Implied risk-neutral default probabilities via conic finance

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Outline

Research question and main result

- Relate risk-neutral default probabilities and credit default swap (CDS) quotes in a two-price economy:
  - Extract risk-neutral default probabilities from bid and ask CDS quotes directly;
  - Work based on [Michielon, Khedher & Spreij, 2021a].
- This is achieved by describing the dynamics of the default process in an economy under conic finance assumptions [Cherny & Madan, 2010];
- Unique solution is guaranteed under non-restrictive assumptions;
- Requires (recursively) solving constrained non-linear systems in two equations and two unknowns.

The product

A CDS:

- Involves two parties, i.e., a *protection buyer* and a *protection seller*;
- The protection seller commits to compensate for the (potential) loss of the counterparty in the case of a credit event, within a given time frame, for a reference entity;
- In exchange, the protection buyer commits to recurring payments until the contract expires or a credit event occurs.

CDS contracts are standardized. The present value of the contract is

$$\mathrm{PV}^{\mathsf{CDS}} = \mathbb{E}^{\mathbb{Q}}\left(\sum_i \mathrm{DCF}_i(\tau)\right),$$

where τ is the random default time.

Standard setup for default probabilities

Reduced-form models common in practice:

- Explicitly specify the functional form of the distribution function of τ which depends on a set of parameters;
- Calibrate the parameters to replicate observed market quantities of interest.

A common way to define the term structure of the default probability, as in the ISDA framework, by:

$$\mathbb{Q}(\tau \leq t) := 1 - e^{-\int_0^t \lambda(s)\, ds},$$

with $\lambda : [0, +\infty) \to (0, +\infty)$ the *hazard rate* (instantaneous default intensity). Given a certain number of quoted CDSs, the hazard rate is assumed to be piecewise-constant (minimal set of assumptions possible).

Bid-ask pricing with distorted expectations I

For a contingent claim X:

- The *ask price* $a(X)$ is the price the market accepts for X, i.e., we can buy X by paying $a(X)$;
- The *bid price* $b(X)$ is the price the market pays for X, i.e., we can sell X by accepting $b(X)$.

For a given underlying asset we assume to know the dynamics under a pricing measure $\mathbb{Q}$. By lowering (increasing) the likelihood of the gains and/or increasing (decreasing) that of losses one would obtain lower (higher) prices, like the bid and the ask.

Bid-ask pricing with distorted expectations II

The risk-neutral price of X can be written, up to discounting, as

$$\mathbb{E}^{\mathbb{Q}}(X) = \int_{-\infty}^{0} \mathbb{Q}(X \geq t) - 1\, dt + \int_{0}^{+\infty} \mathbb{Q}(X \geq t)\, dt.$$

Denote with $\psi(\,\cdot\,)$ a concave function from $[0,1]$ to $[0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$.

$$\mathbb{E}^{\psi(\mathbb{Q})}(X) = \int_{-\infty}^{0} \psi(\mathbb{Q}(X \geq t)) - 1\, dt + \int_{0}^{+\infty} \psi(\mathbb{Q}(X \geq t))\, dt. \qquad (1)$$

(1) is a Choquet integral [Choquet, 1953], [Denneberg, 1994, Sec. 5]. Expressions as (1) are connected to the concepts of coherent risk measure and acceptability index; see [Madan & Schoutens, 2016, Sec. 4].

Bid-ask pricing with distorted expectations III

One obtains that an ask price can be constructed as

$$\mathrm{ask}(X) = \mathrm{DF}(T) \cdot \mathbb{E}^{\psi(\mathbb{Q})}(X).$$

Similarly, for the bid price it holds that

$$\mathrm{bid}(X) = -\mathrm{DF}(T) \cdot \mathbb{E}^{\psi(\mathbb{Q})}(-X).$$

In general, it holds that

$$\mathrm{ask}(X + Y) \leq \mathrm{ask}(X) + \mathrm{ask}(Y)$$

and

$$\mathrm{bid}(X + Y) \geq \mathrm{bid}(X) + \mathrm{bid}(Y).$$

Bid-ask pricing with distorted expectations IV

To reproduce observed bid and ask market prices, one needs to have an "operational" distortion which, depending on a parameter, allows to reproduce the distance between the risk-neutral and the observed market quantities. Consider an increasing family of distortion function $(\psi_\gamma)_{\gamma \geq 0}$. For a given level γ, the γ-dependent ask price is

$$\mathrm{ask}_\gamma(X) = \mathrm{DF}(T) \cdot \mathbb{E}^{\psi_\gamma(\mathbb{Q})}(X).$$

It also results that

$$\mathrm{bid}_\gamma(X) = -\mathrm{DF}(T) \cdot \mathbb{E}^{\psi_\gamma(\mathbb{Q})}(-X).$$

CDS calibration in a two-price economy

At every CDS maturity we need to solve a system of the form

$$\begin{cases} \text{bid}^{\text{CDS}}(\lambda, \gamma) = b \\ \text{ask}^{\text{CDS}}(\lambda, \gamma) = a \end{cases}, \tag{2}$$

with

$$b < \mathrm{PV}^{\text{CDS}}(\lambda) < a.$$

λ represents the *implied hazard rate* for the selected maturity, and γ the corresponding *implied liquidity* in the sense of [Corcuera et al. 2012].
Important: $\mathrm{PV}^{\text{CDS}}(\lambda)$ strictly increasing with respect to λ.

