CREDIT RISK: MODELLING, VALUATION AND HEDGING

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1. VALUE-OF-THE-FIRM APPROACH
2. INTENSITY-BASED APPROACH
3. MODELLING OF DEPENDENT DEFAULTS
4. <u>CREDIT RATINGS AND MIGRATIONS</u>

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CREDIT RATINGS AND MIGRATIONS

1 Model's Inputs
1.1 Term Structure of Credit Spreads
1.1.1 Credit Classes
1.1.2 Credit Spreads
1.1.3 Spot Martingale Measure P^*
1.1.4 Zero-Coupon Bonds
1.1.5 Conditional Dynamics of Bonds Prices
1.2 Recovery Schemes
2 Credit Migration Process
3 Defaultable Term Structures
3.1 Single Credit Ratings Case
3.1.1 Credit Migrations
3.1.2 Martingale Dynamics of a Defaultable ZCB
3.1.3 Risk-Neutral Representations
3.2 Multiple Credit Ratings Case
3.2.1 Credit Migrations
3.2.2 Martingale Dynamics of a Defaultable ZCB
3.2.3 Risk-Neutral Representations
3.3 Statistical Probability
3.3.1 Market Prices for Risks
3.3.2 Statistical Default Intensities

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1. MODEL'S INPUTS

Standard intensity-based approach (as, for instance, in Jarrow and Turnbull (1995) or Jarrow, Lando and Turnbull (1997)) relies on the following assumptions:

- existence of the martingale measure Q^* is postulated,
- the relationship between the statistical probability P and the risk-neutral probability Q^* derived via calibration,
- credit migrations process is modelled as a Markov chain,
- market and credit risk are separated (independent).

The HJM-type model of defaultable term structures with multiple ratings was proposed by Bielecki and Rutkowski (2000) and Schönbucher (2000).

This approach:

- formulates sufficient consistency conditions that tie together credit spreads and recovery rates in order to construct a risk-neutral probability Q^* and the corresponding risk-neutral intensities of credit events,
- shows how the statistical probability P and the risk-neutral probability Q^* are connected via the market price of interest rate risk and the market price of credit risk,
- combines market and credit risks.

1.1 Term Structure of Credit Spreads

We are given a filtered probability space $(\Omega, \mathsf{F}, \mathsf{P})$ endowed with a d-dimensional standard Brownian motion W.

Remark. We may assume that the filtration $\mathsf{F} = \mathsf{F}^{\mathsf{W}}$.

For any fixed maturity $0 < T \leq T^*$ the price of a zero-coupon Treasury bond equals

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right),$$

where the default-free instantaneous forward rate $f(t,T)$ process is subject to the standard HJM postulate.

(HJM) The dynamics of the instantaneous forward rate $f(t,T)$ are given by, for $t \leq T$,

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u$$

for some deterministic function $f(0,\cdot) \,:\, [0,T^*] \to \mathsf{R}$, and some F-adapted stochastic processes

$$\alpha \,:\, A \times \Omega \to \mathsf{R}, \quad \sigma \,:\, A \times \Omega \to \mathsf{R}^d,$$

where $A = \{(u,t) \,|\, 0 \leq u \leq t \leq T^*\}$.

1.1.1 Credit Classes

Suppose there are $K \geq 2$ credit rating classes, where the K^{th} class corresponds to the default-free bond.

For any fixed maturity $0 < T \leq T^*$, the *defaultable* instantaneous forward rate $g_i(t,T)$ corresponds to the rating class $i = 1, \ldots, K-1$. We assume that:

(HJM^i) The dynamics of the instantaneous defaultable forward rates $g_i(t,T)$ are given by, for $t \leq T$,

$$g_i(t,T) = g_i(0,T) + \int_0^t \alpha_i(u,T)\,du + \int_0^t \sigma_i(u,T)\,dW_u$$

for some deterministic functions $g_i(0,\cdot) : [0,T^*] \to \mathsf{R}$, and some F-adapted stochastic processes

$$\alpha_i : A \times \Omega \to \mathsf{R}, \quad \sigma_i : A \times \Omega \to \mathsf{R}^d$$

1.1.2 Credit Spreads

We assume that

$$g_{K-1}(t,T) > g_{K-2}(t,T) > \ldots > g_1(t,T) > f(t,T)$$

for every $t \leq T$.

