20th Winter School on Mathematical Finance

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Constructing & disintegrating p-ty measures. Couplings & transports

Conventions

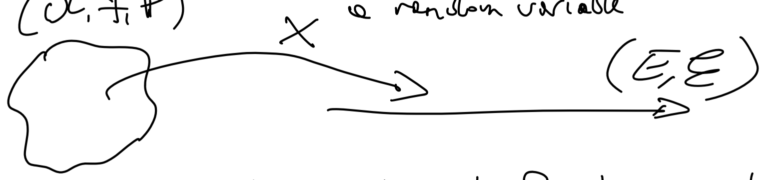
A generic p-ty space will be denoted $(\Omega, \mathcal{F}, \mathbb{P})$. Here $\mathcal{F}$ is a σ-algebra and is not completed, unless specified.

Most of the time, I will work with complete separable metric spaces (Polish spaces). There will always be endowed with their Borel σ-algebra & typically denoted $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, $(\mathcal{Y}, \mathcal{B}(\mathcal{Y}))$ etc. Note that $\mathcal{B}(\mathcal{X})$ is countably generated & (taking finite intersections of complements of the generating open sets $= \mathcal{C}$) countably determined (i.e. μ, ν on $\mathcal{B}(\mathcal{X})$ then $\mu = \nu$ iff $\mu = \nu$ on $\mathcal{C}$).

It follows that $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) = \mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

Pushforward (image) measure

$(\Omega, \mathcal{F}, \mathbb{P})$ — X a random variable, i.e., $X^{-1}(A) \in \mathcal{F}$ $\forall A \in \mathcal{E}$



measurable space

then X naturally pushes $\mathbb{P}$ onto a p-ty measure on $\mathcal{E}$ often called the distribution of X, denoted $\mathcal{L}(X) = \mathbb{P} \circ X^{-1} =: X_{\#}\mathbb{P}$. $\quad \mathcal{L}(X)(A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$, $A \in \mathcal{E}$

Marginals are the pushforward measures of projections on a product space.

Say $(\mathcal{X} \times \mathcal{Y}, \pi)$ is a p-ty space. Let $\mathrm{proj}_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, $(x, y) \to x$ be the projection on the first coordinate (& likewise $\mathrm{proj}_{\mathcal{Y}}$). Then

$\mu := \mathrm{proj}_{\mathcal{X}\#}\pi$ is called the first marginal of π & $\nu := \mathrm{proj}_{\mathcal{Y}\#}\pi$ is the 2nd one.

This is equivalent to saying (i) $\mu(A) = \pi(A \times \mathcal{Y})$ $\quad \forall A \in \mathcal{B}(\mathcal{X})$

$\nu(B) = \pi(\mathcal{X} \times B)$ $\quad B \in \mathcal{B}(\mathcal{Y})$

(ii) $\int_{\mathcal{X}\times\mathcal{Y}} (\varphi(x)+\psi(y))\, d\pi(dx,dy) = \int_{\mathcal{X}} \varphi\, d\mu + \int_{\mathcal{Y}} \psi\, d\nu$ $\forall$ measurable ≥ 0 functions φ on $\mathcal{X}$ & ψ on $\mathcal{Y}$!

|| this really is

$\int_{\mathcal{X}\times\mathcal{Y}} (\tilde{\varphi}+\tilde{\psi})\, d\pi$, for $\tilde{\varphi}(x,y)=\varphi(x)$, $\tilde{\psi}(x,y)=\psi(y)$)

Fubini - constructing measures

Def A p-ty kernel on $\mathcal{X}\times\mathcal{Y}$ is a mapping $\theta : \mathcal{X}\times\mathcal{B}(\mathcal{Y}) \to [0,1]$ such that

- for each $x\in\mathcal{X}$, $\theta(x,\cdot)$ is a p-ty measure on $\mathcal{B}(\mathcal{Y})$
- for each $B\in\mathcal{B}(\mathcal{Y})$, $\theta(\cdot,B)$ is measurable

Thm (Fubini) Let $(\Omega,\mathcal{F}) = (\mathcal{X}\times\mathcal{Y}, \mathcal{B}(\mathcal{X})\times\mathcal{B}(\mathcal{Y}))$, μ be a p-ty measure on $\mathcal{X}$ & θ a p-ty kernel. Then there exists a unique p-ty measure π on $(\Omega,\mathcal{F})$ s.t.

$$\pi(A\times B) = \int_A \mu(dx)\, \theta(x,B), \quad \forall A\in\mathcal{B}(\mathcal{X}),\ B\in\mathcal{B}(\mathcal{Y})$$

