CREDIT RISK: MODELLING, VALUATION AND HEDGING

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INTENSITY-BASED APPROACH

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INTENSITY-BASED APPROACH

Advantages:

- The value of the firm and the default-triggering barrier are not needed. The level of the credit risk is reflected in a single quantity: risk-neutral default intensity.
- The random time of default is unpredictable; default event comes as an almost total surprise.
- Valuation of defaultable claims is rather straightforward; it resembles the valuation of default-free contingent claims in term structure models, through well understood techniques.
- Credit spreads are much easier to quantify and manipulate. Typically, credit spreads are more realistic. Risk premia are easier to handle.

Disadvantages:

- Current data regarding the level of the firm's assets and the firm's leverage are not taken into account.
- Specific features related to safety covenants and debt's seniority are not easy to handle.
- All (important) issues related to the capital structure of a firm are beyond the scope of this approach.
- Most practical approaches to portfolio's credit risk are linked to the value-of-the-firm approach.

1. Hazard Function of Random Time

1.1 Random Time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathsf{P})$, referred to as the *random time*. We assume that $\mathsf{P}(\tau = 0) = 0$ and $\mathsf{P}(\tau > t) > 0$ for any $t \in \mathsf{R}_+$ so that the c.d.f. $F(t) := \mathsf{P}(\tau \le t) < 1$ for every $t \in \mathsf{R}_+$.

We introduce the associated jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we write $\mathsf{H} = (\mathcal{H}_t)_{t \in \mathsf{R}_+}$ to denote the (right-continuous and P-completed) filtration generated by the jump process H. Of course, τ is an H-stopping time.

Conditional expectation. We shall assume throughout that all random variables and processes that are used in what follows satisfy suitable integrability conditions.

Lemma 1 *For any $\mathcal{G}$-measurable r.v. Y we have*

$$\mathsf{E}_{\mathsf{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} \mathsf{E}_{\mathsf{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \frac{\mathsf{E}_{\mathsf{P}}(\mathbb{1}_{\{\tau > t\}} Y)}{\mathsf{P}(\tau > t)}.$$

For any $\mathcal{H}_t$-measurable r.v. Y we have

$$Y = \mathbb{1}_{\{\tau \le t\}} \mathsf{E}_{\mathsf{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \frac{\mathsf{E}_{\mathsf{P}}(\mathbb{1}_{\{\tau > t\}} Y)}{\mathsf{P}(\tau > t)},$$

that is, $Y = h(\tau)$ for a Borel measurable $h : \mathsf{R} \to \mathsf{R}$ which is constant on $]\, t, \infty[$.

1.2 Hazard Function

The notion of the *hazard function* of a random time τ is closely related to the notion of a cumulative distribution function F of τ (or its tail $G(t) = 1 - F(t)$).

Definition 1 The function $\Gamma : \mathsf{R}_+ \to \mathsf{R}_+$ given by the formula

$$\Gamma(t) = -\ln(1 - F(t)) = -\ln G(t), \quad \forall\, t \in \mathsf{R}_+,$$

is called the *hazard function* of a random time τ.

If the distribution function F is an absolutely continuous function, i.e., if

$$F(t) = \int_0^t f(u)\, du$$

for some function $f : \mathsf{R}_+ \to \mathsf{R}_+$ then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du}$$

where

$$\gamma(t) = \frac{f(t)}{1 - F(t)}\,.$$

It is clear that $\gamma : \mathsf{R}_+ \to \mathsf{R}$ is a non-negative function and it satisfies $\int_0^\infty \gamma(u)\, du = \infty$. The function γ is called the *intensity function* or the *hazard rate* of τ.

1.3 Conditional Expectations

In terms of the hazard function Γ of τ we have

$$\mathsf{E}_{\mathsf{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} \mathsf{E}_{\mathsf{P}}(Y \,|\, \tau) + \mathbb{1}_{\{\tau > t\}} \, e^{\Gamma(t)} \, \mathsf{E}_{\mathsf{P}}(\mathbb{1}_{\{\tau > t\}} Y).$$

Corollary 1 *Assume that Y is an $\mathcal{H}_\infty$-measurable r.v. so that $Y = h(\tau)$ for some function $h : \mathsf{R}_+ \to \mathsf{R}$. If Γ is continuous then*

$$\mathsf{E}_{\mathsf{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)} \, d\Gamma(u).$$

If, in addition, the random time τ admits the intensity function γ then we have

$$\mathsf{E}_{\mathsf{P}}(Y \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) \gamma(u) e^{-\int_t^u \gamma(v)\, dv} \, du.$$

In particular, for any $t \le s$ the last formula yields:

$$\mathsf{P}(\tau > s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \, e^{-\int_t^s \gamma(v)\, dv}$$

and

$$\mathsf{P}(t < \tau < s \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} (1 - e^{-\int_t^s \gamma(v)\, dv}).$$

1.4 Associated Martingales

The first two results deal with the general case.

Lemma 2 *The process* M *given by the formula*

$$M_t := \frac{1 - H_t}{1 - F(t)}$$

follows an H*-martingale. Equivalently,*

$$\mathsf{E}_{\mathsf{P}}(H_s - H_t \,|\, \mathcal{H}_t) = \mathbb{1}_{\{\tau > t\}} \frac{F(s) - F(t)}{1 - F(t)}.$$

Lemma 3 *The process* L *given by the formula*

$$L_t := \mathbb{1}_{\{\tau > t\}} e^{\Gamma(t)} = (1 - H_t) e^{\Gamma(t)}$$

is an H*-martingale.*

It suffices to observe that $L_t = M_t$ for every $t \in \mathsf{R}_+$.

In the next lemma, the c.d.f. F of a random time τ is assumed to be continuous.