Definition 1 For every $i = 1, 2, \ldots, K-1$, the *credit spread* equals $s_i(t,T) = g_i(\cdot,T) - f(\cdot,T)$.

1.1.3 Spot Martingale Measure P^*

The following condition excludes arbitrage across default-free bonds for all maturities $T \le T^*$ and the savings account:

(M) There exists an F-adapted R^d-valued process γ such that

$$\mathsf{E}_{\mathsf{P}} \left\{ \exp\left(\int_0^{T^*} \gamma_u \, dW_u - \tfrac{1}{2} \int_0^{T^*} |\gamma_u|^2 \, du \right) \right\} = 1$$

and, for any maturity $T \le T^*$, we have

$$\alpha^*(t,T) = \tfrac{1}{2} |\sigma^*(t,T)|^2 - \sigma^*(t,T)\gamma_t$$

where

$$\alpha^*(t,T) = \int_t^T \alpha(t,u) \, du$$

$$\sigma^*(t,T) = \int_t^T \sigma(t,u) \, du.$$

Let γ be some process satisfying Condition (M). Then the probability measure P^*, given by the formula

$$\frac{d\mathsf{P}^*}{d\mathsf{P}} = \exp\left(\int_0^{T^*} \gamma_u \, dW_u - \tfrac{1}{2} \int_0^{T^*} |\gamma_u|^2 \, du \right), \quad \mathsf{P}\text{-a.s.},$$

is a *spot martingale measure* for the default-free term structure.

1.1.4 Zero-Coupon Bonds

The price of the T-maturity default-free zero-coupon bond (ZCB) is given by the equality

$$B(t,T) := \exp\left(-\int_t^T f(t,u)\,du\right).$$

Formally, the Treasury bond corresponds to credit class K.

"Conditional" value of T-maturity defaultable ZCB belonging at time t to the credit class $i = 1, 2, \ldots, K-1$, equals

$$D_i(t,T) := \exp\left(-\int_t^T g_i(t,u)\,du\right).$$

We consider discounted price processes

$$Z(t,T) = B_t^{-1} B(t,T), \quad Z_i(t,T) = B_t^{-1} D_i(t,T),$$

where B_t is the usual discount factor (savings account)

$$B_t = \exp\left(\int_0^t f(u,u)\,du\right).$$

Let us define a Brownian motion W^* under P^* by setting

$$W_t^* = W_t - \int_0^t \gamma_u\,du, \quad \forall\, t \in [0, T^*].$$

1.1.5 Conditional Dynamics of Bonds Prices

Lemma 1 *Under the spot martingale measure* P^**, for any fixed maturity* $T \leq T^*$*, the discounted price processes* $Z(t,T)$ *and* $Z_i(t,T)$ *satisfy*

$$dZ(t,T) = Z(t,T)b(t,T)\,dW_t^*,$$

where $b(t,T) = -\sigma^*(t,T)$*, and*

$$dZ_i(t,T) = Z_i(t,T)(\lambda_i(t)\,dt + b_i(t,T)\,dW_t^*)$$

where

$$\lambda_i(t) = a_i(t,T) - f(t,t) + b_i(t,T)\gamma_t$$

and

$$a_i(t,T) = g_i(t,t) - \alpha_i^*(t,T) + \tfrac{1}{2}\left|\sigma_i^*(t,T)\right|^2$$

$$b_i(t,T) = -\sigma_i^*(t,T).$$

Remark 1 Observe that usually the process $Z_i(t,T)$ does not follow a martingale under the spot martingale measure P^*. This feature is related to the fact that it does not represent the (discounted) price of a tradable security.

1.2 Recovery Schemes

Let Y denote the cash flow at maturity T and let Z be the *recovery process* (an F-adapted process). We take $K = 2$.