Solution

Assumption 1

The inequalities $\inf_{\lambda>0} \mathrm{PV}^{\mathrm{CDS}}(\lambda) < b$ *and* $\sup_{\lambda>0} \mathrm{PV}^{\mathrm{CDS}}(\lambda) > a$ *hold.*

I.e., we can calibrate the risk-neutral parameter λ to match bid and ask market quotes. This means there exists $[\lambda_b, \lambda_a]$ such that there is equivalence between $b \leq \mathrm{PV}^{\mathrm{CDS}}(\lambda) \leq a$ iff $\lambda \in [\lambda_b, \lambda_a]$.

Assumption 2

For every $\lambda \in [\lambda_b, \lambda_a]$ *there exists* $\gamma > 0$ *such that*
$ask(\lambda, \gamma) - bid(\lambda, \gamma) = a - b$.

I.e., given λ in $[\lambda_b, \lambda_a]$ we can reproduce the bid-ask spread observed.

Theorem

Under Assumptions 1 and 2, there exists a solution of the constrained non-linear system (2)*, and it is unique.*

Example I

In practice:

- Given a partition of Ω as $\bigcup_{i=1}^{M} A_i$ choose, for every i, $x_i \in X(A_i)$, with $x_1 \leq \ldots \leq x_M$;
- Approximate Choquet expectations, see [Wang and Klir, 2009, Sec. 11.5], as

$$\sum_{i=1}^{M}(x_i - x_{i-1}) \cdot \mu\left(\bigcup_{k=i}^{M} A_k\right), \tag{3}$$

  where $x_0 := 0$.

In the case of a CDS, one can then set a grid

$$A_1 := \{\tau \in [0, d_1]\}, \ldots, A_M := \{\tau \in [d_{M-1}, d_M]\},$$

where M is the total number of points (i.e., dates) in the grid, and x_i the sum of the accrued cashflows, assuming $\tau = d_i$, deferred to maturity. Then, apply (3) to $\mathrm{ask}(X) = \mathrm{ask}(X^+) - \mathrm{bid}(X^-)$ and $\mathrm{bid}(X) = \mathrm{bid}(X^+) - \mathrm{ask}(X^-)$.

Example II

Two common choices for the family of distortion function are the *minmaxvar* distortion [Cherny & Madan, 2009]

$$\psi_\gamma(x) \coloneqq 1 - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma}, \tag{4}$$

and the *Wang* distortion [Wang, 2000]

$$\psi_\gamma(x) \coloneqq \Phi\left(\Phi^{-1}(x) + \gamma\right), \tag{5}$$

with $\Phi(\,\cdot\,)$ the cumulative distribution function of a standard normal random variable.

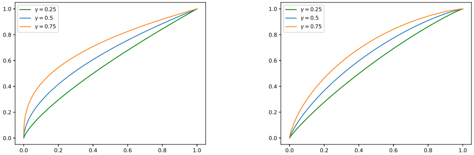


Figure: Minmaxvar, left panel, and Wang, right panel, distortion functions for different levels of the distortion parameter.

Example III

Consider a set of market quotes with maturities 6 months and 1, 2, 3, 4, 5, 7 and 10 years, respectively.

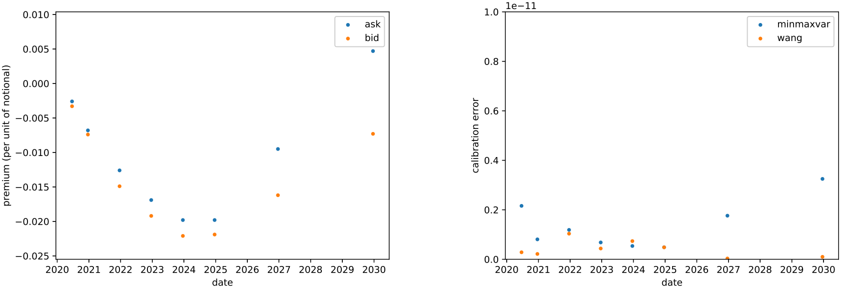
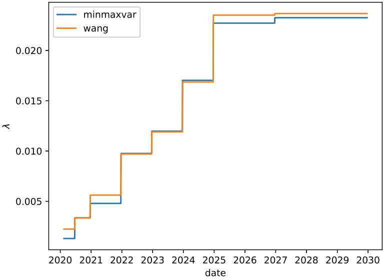


Figure: Bid and ask CDS premia used in the calibration example, left panel, and (aggregate) calibration errors, right panel.

Example IV



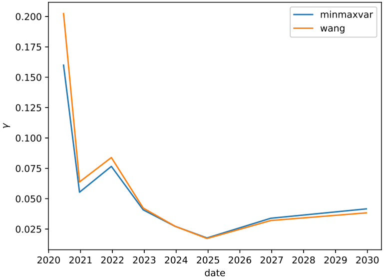


Figure: Implied parameters provided as a result of the calibration procedure to the bid and ask CDS quotes: hazard rates (i.e., λ) are depicted in the left panel, while distortion parameters (i.e., γ) in the right panel.

Conclusion

- Considered the problem of calibrating a CDS model to the available bid and ask quotes within the conic finance framework [Cherny & Madan, 2010];
- Reduced-form model for the default time and standard specifications;
- The bid-ask calibration problem requires to iteratively solve a constrained non-linear system in two equations and two unknowns;
- Under reasonable assumption for practical purposes, the calibration problem admits a unique solution;
- Replication of observed quotes.

Conclusion

Thank you.

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