Lemma 4 *Assume that* F (*and thus also* Γ) *is a continuous function. Then the process*

$$\hat{M}_t = H_t - \Gamma(t \wedge \tau)$$

follows an H*-martingale.*

1.5 Change of a Probability Measure

Let P^* be any probability measure on $(\Omega, \mathcal{H}_\infty)$, which is absolutely continuous with respect to P.

Then there exists a Borel measurable function $h : \mathsf{R}_+ \to \mathsf{R}_+$ which satisfies:

$$\mathsf{E}_{\mathsf{P}}(h(\tau)) = \int_{]0,\infty[} h(u)\, dF(u) = 1$$

and such that the Radon-Nikodým density of P^* with respect to P equals

$$\eta_\infty = \frac{d\mathsf{P}^*}{d\mathsf{P}} = h(\tau) \geq 0 \quad \mathsf{P}\text{-a.s.}$$

Assume that $\mathsf{P}^*\{\tau = 0\} = 0$ and $\mathsf{P}^*\{\tau > t\} > 0$ for $t \in \mathsf{R}_+$.

The first condition is clearly satisfied for any probability measure P^*, which is absolutely continuous with respect to P.
For the second condition to hold, we need to postulate that for every $t \in \mathsf{R}_+$

$$\mathsf{P}^*\{\tau > t\} = 1 - F^*(t) = \int_{]t,\infty[} h(u)\, dF(u) > 0,$$

where F^* is the c.d.f. of τ under P^*:

$$F^*(t) := \mathsf{P}^*\{\tau \leq t\} = \int_{]0,t]} h(u)\, dF(u).$$

Let

$$g(t) = e^{\Gamma(t)}\, \mathsf{E}_{\mathsf{P}}(\mathbb{1}_{\{\tau>t\}} h(\tau)) = e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u)$$

and let $h^* : \mathsf{R}_+ \to \mathsf{R}$ be given by $h^*(t) = h(t)g^{-1}(t)$.

If F (and thus F^*) is continuous, the hazard function Γ^* of τ under P^* satisfies:

$$d\Gamma^*(t) = \frac{dF^*(t)}{1 - F^*(t)}\,.$$

Consequently,

$$d\Gamma^*(t) = \frac{d(1 - e^{-\Gamma(t)} g(t))}{e^{-\Gamma(t)} g(t)} = \frac{g(t) d\Gamma(t) - dg(t)}{g(t)} = h^*(t)\, d\Gamma(t).$$

We have thus established the following partial result in which we denote

$$\kappa(t) = h^*(t) - 1 = h(t)g^{-1}(t) - 1.$$

Proposition 1 *Let* P^* *and* P *be the two equivalent probability measures on* $(\Omega, \mathcal{H}_\infty)$. *If the hazard function* Γ *of* τ *under* P *is continuous, then the hazard function* Γ^* *of* τ *under* P^* *is also continuous and*

$$d\Gamma^*(t) = (1 + \kappa(t))\, d\Gamma(t)$$

where $\kappa(t) = h(t)g^{-1}(t) - 1$.

Radon-Nikodým Density Process

Let us examine the meaning of the function κ. We introduce the non-negative P-martingale η:

$$\eta_t := \frac{d\mathsf{P}^*}{d\mathsf{P}}\,|_{\mathcal{H}_t} = \mathsf{E}_{\mathsf{P}}(\eta_\infty \,|\, \mathcal{H}_t) = \mathsf{E}_{\mathsf{P}}(h(\tau) \,|\, \mathcal{H}_t).$$

The process η is the *Radon-Nikodým density process* of P^* with respect to P. Notice that

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}}\, e^{\Gamma(t)} \int_{]t,\infty[} h(u)\, dF(u),$$

and thus also

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} g(t).$$

If, in addition, F is a continuous function then

$$\eta_t = \mathbb{1}_{\{\tau \le t\}} h(\tau) + \mathbb{1}_{\{\tau > t\}} \int_t^\infty h(u) e^{\Gamma(t) - \Gamma(u)}\, d\Gamma(u).$$

It can also be shown (we omit the proof) that η solves the following SDE:

$$\eta_t = 1 + \int_{]0,t]} \eta_{u-} \kappa(u)\, d\hat{M}_u. \qquad (*)$$

It is not difficult to find an explicit solution to this equation, specifically,

$$\eta_t = (1 + \mathbb{1}_{\{\tau \le t\}} \kappa(\tau)) \exp\big(- \int_0^{t \wedge \tau} \kappa(u)\, d\Gamma(u)\big). \qquad (**)$$

Doléans Exponential

Lemma 5 *Let Y be a process of finite variation. Consider the linear SDE:*

$$Z_t = 1 + \int_{]0,t]} Z_{u-} \, dY_u.$$

The unique solution $Z_t = \mathcal{E}_t(Y)$, called the Doléans exponential of Y, equals

$$\mathcal{E}_t(Y) = e^{Y_t} \prod_{0<u\leq t} (1 + \Delta Y_u) e^{-\Delta Y_u}.$$

Equivalently,

$$\mathcal{E}_t(Y) = e^{Y_t^c} \prod_{0<u\leq t} (1 + \Delta Y_u) \qquad (***)$$

where Y^c is the path-by-path continuous part of Y, i.e.,

$$Y_t^c = Y_t - \sum_{0<u\leq t} \Delta Y_u.$$

Since the process η satisfies $(*)$, it is clear that it can be represented as follows:

$$\eta_t = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa(u) \, d\hat{M}_u \Big).$$

Expression $(**)$ for the random variable η_t can thus also be obtained from $(***)$, upon setting $dY_u = \kappa(u) \, d\hat{M}_u$.

Equality $(***)$ is merely a special case of the general formula for the Doléans exponential (see, e.g., Elliott (1982), Protter (1990), or Revuz and Yor (1999)).