For a non-negative r.v. f on Ω, the function $x\mapsto \int_{\mathcal{Y}} f(x,y)\,\theta(x,dy)$ is measurable &

$$\int f\, d\pi = \int_{\mathcal{X}} \mu(dx) \int_{\mathcal{Y}} f(x,y)\,\theta(x,dy)$$

Rk Note that $\pi(A\times\mathcal{Y}) = \int_A \mu(dx)\, \underbrace{\theta(x,\mathcal{Y})}_{=1} = \mu(A)$ so $\mathrm{proj}_{\mathcal{X}\#}\pi = \mu$.

On the other hand, $\nu := \mathrm{proj}_{\mathcal{Y}\#}\pi$ is given by $\nu(B) = \pi(\mathcal{X}\times B) = \int_{\mathcal{X}} \mu(dx)\,\theta(x,B)$.

Rk A special case is when $\theta(x,\cdot) = \nu(\cdot)$ is a fixed p-ty measure on $\mathcal{Y}$.
π is then called the product measure & denoted $\mu\otimes\nu$.

Couplings & Transports

Def (Coupling) Let $(\mathcal{X},\mu)$, $(\mathcal{Y},\nu)$ be two p-ty measures. A coupling of μ,ν is a p-ty space $(\Omega,\mathcal{F},\mathbb{P})$ with two r.variables $X:\Omega\to\mathcal{X}$ & $Y:\Omega\to\mathcal{Y}$ s.t.
$X_\#\mathbb{P} = \mu$ & $Y_\#\mathbb{P} = \nu$.

Rk We sometimes refer to $\pi = (X,Y)_\#\mathbb{P}$ (which is a p-ty measure on $\mathcal{X}\times\mathcal{Y}$) as the coupling.

The set of such measures is denoted $\Pi(\mu,\nu) = \{\pi\in\mathcal{P}(\mathcal{X}\times\mathcal{Y}) : \mathrm{proj}_{\mathcal{X}\#}\pi = \mu,\ \mathrm{proj}_{\mathcal{Y}\#}\pi = \nu\}$

Rk If no further requirements are given, $(\mathcal{X}\times\mathcal{Y},\ \mu\otimes\nu)$ with $X=\mathrm{proj}_{\mathcal{X}},\ Y=\mathrm{proj}_{\mathcal{Y}}$ gives the independent coupling. So couplings always exists, i.e. $\Pi(\mu,\nu)\neq\emptyset$.

Def A coupling (X,Y) is said to be deterministic if $Y=T(X)$ for some measurable function $T:\mathcal{X}\to\mathcal{Y}$.

This is equivalent to saying that $Y=T(X)$ and $T_{\#}\mu=\nu$.

Or that the distribution of (X,Y) is concentrated on the graph of a measurable function T

Or that T provides a change of variables: $\displaystyle\int_{\mathcal{Y}}\varphi(y)\,\nu(dy)=\int_{\mathcal{X}}\varphi(T(x))\,\mu(dx)\quad\forall\,\varphi\geq 0$

Or that $\mathcal{L}((X,Y))=\pi=(\mathrm{Id},T)_{\#}\mu$.

Such a map T is called a (Monge) transport.

Rk Given $\pi=\mathcal{L}((X,Y))$, such T above is unique μ-a.s. While $\Pi(\mu,\nu)\neq\emptyset$, a transport map may fail to exist (e.g. $\mu=\delta_{\{0\}}$, $\nu=\frac{1}{2}(\delta_{-1}+\delta_{1})$).

Examples: (The increasing rearrangement on $\mathbb{R}$)

For a p-ty measure μ on $\mathbb{R}$, let $F_\mu(x)=\mu((-\infty,x])$ & F_μ^{-1} be its right-cont. inverse $F_\mu^{-1}(t)=\inf\{x\in\mathbb{R}: F_\mu(x)>t\}$. Then on if $X\sim \mathrm{Unif}[0,1]$, $Y=F_\mu^{-1}(X)\sim\mu$ i.e. $(\mathrm{Id},F_\mu^{-1})_{\#}\mathrm{Leb}|_{[0,1]}\in\Pi(\mathrm{Unif}[0,1],\mu)$ & F_μ^{-1} is a transport map.

If μ has no atoms then the reverse works: $X\sim\mu$ gives $Y=F_\mu(X)\sim\mathrm{Unif}[0,1]$ & hence
$$\left(X,\ F_\nu^{-1}(F_\mu(X))\right)$$
is a coupling of (μ,ν).