FRTV: Fractional Recovery of Treasury Value

Fixed recovery at maturity scheme. We set $Z_t = \delta B(t,T)$ and thus

$$Y = \mathbb{1}_{\{\tau > T\}} + \delta\, \mathbb{1}_{\{\tau \le T\}}.$$

FRPV: Fractional Recovery of Par Value

Fixed recovery at time of default. We set $Z_t = \delta$, where δ is a constant. Thus

$$Y = \mathbb{1}_{\{\tau > T\}} + \delta B^{-1}(\tau, T)\, \mathbb{1}_{\{\tau \le T\}}.$$

FRMV: Fractional Recovery of Market Value

The owner of a defaultable ZCB receives at time of default a fraction of the bond's market value just prior to default. We set $Z_t = \delta D(t,T)$, where $D(t,T)$ is the pre-default value of the bond. Thus

$$Y = \mathbb{1}_{\{\tau > T\}} + \delta D(\tau, T) B^{-1}(\tau, T)\, \mathbb{1}_{\{\tau \le T\}}.$$

2 CREDIT MIGRATION PROCESS

We assume that the set of rating classes is $\mathcal{K} = \{1, \ldots, K\}$, where the class K corresponds to default. The *migration process* C will be constructed as a (nonhomogeneous) conditionally Markov process on $\mathcal{K}$. Moreover, the state K will be the unique *absorbing* state for this process.

Let us denote by $\mathcal{F}_t^C$ the σ-field generated by C up to time t. A process C is *conditionally Markov* with respect to the reference filtration F if for arbitrary $s > t$ and $i, j \in \mathcal{K}$ we have

$$\mathsf{Q}^*\left(C_{t+s} = i \,|\, \mathcal{F}_t \vee \mathcal{F}_t^C\right) = \mathsf{Q}^*\left(C_{t+s} = i \,|\, \mathcal{F}_t \vee \{C_t = j\}\right).$$

The probability measure Q^* is the extended spot martingale measure.

The formula above will provide the risk-neutral conditional probability that the defaultable bond is in class i at time $t + s$, given that it was in the credit class C_t at time t.

We introduce the default time τ by setting

$$\tau = \inf\{t \in \mathsf{R}_+ \,:\, C_t = K\}.$$

For any date $t,$ we denote by $\hat{C}_t$ the previous bond's rating.

3 DEFAULTABLE TERM STRUCTURE

3.1 Single Rating Class ($K = 2$)

We assume the FRTV scheme (other recovery schemes can also be covered, though).

Our first goal is to derive the equation that is satisfied by the risk-neutral intensity of default time.

Intensity of Default Time

We introduce the *risk-neutral default intensity* $\lambda_{1,2}$ as a solution to the *no-arbitrage equation*

$$(Z_1(t,T) - \delta Z(t,T))\lambda_{1,2}(t) = Z_1(t,T)\lambda_1(t).$$

It is interesting to notice that for $\delta = 0$ (zero recovery) we have simply

$$\lambda_{1,2}(t) = \lambda_1(t), \quad \forall\, t \in [0,T].$$

On the other hand, if we take $\delta > 0$ then the process $\lambda_{1,2}$ is strictly positive provided that

$$D(t,T) > \delta B(t,T), \quad \forall\, t \in [0,T].$$

Recall that we have assumed that $D(t,T) < B(t,T)$.

3.1.1 Credit Migrations

Since $K = 2$, the migration process C lives on two states. The state 1 is the *pre-default state,* and the state 2 is the absorbing *default state.* We may and do assume that $C_0 = 1$.

We postulate that the conditional intensity matrix for the process C is given by the formula

$$\Lambda_t = \begin{pmatrix} -\lambda_{1,2}(t) & \lambda_{1,2}(t) \\ 0 & 0 \end{pmatrix}.$$

For $\delta = 0$, the matrix Λ takes the following simple form

$$\Lambda_t = \begin{pmatrix} -\lambda_1(t) & \lambda_1(t) \\ 0 & 0 \end{pmatrix}.$$

The default time τ now equals

$$\tau = \inf \{ t \in \mathsf{R}_+ \, : \, C_t = 2 \}.$$

It is defined on an enlarged probability space

$$(\Omega^*, \mathcal{F}_{T^*}, \mathsf{Q}^*) := (\Omega \times \hat{\Omega}, \mathcal{F}_{T^*} \otimes \hat{\mathcal{F}}, \mathsf{P}^* \otimes \mathsf{Q})$$

where the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \mathsf{Q})$ is large enough to support a unit exponential random variable, η say. Then

$$\tau = \inf \{ t \in \mathsf{R}_+ \, : \, \textstyle\int_0^t \lambda_{1,2}(u) \, du \geq \eta \}.$$

Hypotheses (H)

All processes and filtrations may always be extended past the horizon date T^* "by constancy."