1.5.1 Girsanov's Theorem

Proposition 2 *Assume that F is continuous. Let P^* be any probability measure on $(\Omega, \mathcal{H}_\infty)$ equivalent to P, so that*

$$\eta_\infty = \frac{d\mathsf{P}^*}{d\mathsf{P}} = h(\tau) > 0 \quad \mathsf{P}\text{-a.s.}$$

for some Borel measurable function $h : \mathsf{R}_+ \to \mathsf{R}_+$. Then:
(i) *The Radon-Nikodým density process η of P^* with respect to P satisfies*

$$\eta_t := \frac{d\mathsf{P}^*}{d\mathsf{P}}|_{\mathcal{H}_t} = \mathcal{E}_t\Big(\int_{]0,\,\cdot\,]} \kappa(u)\, d\hat{M}_u\Big)$$

where

$$\kappa(t) = h(t) g^{-1}(t) - 1$$

and

$$g(t) = e^{\Gamma(t)} \int_t^\infty h(u)\, dF(u).$$

(ii) *The hazard function Γ^* equals $\Gamma^*(t) = g^*(t)\Gamma(t)$ with*

$$g^*(t) = \frac{\ln\big(\int_{]t,\infty[} h(u)\, dF(u)\big)}{\ln(1 - F(t))}.$$

If Γ is continuous, then $d\Gamma^*(t) = (1 + \kappa(t)) d\Gamma(t)$. In particular, $\gamma^*(t) = (1 + \kappa(t))\gamma(t)$ if the intensity γ is well defined.

1.6 Martingale Hazard Function

Definition 2 A function $\Lambda : \mathsf{R}_+ \to \mathsf{R}$ is called a *martingale hazard function* of a random time τ with respect to the filtration H if and only if the process $H_t - \Lambda(t \wedge \tau)$ follows an H-martingale.

Proposition 3 (i) *The unique martingale hazard function of τ with respect to the filtration H is the right-continuous increasing function Λ given by the formula*

$$\Lambda(t) = \int_{]0,t]} \frac{dF(u)}{1 - F(u-)} = \int_{]0,t]} \frac{d\mathsf{P}(\tau \leq u)}{1 - \mathsf{P}(\tau < u)}.$$

(ii) *The martingale hazard function Λ is continuous if and only if F is continuous. In this case, $\Lambda(t) = -\ln(1 - F(t))$.*

(iii) *The martingale hazard function Λ coincides with the hazard function Γ if and only if F is a continuous function.*
In general

$$e^{-\Gamma(t)} = e^{-\Lambda^c(t)} \prod_{0 \leq u \leq t} (1 - \Delta\Lambda(u)),$$

where

$$\Lambda^c(t) = \Lambda(t) - \sum_{0 \leq u \leq t} \Delta\Lambda(u)$$

and $\Delta\Lambda(u) = \Lambda(u) - \Lambda(u-)$.

2 Valuation of Defaultable Claims

A defaultable claim consists of:

- the *promised contingent claim* X, representing the payoff received by the owner of the claim at time T, if there was no default prior to or at time T,
- the process C representing the *promised dividends* – that is, the stream of (continuous or discrete) cash flows received by the owner of the claim prior to default,
- the *recovery process* Z, representing the recovery payoff at time of default, if default occurs prior to or at time T,
- the *recovery claim* $\tilde{X}$, which represents the recovery payoff at time T if default occurs prior to or at the maturity date T.

Definition 3 The *dividend process* D of a defaultable claim $(X, C, \tilde{X}, Z, \tau)$ equals

$$D_t = X^d(T)\, \mathbb{1}_{\{t \geq T\}} + \int_{]0,t]} (1 - H_u)\, dC_u + \int_{]0,t]} Z_u\, dH_u$$

where $X^d(T) = X\, \mathbb{1}_{\{\tau > T\}} + \tilde{X}\, \mathbb{1}_{\{\tau \leq T\}}$.

Definition 4 The *ex-dividend price process* S of a defaultable claim $(X, C, \tilde{X}, Z, \tau)$ which settles at time T is given as

$$S_t = B_t\, \mathsf{E}_{\mathsf{Q}^*}\Big(\int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t \Big)$$

where Q^* is the *martingale measure* for our model.

2.1 Hazard Process of a Random Time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathsf{Q}^*)$. Assume that $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for some reference filtration F. We shall write $\mathsf{G} = \mathsf{H} \vee \mathsf{F}$.

We denote $F_t = \mathsf{Q}^*(\tau \leq t \,|\, \mathcal{F}_t)$, so that

$$G_t := 1 - F_t = \mathsf{Q}^*(\tau > t \,|\, \mathcal{F}_t)$$

is the *conditional survival probability.* It is easily seen that F is a bounded, non-negative, F-submartingale.

Assume that $F_t < 1$ for every $t \in \mathsf{R}_+$. The F-*hazard process* Γ of τ is defined through the equality $1 - F_t = e^{-\Gamma_t}$.

2.1.1 Valuation of the Terminal Payoff

To value the *terminal payoff* $X^d(T)$ we shall use:

Lemma 6 *For any $\mathcal{G}$-measurable integrable random variable Y we have*

$$\mathsf{E}_{\mathsf{Q}^*}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathsf{E}_{\mathsf{Q}^*}(\mathbb{1}_{\{\tau > t\}} Y \,|\, \mathcal{F}_t)}{\mathsf{Q}^*(\tau > t \,|\, \mathcal{F}_t)}.$$

If, in addition, Y is $\mathcal{F}_s$-measurable where $s \geq t$ then

$$\mathsf{E}_{\mathsf{Q}^*}(\, \mathbb{1}_{\{\tau > s\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, \mathsf{E}_{\mathsf{Q}^*}(e^{\Gamma_t - \Gamma_s} Y \,|\, \mathcal{F}_t).$$

2.1.2 Valuation of Recovery Process and Promised Dividends

The following result appears to be useful in the valuation of the *recovery payoff* Z_τ which occurs at time τ.