- Optimal coupling / transport

Consider a cost function $c:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ & look for

$$\inf_{T:\,T_{\#}\mu=\nu}\int_{\mathcal{X}}c(x,T(x))\,\mu(dx)\quad\text{or}\quad\inf_{\pi\in\Pi(\mu,\nu)}\int_{\mathcal{X}\times\mathcal{Y}}c(x,y)\,d\pi=:\mathcal{P}(\mu,\nu)=\mathcal{P}_c(\mu,\nu).$$

transport maps / plans

$\|\mu-\nu\| = \sup |\mu(A)-\nu(A)| = \inf \int 1 \, d\pi$ [proof to follow when we work on

Disecting measures (disintegration)

We saw above how to use μ & a kernel Θ to obtain a p-ty measure on $\mathcal{X}\times\mathcal{Y}$. We now want to reverse this procedure. This is called disintegration in analysis & regular conditioning in p-ty.

Consider $(\Omega,\mathcal{F},\mathbb{P})$ & a sub-σ-algebra $\mathcal{G}$. We know that conditional expectations exist so that $\forall_{A\in\mathcal{F}}$ $\mathbb{E}[\mathbb{1}_A \mid \mathcal{G}]$ is a $\mathcal{G}$-measurable r.v. It is defined $\mathbb{P}$-a.s., i.e. outside of some $\mathbb{P}$-null set ... which may depend on A. As $A\in\mathcal{F}$ varies I could easily be left with no $\omega\in\Omega$ on which my objects are jointly identified.

Def $\Theta:\Omega\times\mathcal{F}\to[0,1]$ is called a regular conditional p-ty for $\mathcal{F}$ given $\mathcal{G}$, if

- $\forall_{A\in\mathcal{F}}$, $\Theta(\cdot,A):\Omega\to[0,1]$ is $\mathcal{G}$-measurable ; • $\forall_{\omega\in\Omega}$ $\Theta(\omega,\cdot)$ is a p-ty measure on $(\Omega$
- $\forall_{A\in\mathcal{F}}$, $\Theta(\cdot,A)=\mathbb{E}[\mathbb{1}_A \mid \mathcal{G}]$ $\mathbb{P}$-a.s.

Thm If Ω is a complete separable metric space & $\mathcal{F}=\mathcal{B}(\Omega)$, then a regular cond. p-ty for $\mathcal{F}$ given $\mathcal{G}$ exists & is unique. Furthermore, if $\mathcal{H}\subseteq\mathcal{G}$ is a countably determined σ-algebra then $\exists\, N\in\mathcal{G}$, $\mathbb{P}(N)=0$,

$$\Theta(\omega,A)=\mathbb{1}_A(\omega)\quad,\ \forall_{A\in\mathcal{H}}\ ,\ \forall_{\omega\in\Omega\setminus N}.$$

$\Rightarrow$ If X is a $\mathcal{G}$-m. r.v. taking values in $(\mathcal{X},\mathcal{B}(\mathcal{X}))$, $\mathcal{H}=\sigma(X)$ then

$$\Theta(\omega,\{\omega'\in\Omega: X(\omega)=X(\omega')\})=1\quad \mathbb{P}\text{-a.s.}$$

Rk This is often stated for $\mathcal{G}=\sigma(\xi)$ & then "$\Theta(x,A)=\mathbb{E}[\mathbb{1}_A \mid \xi=x]$" & $\Theta: S\times\mathcal{F}\to[0,1]$ for $\xi:\Omega\to S$.

/See KS (ch 5) & Parthasarathy ('67).

Let us apply this to the particular case of $(\Omega,\mathcal{F})=(\mathcal{X}\times\mathcal{Y},\ \mathcal{B}(\mathcal{X})\otimes\mathcal{B}(\mathcal{Y}))$. Let $\pi=\mathbb{P}$ & $\mathcal{G}=\mathcal{H}=\sigma(\mathrm{proj}_\mathcal{X})$. Note that $\Theta(\cdot,A)$ being $\mathcal{G}$-measurable means it is a function composition $\Theta(\cdot,A)=f(\mathrm{proj}_\mathcal{X}(\omega))$, i.e., is $\mathcal{B}(\mathcal{X})$-measurable.