We set $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathbb{H}$ the filtration generated by the process H:

$$\mathcal{H}_t = \sigma(H_u : u \leq t).$$

In other words, $\mathbb{H}$ is the filtration associated with the observations of the default time.

It is clear that in the present setup

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}.$$

It is not difficult to check that the hypotheses (H.1)-(H.3) hold in the present context.

In the general case of a model with multiple ratings, the filtration $\mathbb{H}$ will be generated by the migrations process C, that is, we shall set

$$\mathcal{H}_t = \sigma(C_u : u \leq t).$$

Due to the judicious construction of the migration process C, the hypotheses (H.1)-(H.2) remain valid in the case of multiple ratings.

3.1.2 Martingale Dynamics of a Defaultable ZCB

Thanks to the consistency equation, the process

$$M_{1,2}(t) := H_t - \int_0^t \lambda_{1,2}(u)(1 - H_u)\, du$$

is a martingale under $\mathbb{Q}^*$ relative to the enlarged filtration $\mathbb{G}$.

Recall that for any $t \in [0, T]$ we have

$$D(t,T) = \exp\Big(-\int_t^T g(t,u)\, du\Big)$$

and that $D(t,T)$ is interpreted as the pre-default value of a T-maturity defaultable ZCB that is subject to the FRTV scheme.

In other words, $D(t,T)$ is understood as the value of a T-maturity defaultable ZCB conditioned on the event: the bond has not defaulted by the time t.

Recall that

$$Z_1(t,T) = B_t^{-1} D(t,T)$$

and

$$Z(t,T) = B_t^{-1} B(t,T).$$

Auxiliary Process $\hat{Z}(t,T)$

We introduce an auxiliary process $\hat{Z}(t,T), t \in [0,T]$,

$$\hat{Z}(t,T) = \mathbb{1}_{\{\tau>t\}} Z_1(t,T) + \delta\, \mathbb{1}_{\{\tau\le t\}} Z(t,T).$$

It can be shown that $\hat{Z}(t,T)$ satisfies the SDE (A)

$$\begin{aligned} d\hat{Z}(t,T) \;&=\; Z_1(t,T) b_1(t,T)\, \mathbb{1}_{\{\tau>t\}}\, dW_t^* \\ &+\; \delta Z(t,T) b(t,T)\, \mathbb{1}_{\{\tau\le t\}}\, dW_t^* \\ &+\; (\delta Z(t,T) - Z_1(t,T))\, dM_{1,2}(t). \end{aligned}$$

Notice that $\hat{Z}(t,T)$ follows a G-martingale under Q*.

This leads to construction of an arbitrage–free model of the defaultable term structure and to risk-neutral representation for the price of the defaultable bond.

We introduce the price process through the following definition.

Definition 2 The *price process* $D_C(t,T)$ of a T-maturity ZCB is given by

$$D_C(t,T) = B_t \hat{Z}(t,T).$$

3.1.3 Risk-Neutral Representations

Proposition 1 *The price $D_C(t,T)$ of a defaultable ZCB satisfies*

$$D_C(t,T) = \mathbb{1}_{\{\tau>t\}} D(t,T) + \delta\, \mathbb{1}_{\{\tau\le t\}} B(t,T).$$

$$D_C(t,T) = \mathbb{1}_{\{C_t=1\}} \exp\big(-\textstyle\int_t^T g(t,u)\,du\big)$$

$$+\delta\, \mathbb{1}_{\{C_t=2\}} \exp\big(-\textstyle\int_t^T f(t,u)\,du\big).$$

Moreover, the risk-neutral valuation formula holds

$$D_C(t,T) = B_t\, \mathsf{E}_{\mathsf{Q}^*}(\delta B_T^{-1}\, \mathbb{1}_{\{\tau\le T\}} + B_T^{-1}\, \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t).$$

Furthermore

$$D_C(t,T) = B(t,T)\, \mathsf{E}_{\mathsf{Q}_T}(\delta\, \mathbb{1}_{\{\tau\le T\}} + \mathbb{1}_{\{\tau>T\}} \,|\, \mathcal{G}_t)$$

where Q_T is the T-forward measure associated with Q^*.