Proposition 4 *Let Γ be a continuous process and let Z be an* F*-predictable process. Then for any $t \le s$ we have*

$$\mathsf{E}_{\mathsf{Q}^*}(Z_\tau \mathbb{1}_{\{t<\tau\le s\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \mathsf{E}_{\mathsf{Q}^*}\big(\textstyle\int_t^s Z_u e^{\Gamma_t - \Gamma_u} \, d\Gamma_u \,|\, \mathcal{F}_t\big).$$

To value the *promised dividends* C that are paid prior to τ we shall make use of the following result.

Proposition 5 *Let Γ be a continuous process and let C be an* F*-predictable bounded process of finite variation. Then for every $t \le s$*

$$\mathsf{E}_{\mathsf{Q}^*}\big(\textstyle\int_{]t,s]} (1 - H_u) \, dC_u \,|\, \mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}} \mathsf{E}_{\mathsf{Q}^*}\big(\textstyle\int_{]t,s]} e^{\Gamma_t - \Gamma_u} \, dC_u \,|\, \mathcal{F}_t\big).$$

Remark: Of course, in order to value a defaultable claim we need also to specify a discount factor (a numeraire). For the sake of simplicity, we shall take the savings account as a numeraire (cf. Definition 4).

2.2 Defaultable Bonds

We assume that:

(i) the default time admits the intensity function γ,

(ii) the short-term interest rate r is deterministic.

In view of the latter assumption, at time t the price of a unit default-free zero-coupon bond (ZCB) of maturity T equals

$$B(t,T) = e^{-\int_t^T r(v)\,dv}.$$

2.2.1 Zero Recovery Scheme

Let us first consider a corporate ZCB with *zero recovery* at default. The pre-default value $D^0(t,T)$ of such a bond equals

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}} e^{-\int_t^T (r(v)+\gamma(v))dv} = \mathbb{1}_{\{\tau>t\}} B(t,T) e^{-\int_t^T \gamma(v)dv}.$$

A corporate ZCB becomes worthless as soon as default occurs.

2.2.2 Fractional Recovery of Par Value – FRPV

Let the Z satisfy $Z_t = \delta$ for some constant *recovery rate* $0 \le \delta \le 1$. The pre-default value $\tilde{D}^\delta(t,T)$ of a unit corporate ZCB equals

$$\tilde{D}^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} \left(\delta \int_t^T e^{-\int_t^u \tilde{r}(v)\,dv} \gamma(u)\,du + e^{-\int_t^T \tilde{r}(v)\,dv}\right).$$

where $\tilde{r} = r + \gamma$ is the default-risk-adjusted interest rate.

2.2.3 Fractional Recovery of Treasury Value – FRTV

Assume that the recovery process equals

$$Z_t = \delta B(t,T).$$

Let us denote by $D^\delta(t,T)$ the pre-default value of a unit corporate bond subject to FRTV.

Then

$$D^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} \left(\int_t^T \delta B(t,T) e^{-\int_t^u \gamma(v)dv} \gamma(u)\, du + e^{-\int_t^T \tilde{r}(v)dv} \right)$$

that is,

$$D^\delta(t,T) = \mathbb{1}_{\{\tau>t\}} B(t,T) \big(\delta (1 - e^{-\int_t^T \gamma(v)\, dv}) + e^{-\int_t^T \gamma(v)\, dv}\big).$$

The price $D^\delta(t,T)$ can also be expressed as follows

$$D^\delta(t,T) = B(t,T) \left(\delta\, \mathsf{Q}^*(t < \tau \le T \,|\, \mathcal{G}_t) + \mathsf{Q}^*(\tau > T \,|\, \mathcal{G}_t) \right).$$

Remark: Similar representations can be derived under the assumption that *market risk* and *credit risk* are independent:

(i) the default time admits the F-intensity process γ,

(ii) the short-term interest rate r follows a stochastic process independent of the filtration F.

2.3 Hedging of Credit Derivatives

1) Specification of essential contractual features of a credit-risk-sensitive contract under study.

2) Identification of risks (market and credit) involved.

3) Choice of the most convenient and adequate model.

4) Arbitrage-free valuation of a considered contract.

5) Identification of a family of traded (liquid) instruments that can be used to construct a hedging strategy.

6) Construction of a self-financing strategy that replicates the value of a contract up to and including time τ.

7) Calibration of the model to market prices.

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Y. Greenfield (2000) Hedging of credit risk embedded in derivative transactions. Thesis, Carnegie Mellon University.

2.3.1 Practical Approach versus Theoretical Approach

Practical Approach. The following simplifying assumptions are common:

1) Only a pure credit risk instrument (e.g., a basic credit default swap) is considered.

2) One deals with a one-sided counterparty risk with a fixed recovery rate (the same for a derivative product and for a corporate bond).

3) The mark-to-market value of the contract is assumed to be nonnegative to a non-defaultable counterparty (thus defaultable loans and bonds or vulnerable options are covered, but defaultable swaps are not).

4) Independence of market and credit risks is frequently postulated.

5) Existence of a non-defaultable version of the contract and of a liquid market in corporate bonds is assumed.

Theoretical Approach. A suitable version of a predictable representation theorem with respect to martingales associated with default event or with credit migrations (see, e.g., Wong (1999) or Blanchet-Scalliet and Jeanblanc (2001)). Unfortunately, the general formulae seem to be very difficult to implement.

