mu_0(t-)}{\sigma_0(t-)}, \\ \psi_j(t) = \dfrac{r(t-) - \mu_j(t-)}{\sigma_j(t-)\lambda_j(t)}, \quad j = 1, \ldots, N. \end{cases}$$

Then, the discounted price processes of the securities, in the enlarged Itô-Markov additive market, are martingales under $\mathbb{Q}$ and this market is asymptotic-arbitrage-free.

We denote a portfolio strategy by

$$\pi(t) = (\pi_0(t), \pi_1(t), \ldots, \pi_N(t), \pi^{(2)}(t), \ldots, \pi_1^{(1)}(t), \pi_2^{(1)}(t), \ldots).$$

The wealth process R_π^K for the first K assets is governed by:

$$\frac{\mathrm{d}R_\pi^K(t)}{R_\pi^K(t-)} := \Bigg(r(t) + \sum_{j=0}^{N} \pi_j(t)\big(\mu_j(t) - r(t)\big) + \sum_{k=2}^{K} \pi^{(k)}(t)\big(\mu^{(k)}(t) - r(t)\big)$$

$$+ \sum_{i=1}^{N}\sum_{l=1}^{K} \pi_i^{(l)}(t)\big(\mu_i^{(l)}(t) - r(t)\big)\Bigg)\mathrm{d}t + \pi_0(t)\sigma_0(t-)\mathrm{d}W(t)$$

$$+ \int_{\mathbb{R}} \Bigg(\pi_0(t)\gamma(t-,x) + \sum_{k=2}^{K} \pi^{(k)}(t)\sigma^{(k)}(t-)\gamma^k(t-,x)\Bigg)\bar{\Pi}(\mathrm{d}t,\mathrm{d}x)$$

$$+ \sum_{i=1}^{N}\int_{\mathbb{R}} \Bigg(x\pi_0(t) + \sum_{l=1}^{K} x^l \pi_i^{(l)}(t)\sigma_i^{(l)}(t-)\Bigg)\bar{\Pi}_U^i(\mathrm{d}t,\mathrm{d}x)$$

$$+ \sum_{j=1}^{N} \pi_j(t)\sigma_j(t-)\mathrm{d}\overline{\Phi}_j(t).$$

Let U denote a utility function of the investor.
Then the value function of the investor's portfolio selection problem is defined by

$$V(t, z, \mathbf{e}_i) := \sup_{\pi \in \mathcal{A}} V^{\pi}(t, z, \mathbf{e}_i) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_{t,z,i}\left[U(R_\pi(T))\right]$$

where $\mathbb{E}_{t,z,i}$ is the conditional expectation given $R_\pi(t) = z$ and $J(t) = \mathbf{e}_i$ under $\mathbb{P}$.

Logarithmic utility $U(z) = \log(z)$

Theorem 4. The optimal portfolio strategy for the portfolio selection problem with logarithmic utility function of wealth satisfy following equations:

$$
\begin{aligned}
r(t-) - \mu_0(t-) &= \pi_0^\star(t)\sigma_0^2(t-) + \sum_{i=1}^{N} \frac{r(t-) - \mu_i^{(1)}(t-)}{\sigma_i^{(1)}(t-)} + \int_{\mathbb{R}} \gamma(t-,x)\Bigg(\bigg(1 \\
&\quad + \pi_0^\star(t)\gamma(t-,x) + \sum_{k=2}^{\infty} \pi^{(k)\star}(t)\sigma^{(k)}(t-)\gamma^k(t-,x)\bigg)^{-1}\Bigg)\nu(\mathrm{d}x), \\
\pi_j^\star(t) &= \frac{\mu_j(t-) - r(t-)}{\big(r(t-) - \mu_j(t-)\big)\sigma_j(t-) + \lambda_j(t)\sigma_j^2(t-)}, \\
\frac{r(t-) - \mu^{(k)}(t-)}{\sigma^{(k)}(t-)} &= \int_{\mathbb{R}} \gamma^k(t-,x)\Bigg(\bigg(1 + \pi_0^\star(t)\gamma(t-,x) \\
&\quad + \sum_{k=2}^{\infty} \pi^{(k)\star}(t)\sigma^{(k)}(t-)\gamma^k(t-,x)\bigg)^{-1} - 1\Bigg)\nu(\mathrm{d}x), \\
\frac{r(t-) - \mu_i^{(l)}(t-)}{\sigma_i^{(l)}(t-)} &= \int_{\mathbb{R}} x^l\Bigg(\bigg(1 + x\pi_0^\star(t) + \sum_{l=1}^{\infty} \pi_i^{(l)\star}(t)\sigma_i^{(l)}(t-)x^l\bigg)^{-1} - 1\Bigg)\eta(\mathrm{d}x).
\end{aligned}
$$

Theorem 5. The optimal portfolio strategy for the portfolio selection problem with power utility function of wealth satisfy following equations:

$$r(t-) - \mu_0(t) = (\alpha - 1)\pi_0^\star(t)\sigma_0^2(t-) + \sum_{i=1}^{N} \frac{\mu_i^{(1)}(t-) - r(t-)}{\sigma_i^{(1)}(t-)} + \int_{\mathbb{R}} \Bigg(\bigg(\pi_0^\star(t) + \sum_{k=2}^{\infty} \pi^{(k)\star}(t)\sigma^{(k)}(t-)\gamma^k(t-,x) \bigg)^{\alpha-1} - 1 \Bigg) \nu(\mathrm{d}x),$$

$$\pi_j^\star(t) = \frac{\left(1 - \frac{\mu_j(t-)-r(t-)}{\lambda_i(t)\sigma_j(t-)}\right)^{\frac{1}{\alpha-1}} - 1}{\sigma_j(t-)},$$

$$r(t-) - \mu^{(k)}(t) = \int_{\mathbb{R}} \sigma^{(k)}(t-)\gamma^k(t-,x) \Bigg(\bigg(1 + \pi_0^\star(t)\gamma(t-,x) + \sum_{k=2}^{\infty} \pi^{(k)\star}(t)\sigma^{(k)}(t-)\gamma^k(t-,x) \bigg)^{\alpha-1} - 1 \Bigg) \nu(\mathrm{d}x),$$

$$r(t-) - \mu_i^{(l)}(t) = \int_{\mathbb{R}} \sigma_i^{(l)}(t-)x^l \Bigg(\bigg(\sum_{l=1}^{\infty} \pi_i^{\star(l)}(t)x^l\sigma_i^{(l)}(t-) \bigg)^{\alpha-1} - 1 \Bigg) \lambda_i(t)\eta(\mathrm{d}x).$$

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Thank you for your attention!