Special cases:

- For $\delta = 0$, we obtain $D_C(t,T) = \mathbb{1}_{\{\tau>t\}} D(t,T)$.
- For $\delta = 1$, we have, as expected, $D_C(t,T) = B(t,T)$.

Default-Risk-Adjusted Discount Factor

The *default-risk-adjusted discount factor* equals

$$\hat{B}_t = \exp\big(\int_0^t (r_u + \lambda_{1,2}(u))\, du \big)$$

and we set

$$\hat{B}(t,T) = \hat{B}_t \, \mathsf{E}_{\mathsf{P}^*}(\hat{B}_T^{-1} \,|\, \mathcal{F}_t).$$

We consider a bond with FRTV.

Proposition 2 *We have*

$$D_C(t,T) = \delta B(t,T) + (1-\delta)\, \mathbb{1}_{\{\tau > t\}} \hat{B}(t,T)$$

and thus

$$D_C(t,T) = B(t,T) - (1-\delta) \Big(B(t,T) - \mathbb{1}_{\{\tau > t\}} \hat{B}(t,T) \Big).$$

Interpretation:

- A decomposition of $D_C(t,T)$ of the price of a defaultable ZCB into its predicted *post-default value* $\delta B(t,T)$ and the *pre-default premium* $D_C(t,T) - \delta B(t,T)$.
- A decomposition $D_C(t,T)$ as the difference between its *default-free value* $B(t,T)$ and the *expected loss in value* due to the credit risk. From the buyer's perspective: the price $D_C(t,T)$ equals the price of the default-free bond minus a compensation for the credit risk.

3.2 Multiple Credit Ratings Case

We work under the FRTV scheme. To each credit rating $i = 1, \dots, K-1$, we associate the recovery rate $\delta_i \in [0,1)$, where δ_i is the fraction of par paid at bond's maturity, if a bond belonging to the i^{th} class defaults.

As we shall see shortly, the notation $\hat{C}_\tau$ indicates the rating of the bond just prior to default. Thus, the cash flow at maturity is

$$X = \mathbb{1}_{\{\tau > T\}} + \delta_{\hat{C}_\tau} \mathbb{1}_{\{\tau \le T\}}.$$

To simplify presentation we let $K = 3$ (two different credit classes) and we let $\delta_i \in [0,1)$ for $i = 1, 2$. The results carry over to the general case of $K \ge 2$.

3.2.1 Credit Migrations

Risk-neutral intensities of credit migrations $\lambda_{1,2}(t)$, $\lambda_{1,3}(t)$, $\lambda_{2,1}(t)$ and $\lambda_{2,3}(t)$ are specified by the *no-arbitrage condition*:

$$\begin{aligned}
\lambda_{1,2}(t)(Z_2(t,T) \; &- \; Z_1(t,T)) + \lambda_{1,3}(t)(\delta_1 Z(t,T) - Z_1(t,T)) \\
&+ \; \lambda_1(t) Z_1(t,T) = 0, \\
\lambda_{2,1}(t)(Z_1(t,T) \; &- \; \hat{Z}_2(t,T)) + \lambda_{2,3}(t)(\delta_2 Z(t,T) - Z_2(t,T)) \\
&+ \; \lambda_2(t) Z_2(t,T) = 0.
\end{aligned}$$

If the processes $\lambda_{1,2}(t)$, $\lambda_{1,3}(t)$, $\lambda_{2,1}(t)$ and $\lambda_{2,3}(t)$ are non-negative, we construct a migration process C, on some enlarged probability space $(\Omega^*, \mathsf{G}, \mathsf{Q}^*)$, with the conditional intensity matrix

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \lambda_{1,3}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \lambda_{2,3}(t) \\ 0 & 0 & 0 \end{pmatrix}$$

where $\lambda_{i,i}(t) = -\Sigma_{j \neq i} \lambda_{i,j}(t)$ for $i = 1, 2$. Notice that the transition intensities $\lambda_{i,j}$ follow F-adapted stochastic processes. The default time τ is given by the formula

$$\tau = \inf\{\, t \in \mathsf{R}_+ \,:\, C_t = 3 \,\}.$$

3.2.2 Martingale Dynamics of a Defaultable ZCB

We set $H_i(t) = \mathbb{1}_{\{C_t = i\}}$ for $i = 1, 2$, and we let $H_{i,j}(t)$ represent the number of transitions from i to j by C over the time interval $(0, t]$.