2.3.2 Predictable Representation Theorem

We focus on the special case, an we consider an H-martingale $M^h_t = \mathsf{E}_{\mathsf{P}}(h(\tau) \,|\, \mathcal{H}_t)$ for some function $h : \mathsf{R}_+ \to \mathsf{R}$.

Denote by $g(t)$ the conditional expected value of the future "payoff" on the set $\{\tau > t\}$

$$g(t) = e^{\Gamma(t)} \mathsf{E}_{\mathsf{P}}(\mathbb{1}_{\{\tau > t\}} h(\tau)).$$

Proposition 6 *Assume that F is a continuous function. Then*

$$M^h_t = M^h_0 + \int_{]0,t]} \hat{h}(u) \, d\hat{M}_u,$$

where $\hat{M}_t = H_t - \Gamma(t \wedge \tau)$ and $\hat{h} = h - g$.

Notice that we also have

$$M^h_t = M^h_0 + \int_{]0,t]} (h(u) - M^h_{u-}) \, d\hat{M}_u.$$

The latter equality has a nice financial interpretation.

Remark: In a more general setup, the integral representation of a H-martingale involves two (or more) integrals with respect to continuous/pure jump basic martingales. This representation needs to be translated into self-financing trading strategies.

2.4 Martingale Hazard Process

The next result is valid for any F-hazard process Γ.

Lemma 7 *The process*

$$L_t = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} = (1 - H_t) e^{\Gamma_t}$$

follows a G*-martingale.*

If the process Γ is continuous, it defines the compensator of the stopping time τ.

Proposition 7 *Assume that the* F*-hazard process* Γ *of a random time* τ *follows a continuous process of finite variation. Then the process*

$$\hat{M}_t = H_t - \Gamma_{t \wedge \tau}$$

follows a G*-martingale.*

Definition 5 An F-predictable right-continuous increasing process Λ is called an F-*martingale hazard process* of a random time τ if and only if the process $H_t - \Lambda_{t \wedge \tau}$ follows a G-martingale. In addition, $\Lambda_0 = 0$.

In the *martingale approach*, the martingale hazard process Λ, rather than the hazard process Γ, is used. An important issue thus arises: provide sufficient conditions for the equality $\Lambda = \Gamma$.

2.4.1 Properties of Λ

Condition (G) The process $F_t = \mathsf{Q}^*(\tau \le t \,|\, \mathcal{F}_t)$ admits a modification with increasing sample paths.

Proposition 8 *Assume that* (G) *holds. If the process* Λ

$$\Lambda_t = \int_{]0,t]} \frac{dF_u}{1 - F_{u-}} = \int_{]0,t]} \frac{d\mathsf{Q}^*(\tau \le u \,|\, \mathcal{F}_u)}{1 - \mathsf{Q}^*(\tau < u \,|\, \mathcal{F}_u)}$$

is F*-predictable, then* Λ *is the* F*-martingale hazard process of the random time* τ*.*

If condition (G) is not postulated, we have:

Proposition 9 *The* F*-martingale hazard process of* τ *equals*

$$\Lambda_t = \int_{]0,t]} \frac{d\tilde{F}_u}{1 - F_{u-}}$$

where $\tilde{F}$ *denotes the* F*-compensator of the* F*-submartingale* F*; that is, the unique* F*-predictable, increasing process, such that* $\tilde{M} = F - \tilde{F}$ *is an* F*-martingale.*

A counter-example in Elliott et al. (2000) shows that if condition (G) is not assumed, the continuity of processes Γ and Λ is not sufficient for the equality $\Gamma = \Lambda$.

3 Martingale Approach

The *martingale approach* is that version of the intensity-based approach in which we work directly with the martingale hazard process of a default time.

References:

D. Duffie, M. Schroder, C. Skiadas (1996) Recursive valuation of defaultable securities and the timing of resolution of uncertainty. *Ann. Appl. Probab.* 6, 1075–1090.

D. Duffie (1998) Defaultable term structure models with fractional recovery of par. Working paper, Stanford University, 1998.

D. Duffie, K. Singleton (1999) Modeling term structures of defaultable bonds. *Rev. Fin. Studies* 12, 687–720.

3.1 Basic Setup

- Martingale measure:

(A.1) We are given a probability space $(\Omega, \mathsf{F}, \mathsf{Q}^*)$, with Q^* interpreted as a spot martingale measure. An F-adapted process r represents the short-term interest rate, and the process $B_t = \exp\left(\int_0^t r_u\, du\right)$ models the money market account, which plays here the role of the numeraire asset.

- Promised claim:

(A.2) An $\mathcal{F}_T$-measurable random variable X represents the *promised claim*, that is, the amount of cash which the owner of a defaultable claim is entitled to receive at time T, provided that the default has not occurred prior to T.

- Default time:

(A.3) The *default time* τ is a random time. If $H_t = \mathbb{1}_{\{\tau \leq t\}}$ then the process

$$\hat{M}_t = H_t - \int_0^{t \wedge \tau} \lambda_u \, du$$

follows a G-martingale under Q*, where G $=$ F $\vee$ H and $\mathcal{H}_t = \sigma(H_u : u \leq t)$. The process λ is the F-intensity of τ under Q*.

- Recovery process:

(A.4) An F-predictable process Z, called *recovery process*, models the payoff which is actually received by the owner of a defaultable claim in case the default occurs prior to the claim's maturity T.

Definition 6 A *defaultable claim* is formally represented by a triplet (X, Z, τ).

This means that, for the sake of simplicity, we take $C \equiv 0$ (promised dividends are zero) and $\tilde{X} = 0$ (recovery payoff at T equals 0).