By restricting $\Theta(x,\cdot)$ to $\mathcal{B}(\mathcal{Y})$ we obtain a p-ty kernel (still denoted Θ) s.t.

$$\Theta(x,B)=\mathbb{E}_\pi[\mathbb{1}_B \mid \mathcal{G}](x)\quad d\pi\text{-}x\text{ a.e.}\quad ,\text{or}$$

$$\forall_{A\in\mathcal{G}}\quad \int_{A\times\Omega}\Theta(x,B)\,d\pi=\int_{A\times\Omega}\mathbb{1}_B\,d\pi=\int_{A\times B}d\pi$$

$\mu : \mathcal{B}(\mathcal{X}) \to [0,1]$ & $\Theta : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \mapsto [0,1]$ uniquely determine π on $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$.

We may write $\pi = \mu \otimes \Theta$ & say this is the disintegration of π along its first marginal.

<u>Example</u>. (Martingale couplings) We say that a coupling (X, Y) of (μ, ν), p-ties on $\mathbb{R}$, is a martingale coupling if $\mathbb{E}[Y \mid \sigma(X)] = X$ a.s. & we write $\mathcal{L}((X,Y)) \in \mathcal{M}(\mu, \nu)$.

Let $\pi = \mathcal{L}((X,Y))$. Note that $\mathbb{E}[Y \mid \sigma(X)](x) = \int_{\mathcal{Y}} y \, \Theta(x, dy)$ $\mu(dx)$-a.e.

Indeed, by measurability of Θ, $x \mapsto \int y \, \Theta(x, dy)$ is Borel-measurable & $\forall A \in \mathcal{B}(\mathbb{R})$

$$\mathbb{E}[\mathbb{1}_{X \in A} Y] = \iint \mathbb{1}_A(x)\, y \, d\pi = \int_A \mu(dx) \int_{\mathbb{R}} y \, \Theta(x, dy) \quad \text{as required.}$$

So the coupling is a martingale one <u>iff</u> $\int_{\mathbb{R}} y \, \Theta(x, dy) = x$ $\mu(dx)$-a.e., i.e. the kernel Θ is barycentre-preserving.

<u>Example</u> (Knothe-Rosenblatt rearrangement in $\mathbb{R}^n$). Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ with $\mu \ll \mathrm{Leb}$.

1) Let $\mu_1 = \mathrm{proj}_1 \# \mu$ & $y_1 = T_1(x_1)$ for $T_1 = F^{-1}_{\nu_1} \circ F_{\mu_1}$. This gives a coupling of (μ_1, ν_1)

2) Let $\mu_{12} = \mathrm{proj}_{12} \# \mu$ & $\mu_{12} = \mu_1 \otimes \Theta_1$, $\nu_{12} = \nu_1 \otimes \eta_1$. Let $y_2 = T_2(x_2; x_1)$ where $T_2 = F^{-1}_{\eta_1(y_1; \cdot)} \circ F_{\Theta_1(x_1; \cdot)}$

i.e. for x_1 (& y_1) fixed, we transport the cond distrib'n $\mathcal{L}(x_2 \mid x_1) \rightsquigarrow \mathcal{L}(y_2 \mid y_1)$.

3) Let $\mu_{13} = \mu_{12} \otimes \Theta_2$, $\nu_{13} = \nu_{12} \otimes \eta_2$ ... etc. $\rightsquigarrow$ map T.

Note that Jacobian matrix of this change of variables T is upper triangular with positive entries on the diagonal.

<u>Example</u> (Gluing) Let $(\mathcal{X}_i, \mu_i)$ $i = 1,2,3$ be Polish p-ty spaces & $\pi_{12} \in \Pi(\mu_1, \mu_2)$ & $\pi_{23} \in \Pi(\mu_2, \mu_3)$.

Then there exists $\pi \in \Pi(\mu_1, \mu_2, \mu_3)$ with $\mathrm{proj}_{12} \# \pi = \pi_{12}$ & $\mathrm{proj}_{23} \# \pi = \pi_{23}$.

<u>Proof</u> (sketch). Disintegrate $\pi_{12} = \Theta_{12} \otimes \mu_2$, $\pi_{23} = \mu_2 \otimes \Theta_{23}$ & glue along their common marginal:

$$\pi(dx_1, dx_2, dx_3) = \Theta_{12}(x_2, dx_1)\, \mu_2(dx_2)\, \Theta_{23}(x_2, dx_3).$$

$\boxtimes$

Remarks on the literature / sources:

As advertised, these notes follow & borrow from

- Villani '03 "Topics in Optimal Transportation"
- Villani '09 "Optimal transport. Old and New."
- Santambrogio '15 "Optimal Transport for Applied Mathematicians"

these are all wonderful books!