It can be shown that the process

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(s) H_i(s)\, ds, \quad \forall\, t \in [0, T],$$

for $i = 1, 2$ and $j \neq i$, is a martingale on the enlarged probability space $(\Omega^*, \mathsf{G}, \mathsf{Q}^*)$.

Auxiliary Process $\hat{Z}(t,T)$

We introduce the process $\hat{Z}(t,T)$ as a solution to the following SDE (A)

$$
\begin{aligned}
d\hat{Z}(t,T) &= (Z_2(t,T) - Z_1(t,T))\, dM_{1,2}(t) \\
&\quad + (Z_1(t,T) - Z_2(t,T))\, dM_{2,1}(t) \\
&\quad + (\delta_1 Z(t,T) - Z_1(t,T))\, dM_{1,3}(t) \\
&\quad + (\delta_2 Z(t,T) - Z_2(t,T))\, dM_{2,3}(t) \\
&\quad + H_1(t) Z_1(t,T) b_1(t,T)\, dW_t^* \\
&\quad + H_2(t) Z_2(t,T) b_2(t,T)\, dW_t^* \\
&\quad + (\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)) Z(t,T) b(t,T)\, dW_t^*,
\end{aligned}
$$

with the initial condition

$$\hat{Z}(0,T) = H_1(0) Z_1(0,T) + H_2(0) Z_2(0,T).$$

The process $\hat{Z}(t,T)$ follows a martingale on $(\Omega^*, \mathsf{G}, \mathsf{Q}^*)$, and thus Q^* is called the *extended spot martingale measure.*

The proof of the next result employs the no-arbitrage condition.

Lemma 2 *For any maturity $T \le T^*$, we have*

$$\hat{Z}(t,T) = \mathbb{1}_{\{C_t \neq 3\}}\, Z_{C_t}(t,T) + \mathbb{1}_{\{C_t = 3\}}\, \delta_{\hat{C}_t}\, Z(t,T)$$

for every $t \in [0,T]$.

Price of a Defaultable ZCB

We introduce the price process of a T-maturity defaultable ZCB by setting $D_C(t,T) = B_t \hat{Z}(t,T)$ for any $t \in [0,T]$.

In view of Lemma 2, the price of a defaultable ZCB equals

$$D_C(t,T) = \mathbb{1}_{\{C_t \neq 3\}} D_{C_t}(t,T) + \mathbb{1}_{\{C_t = 3\}} \delta_{\hat{C}_t} B(t,T)$$

with some initial condition $C_0 \in \{1,2\}$. An analogous formula can be established for an arbitrary number K of rating classes, namely,

$$D_C(t,T) = \mathbb{1}_{\{C_t \neq K\}} D_{C_t}(t,T) + \mathbb{1}_{\{C_t = K\}} \delta_{\hat{C}_t} B(t,T).$$

Properties of $D_C(t,T)$:

- $D_C(t,T)$ follows a $(\mathsf{Q}^*, \mathsf{G})$-martingale, when discounted by the savings account.
- In contrast to the "conditional" price processes $D_i(t,T)$, the process $D_C(t,T)$ admits discontinuities, associated with changes in credit quality.
- It represents the price process of a tradable security: the defaultable ZCB of maturity T.

3.2.3 Risk-Neutral Representations

Recall that $\delta_i \in [0,1)$ is the recovery rate for a bond which is in the i^{th} rating class prior to default.

The price process $D_C(t,T)$ of a T-maturity defaultable ZCB equals

$$\begin{aligned} D_C(t,T) \;=\;& \mathbb{1}_{\{C_t \neq 3\}}\, B(t,T)\, \exp\big(-\textstyle\int_t^T s_{C_t}(t,u)\,du\big) \\ &+\; \mathbb{1}_{\{C_t = 3\}}\, \delta_{\hat{C}_t}\, B(t,T) \end{aligned}$$

where $s_i(t,u) = g_i(t,u) - f(t,u)$ is the i^{th} credit spread.