3.2 Valuation of Defaultable Claims

We postulate that the value S_t at time t of a defaultable claim (X, Z, τ) equals

$$S_t := B_t \, \mathsf{E}_{\mathsf{Q}^*}\big(\int_{]t,T]} B_u^{-1} \, dD_u \,|\, \mathcal{G}_t \big)$$

where D is the *dividend process.* More explicitly

$$S_t = B_t \, \mathsf{E}_{\mathsf{Q}^*}\big(B_\tau^{-1} Z_\tau \, \mathbb{1}_{\{t<\tau\leq T\}} + B_T^{-1} X \, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t\big).$$

In particular, $S_T = X \, \mathbb{1}_{\{\tau>T\}}$.

First step. Let $h_t = \lambda_t \, \mathbb{1}_{\{\tau\geq t\}}$. Recall that λ is the F-intensity under Q^* of τ (it is given in advance). Since

$$\hat{M}_t = H_t - \int_0^t h_u \, du$$

is a G-martingale, the process $A_t = \int_0^t h_u \, du$ is the compensator of the bounded G-submartingale H.

Lemma 8 *The value process S satisfies*

$$S_t = \mathsf{E}_{\mathsf{Q}^*}\big(\int_t^T (Z_u h_u - r_u S_u) \, du + X \, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t\big).$$

Second step. We introduce an auxiliary process V by setting

$$V_t = \tilde{B}_t \, \mathsf{E}_{\mathsf{Q}^*}\big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,|\, \mathcal{G}_t \big)$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-risk-adjusted interest rate $\tilde{r}_t = r_t + \lambda_t$

$$\tilde{B}_t = \exp\big(\int_0^t (r_u + \lambda_u) \, du \big).$$

Proposition 10 *The value process S satisfies*

$$S_t = \mathbb{1}_{\{\tau > t\}} \{ V_t - B_t \, \mathsf{E}_{\mathsf{Q}^*}(B_\tau^{-1} \, \mathbb{1}_{\{\tau \le T\}} \Delta V_\tau \,|\, \mathcal{G}_t) \}.$$

Third step. Assume that

$$\mathsf{E}_{\mathsf{Q}^*}(B_\tau^{-1} \, \mathbb{1}_{\{\tau \le T\}} \Delta V_\tau \,|\, \mathcal{G}_t) = 0. \qquad (*)$$

Then

$$S_t = \mathbb{1}_{\{\tau > t\}} \tilde{B}_t \, \mathsf{E}_{\mathsf{Q}^*}\big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,|\, \mathcal{G}_t \big).$$

Since $S_t = \mathbb{1}_{\{\tau > t\}} V_t$, the process V is called the $\textit{pre-default value}$ of a defaultable claim.

Condition $(*)$ is not easy to check. It depends on the choice of a claim, in general.

3.3 Martingale Hypothesis

We shall now introduce specific assumptions related to the conditional independence of the two filtrations F and H.

(H.1) For any $t \in \mathsf{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. Equivalently, for any $t \in \mathsf{R}_+$ and any bounded $\mathcal{F}_\infty$-measurable r.v. ξ we have

$$\mathsf{E}_{\mathsf{Q}^*}(\xi \,|\, \mathcal{G}_t) = \mathsf{E}_{\mathsf{Q}^*}(\xi \,|\, \mathcal{F}_t).$$

Conditions (H.2) and (H.3) are equivalent to (H.1).

(H.2) For any $t \in \mathsf{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$.

(H.3) For any $t \in \mathsf{R}_+$ and any $u \leq t$ we have

$$\mathsf{Q}^*(\tau \leq u \,|\, \mathcal{F}_t) = \mathsf{Q}^*(\tau \leq u \,|\, \mathcal{F}_\infty).$$

Definition 7 We say that a filtration F has the $martingale$ $invariance$ $property$ with respect to a filtration G if every F-martingale is also a G-martingale.

Lemma 9 *A filtration* F *has the martingale invariance property with respect to a filtration* G *if and only if condition* (H.1) *is satisfied.*

3.3.1 Application of the Martingale Hypothesis

If (H.1) holds, condition $(*)$ is satisfied on the pre-default set.

Lemma 10 *Under* (A.1)-(A.4) *and* (H.1), *we have on the set* $\{\tau > t\}$

$$\mathsf{E}_{\mathsf{Q}^*}(B_\tau^{-1}\,\mathbb{1}_{\{\tau \le T\}}\Delta V_\tau \,|\, \mathcal{G}_t) = 0.$$

Combining the last result with Proposition 10 we obtain the following corollary.

Corollary 2 *Under* (A.1)-(A.4) *and* (H.1), *we have*

$$S_t = \mathbb{1}_{\{\tau > t\}}\tilde{B}_t\,\mathsf{E}_{\mathsf{Q}^*}\big(\textstyle\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u\,du + \tilde{B}_T^{-1} X \,|\, \mathcal{F}_t\big)$$

where $\tilde{B}$ *is the default-risk-adjusted savings account*

$$\tilde{B}_t = \exp\big(\textstyle\int_0^t (r_u + \lambda_u)\,du\big).$$

Remark. It is interesting to observe that conditions (H.1)-(H.3) are not invariant with respect to an equivalent change of a probability measure. For a simple counterexample, see S. Kusuoka (1999) A remark on default risk models. *Advances in Mathematical Economics* 1, 69–82.

3.3.2 Counter-example: Kusuoka (1999)

The following example shows that the martingale invariance property is not preserved, in general, under an equivalent change of a probability measure.

Under the original probability measure Q the random times $\tau_i,\ i=1,2$ are mutually independent random variables, with exponential laws with parameters λ_1 and λ_2, respectively.

For a fixed $T>0$, we introduce an equivalent probability measure Q^* on $(\Omega,\mathcal{G})$ by setting

$$\frac{d\mathsf{Q}^*}{d\mathsf{Q}} = \eta_T \quad \mathsf{Q}\text{-a.s.}$$

where $\eta_t,\ t\in[0,T],$ satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-}\kappa_u^i \, d\hat{M}_u^i,$$

$$\hat{M}_t^i = H_t^i - \int_0^{t\wedge\tau_i} \lambda_i \, du,$$

where $H_t^i = \mathbb{1}_{\{\tau_i \le t\}}$, and processes κ^1 and κ^2 satisfy:

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2<t\}}\left(\frac{\alpha_1}{\lambda_1}-1\right), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1<t\}}\left(\frac{\alpha_2}{\lambda_2}-1\right).$$

Notice that the process κ^1 (κ^2, respectively) is H^2-predictable (H^1-predictable, respectively).

It is easily seen that $\Lambda_t^{i*} = \int_0^t \lambda_u^{i*}\, du$ for $i = 1, 2$, where

$$\lambda_t^{*1} = \lambda_1(1 - H_t^2) + \alpha_1 H_t^2 = \lambda_1 \, \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \, \mathbb{1}_{\{\tau_2 \le t\}},$$

and

$$\lambda_t^{*2} = \lambda_2(1 - H_t^1) + \alpha_2 H_t^1 = \lambda_2 \, \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \, \mathbb{1}_{\{\tau_1 \le t\}}.$$

This means that the H^2-martingale intensity λ_1^* of default time τ_1 under Q^* jumps from λ_1 to α_1 after τ_2. The second default time has an analogous property.

It appears that the following inequality holds

$$\mathsf{Q}^*(\tau_1 > s \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2) \neq \mathbb{1}_{\{\tau_1 > t\}} \mathsf{E}_{\mathsf{Q}^*}(e^{\Lambda_t^{1*} - \Lambda_s^{1*}} \,|\, \mathcal{H}_t^2).$$

The martingale invariance property can now be restated as follows: for any bounded $\mathcal{H}_\infty^2$-measurable random variable ξ, the equality

$$\mathsf{E}_{\mathsf{Q}^*}(\xi \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2) = \mathsf{E}_{\mathsf{Q}^*}(\xi \,|\, \mathcal{H}_t^2)$$

is valid for arbitrary $t \in \mathsf{R}_+$.

It is possible to check directly, that the last condition fails to hold in Kusuoka's example. This example is closely related to the valuation of *basket derivatives*, for instance, the *first-to-default claims*.

3.4 Canonical Construction

A random time obtained through the *canonical construction* has certain specific features that are not necessarily shared by all random times with a given F-hazard process Γ.

Assume that we are given an F-adapted, right-continuous, increasing process Γ defined on a probability space $(\tilde{\Omega}, \mathsf{F}, \mathsf{P}^*)$ such that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$.

To construct a random time τ such that Γ is the F-hazard process of τ, we enlarge the underlying probability space $\tilde{\Omega}$. This means that Γ will be the F-hazard process of τ under a suitable extension Q^* of P^*.

Let ξ be a r.v. defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathsf{Q}})$ uniformly distributed on the interval $[0, 1]$ under $\hat{\mathsf{Q}}$. We consider the product space $\Omega = \tilde{\Omega} \times \hat{\Omega}$ with the σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and the probability measure $\mathsf{Q}^* = \mathsf{P}^* \otimes \hat{\mathsf{Q}}$. The latter equality means that for any events $A \in \mathcal{F}_\infty$ and $B \in \hat{\mathcal{F}}$ we have $\mathsf{Q}^*(A \times B) = \mathsf{P}^*(A)\hat{\mathsf{Q}}(B)$.

Define the random time $\tau : \Omega \to \mathsf{R}_+$ by setting

$$\tau = \inf\{\, t \in \mathsf{R}_+ : e^{-\Gamma_t} \leq \xi \,\} = \inf\{\, t \in \mathsf{R}_+ : \Gamma_t \geq \eta \,\}$$

where $\eta = -\ln \xi$ has a unit exponential law under Q^*.

Let us find the process

$$F_t = \mathsf{Q}^*(\tau \le t \,|\, \mathcal{F}_t).$$

Since $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathsf{Q}^*(\tau > t \,|\, \mathcal{F}_\infty) = \mathsf{Q}^*(\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty) = \hat{\mathsf{Q}}(\xi < e^x)_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathsf{Q}^*(\tau > t \,|\, \mathcal{F}_t) = \mathsf{E}_{\mathsf{Q}^*}(\mathsf{Q}^*(\tau > t \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t) = e^{-\Gamma_t}$$

and thus F is an $\mathbb{F}$-adapted, right-continuous and increasing. Thus Γ is the $\mathbb{F}$-hazard process of τ under Q^*.

In addition, we obtain the following property of the canonical construction:

$$\mathsf{Q}^*(\tau \le t \,|\, \mathcal{F}_\infty) = \mathsf{Q}^*(\tau \le t \,|\, \mathcal{F}_t).$$

Consequently, for any two dates $0 \le u \le t$

$$\mathsf{Q}^*(\tau \le u \,|\, \mathcal{F}_\infty) = \mathsf{Q}^*(\tau \le u \,|\, \mathcal{F}_t) = \mathsf{Q}^*(\tau \le u \,|\, \mathcal{F}_u) = e^{-\Gamma_u}.$$

The latter equality proves the conditional independence under Q^* of the σ-fields $\mathcal{H}_t$ and $\mathcal{F}_t$ given $\mathcal{F}_\infty$. We conclude that (H.3), and thus also (H.1) and (H.2), are valid.

3.5 Defaultable Bonds

Consider a defaultable ZCB with par value 1 and maturity T. We shall examine three recovery schemes:

– Fractional Recovery of Par Value.

– Fractional Recovery of Treasury Value.