Proposition 3 *The price process $D_C(t,T)$ satisfies the risk-neutral valuation formula*

$$D_C(t,T) = B_t\, \mathsf{E}_{\mathsf{Q}^*}(\delta_{\hat{C}_T}\, B_T^{-1}\, \mathbb{1}_{\{\tau \le T\}} + B_T^{-1}\, \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t).$$

It is also clear that

$$D_C(t,T) = B(t,T)\, \mathsf{E}_{\mathsf{Q}_T}(\delta_{\hat{C}_T}\, \mathbb{1}_{\{\tau \le T\}} + \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t)$$

where Q_T stands for the T-forward measure associated with the extended spot martingale measure Q^*.

3.3 Statistical Probability

We shall now change, using a suitable generalization of Girsanov's theorem, the measure Q^* to the equivalent probability measure Q.

In the financial interpretation, the probability measure Q will play the role of the *statistical probability*.

It is thus natural to postulate that the restriction of Q to the original probability space Ω necessarily coincide with the statistical probability P for the default-free market.

Condition (L): We set

$$\frac{d\mathsf{Q}}{d\mathsf{Q}^*} = L_{T^*}, \quad \mathsf{Q}^*\text{-a.s.},$$

where the Q^*-local positive martingale L is given by the formula

$$dL_t = -L_t \gamma_t \, dW_t^* + L_{t-} \, dM_t, \quad L_0 = 1,$$

and the Q^*-local martingale M equals

$$\begin{aligned} dM_t &= \sum_{i \neq j} \kappa_{i,j}(t) \, dM_{i,j}(t) \\ &= \sum_{i \neq j} \kappa_{i,j}(t) \, (dH_{i,j}(t) - \lambda_{i,j}(t) H_i(t) \, dt) \end{aligned}$$

for some processes $\kappa_{i,j} > -1$.

3.3.1 Prices for Market and Credit Risks

For any $i \neq j$ we denote by $\kappa_{i,j} > 1$ an arbitrary nonnegative F-predictable process such that

$$\int_0^{T^*} (\kappa_{i,j}(t) + 1)\lambda_{i,j}(t)\,dt < \infty, \quad \mathsf{Q}^*\text{-a.s.}$$

We assume that $\mathsf{E}_{\mathsf{Q}^*}(L_{T^*}) = 1$, so that the probability measure Q is well defined on $(\Omega^*, \mathcal{G}_{T^*})$.

Financial interpretations:

- The process γ corresponds to the *market price of interest rate risk.*
- Processes $\kappa_{i,j}$ represent the *market prices of credit risk.*

Let us define processes $\lambda^{\mathsf{Q}}_{i,j}$ by setting for $i \neq j$

$$\lambda^{\mathsf{Q}}_{i,j}(t) = (\kappa_{i,j}(t) + 1)\lambda_{i,j}(t)$$

and

$$\lambda^{\mathsf{Q}}_{i,i}(t) = -\sum_{j \neq i} \lambda^{\mathsf{Q}}_{i,j}(t).$$

3.3.2 Statistical Default Intensities

Proposition 4 *Under an equivalent probability* Q, *given by Condition (L), the process* C *is a conditionally Markov process. The matrix of conditional intensities of* C *under* Q *equals*

$$\Lambda_t^{\mathsf{Q}} = \begin{pmatrix} \lambda_{1,1}^{\mathsf{Q}}(t) & \dots & \lambda_{1,K}^{\mathsf{Q}}(t) \\ \cdot & \dots & \cdot \\ \lambda_{K-1,1}^{\mathsf{Q}}(t) & \dots & \lambda_{K-1,K}^{\mathsf{Q}}(t) \\ 0 & \dots & 0 \end{pmatrix}.$$

If the market price for the credit risk depends only on the current rating i (and not on the rating j after jump), so that

$$\kappa_{i,j} = \kappa_{i,i} =: \kappa_i \quad \text{for every} \quad j \neq i$$

then $\Lambda_t^{\mathsf{Q}} = \Phi_t \Lambda_t$, where $\Phi_t = \text{diag}\,[\phi_i(t)]$ with $\phi_i(t) = \kappa_i(t) + 1$ is the diagonal matrix (see, e.g., Jarrow, Lando and Turnbull (1997).

Important issues:

- Valuation of defaultable coupon-bonds.
- Modelling of correlated defaults (dependent migrations).
- Valuation and hedging of credit derivatives.
- Calibration to liquid instruments.