– Fractional Recovery of Market Value.

3.5.1 Fractional Recovery of Par Value

If a fixed fraction of bond's face value is paid to the bondholders at time τ, the scheme is referred to as the *fractional recovery of par value.*

We deal with a defaultable claim (X, Z, τ), which settles at time T, with the promised payoff $X = 1$ and the recovery process $Z = \delta$.

If δ is constant, the pre-default value $\tilde{D}^{\delta}(t,T)$ of a defaultable ZCB equals

$$\tilde{D}^{\delta}(t,T) = B_t \, \mathsf{E}_{\mathsf{Q}^*}(\delta B_{\tau}^{-1} \, \mathbb{1}_{\{t<\tau\leq T\}} + B_T^{-1} \, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t).$$

If $\Delta V_{\tau} = 0$ then we also have

$$\tilde{D}^{\delta}(t,T) = \mathbb{1}_{\{\tau>t\}} \tilde{B}_t \, \mathsf{E}_{\mathsf{Q}^*}(\delta \int_t^T \tilde{B}_u^{-1} \lambda_u \, du + \tilde{B}_T^{-1} \,|\, \mathcal{F}_t).$$

3.5.2 Fractional Recovery of Treasury Value

If, in the case of default, the fixed fraction of bond's face value is paid to bondholders at maturity date T, the recovery scheme is termed the *fractional recovery of Treasury value.*

A bond is now a defaultable claim (X, Z, τ) where

$$X = 1, \quad Z_t = \delta B(t, T)$$

and $B(t, T)$ denotes the price at time t of a unit zero-coupon Treasury bond that matures at time T.

Hence, it is equivalent to a default-free contingent claim Y, which settles at time T, and equals

$$Y = \mathbb{1}_{\{\tau > T\}} + \delta \, \mathbb{1}_{\{\tau \le T\}}.$$

We have on the set $\{\tau > t\}$

$$D^\delta(t, T) = B_t \, \mathsf{E}_{\mathsf{Q}^*} \left(B_T^{-1} (\delta \, \mathbb{1}_{\{\tau \le T\}} + \mathbb{1}_{\{\tau > T\}}) \,|\, \mathcal{G}_t \right).$$

Equivalently, the pre-default value $D^\delta(t, T)$ of a defaultable bond equals

$$D^\delta(t, T) = B_t \, \mathsf{E}_{\mathsf{Q}^*} \left(\delta B_\tau^{-1} B(\tau, T) \, \mathbb{1}_{\{t < \tau \le T\}} + B_T^{-1} \, \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t \right).$$

3.5.3 Fractional Recovery of Market Value

Under this scheme, the bondholders receive at time τ a fraction of the pre-default market value of a bond.

Assume that the recovery process satisfies: $Z_t = K_t S_{t-}$ where K is a given F-predictable process and S is the pre-default value of the bond. Let the process V solve

$$V_t = \tilde{B}_t \, \mathsf{E}_{\mathsf{Q}^*}\big(\int_t^T \tilde{B}_u^{-1} K_u V_u \lambda_u \, du + \tilde{B}_T^{-1} X \,|\, \mathcal{G}_t \big).$$

Lemma 11 *The solution V is unique and is given by the formula*

$$V_t = \hat{B}_t \, \mathsf{E}_{\mathsf{Q}^*}(\hat{B}_T^{-1} X \,|\, \mathcal{G}_t)$$

with

$$\hat{B}_t = \exp\big(\int_0^t (r_u + (1 - K_u)\lambda_u) \, du \big).$$

If $\Delta V_\tau = 0$ the value of a defaultable bond is $S_t = \mathbb{1}_{\{\tau > t\}} V_t$.

We write $S_t = \hat{D}^K(t,T)$. When $K \equiv \delta$, where δ is a constant, we obtain

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{\tau > t\}} \, \mathsf{E}_{\mathsf{Q}^*} \left(e^{-\int_t^T (r_u + (1-\delta)\lambda_u) du} \,|\, \mathcal{F}_t \right).$$

Backward SDE Approach

Assume that the recovery process Z is defined through the formula $Z_t = p(t, S_{t-})$, where the function $p(t,s)$ is Lipschitz continuous with respect to s and satisfies $p(t,0) = 0$.

Let S be the unique solution to the BSDE

$$S_t = B_t \, \mathsf{E}_{\mathsf{Q}^*}(B_\tau^{-1} p(\tau, S_{\tau-}) \, \mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1} X \, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t),$$

or equivalently, to the equation

$$S_t = \mathsf{E}_{\mathsf{Q}^*}(\textstyle\int_t^T (p(u, S_u)h_u - r_u S_u) \, du + X \, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t).$$

Let V be the unique solution to the BSDE

$$V_t = \tilde{B}_t \, \mathsf{E}_{\mathsf{Q}^*}(\textstyle\int_t^T \tilde{B}_u^{-1} p(u, V_u) \lambda_u \, du + \tilde{B}_T^{-1} X \,|\, \mathcal{G}_t),$$

or equivalently, to the equation

$$V_t = \mathsf{E}_{\mathsf{Q}^*}(\textstyle\int_t^T (p(u, V_u)\lambda_u - (r_u + \lambda_u)V_u) \, du + X \,|\, \mathcal{G}_t).$$

Proposition 11 *If $\Delta V_\tau = 0$ then $S_t = \mathbb{1}_{\{\tau>t\}} V_t$. In general, S is given by formula*

$$S_t = \mathbb{1}_{\{\tau>t\}} \{ V_t - B_t \, \mathsf{E}_{\mathsf{Q}^*}(B_\tau^{-1} \, \mathbb{1}_{\{\tau\le T\}} \Delta V_\tau \,|\, \mathcal{G}_t) \}.$$