Numeraire-invariant option pricing

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A conceptual definition of an option

- An *Option* can be viewed as a pair $\mathcal{O} = (T, O)$ consisting of an *Expiry* T and a *Payoff* O (in base currency) paid at T.

- For a European option, the expiry T is deterministic.

- In general, T is a stopping time bounded by some $m > 0$.

  - American and Bermudan options: T is the optimal exercise time.

  - Barrier options with rebate: T is the first passage time to the barrier.

  - Credit derivatives with recovery : T is the default time.

- The payoff O is known at time T: random variable O is measurable w.r. to σ-algebra $\mathcal{F}_T$ of events at or before T:
$$\mathcal{F}_T := \{\Lambda \in \mathcal{F} : \Lambda \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

A conceptual definition of an option $\cdots$

- The definition so far depends only on the underlying filtration $(\mathcal{F}_t)$. To talk about the option price, a probability measure $\mathbb{P}$ and an integrability condition are required.

- Think of a *Numeraire* β as a claim which pays no dividend and has a positive price $\beta_t > 0$ at all times $t \leq m$, e.g., a zero-dividend stock, or the m-maturity zero-coupon bond.

- To each numeraire β, there is associated a *Numeraire Measure* $\mathbb{P}^\beta$, characterized by the property that if B is any other numeraire, then the relative price process (B_t/β_t) is a (right-continuous) $\mathbb{P}^\beta$-martingale.

- The required integrability condition on option $\mathcal{O} = (T, O)$ is this: O/β_T is $\mathbb{P}^\beta$-integrable for some numeraire β.

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Numeraire invariance of the option definition

The important aspect of this integrability condition is that if it holds for some numeraire, then it holds for *all* numeraires:

$$\frac{B_0}{\beta_0} \mathbb{E}^B\left[\frac{O}{B_T}\right] = \mathbb{E}^\beta\left[\frac{O}{B_T}\frac{B_m}{\beta_m}\right] \qquad \text{(change of numeraire)}$$

$$= \mathbb{E}^\beta\left[\mathbb{E}^\beta\left[\frac{O}{B_T}\frac{B_m}{\beta_m}\mid \mathcal{F}_T\right]\right] \qquad \text{(iterating expectation)}$$

$$= \mathbb{E}^\beta\left[\frac{O}{B_T}\mathbb{E}^\beta\left[\frac{B_m}{\beta_m}\mid \mathcal{F}_T\right]\right] \qquad \text{(by } \mathcal{F}_T \text{ measurability of } \frac{O}{B_T})$$

$$= \mathbb{E}^\beta\left[\frac{O}{B_T}\frac{B_T}{\beta_T}\right] \qquad \text{(optional sampling theorem)}$$

$= \mathbb{E}^\beta[O/\beta_T] < \infty.$

Price process $\mathcal{O}_t$ of an option

Consider investing the option payoff O at expiry T in a numeraire β and holding this position until the terminal date m. We end up with a claim that pays $\beta_m O / \beta_T$ at time m. At a time and state before expiry (i.e., $t < T(\omega)$), the option has not yet been invested in the numeraire, so this claim is identical with the option itself. We are thus forced to define the option price $\mathcal{O}_t$ to be the price of this claim, i.e., $\beta_t \mathbb{E}^\beta [O/\beta_T \mid \mathcal{F}_t]$.

After expiry, the option has ceased to exist and has no price. In this case it is convenient to define the option price to be zero. We thus arrive at the following definition of an option price:

$$\mathcal{O}_t := 1_{t \le T} \beta_t \mathbb{E}^\beta [\frac{O}{\beta_T} \mid \mathcal{F}_t].$$

As before, *this definition is independent of choice of numeraire.*

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Price Transitivity Law

Using the optional sampling theorem, the option price at any stopping time τ (bounded by m) is given by

$$\mathcal{O}_\tau = 1_{\tau \leq T} \beta_\tau \mathbb{E}^\beta\left[\frac{O}{\beta_T} \mid \mathcal{F}_\tau\right] \text{ a.s.}$$

The pair $(\tau, \mathcal{O}_\tau)$ is a τ-expiry option with payoff $\mathcal{O}_\tau$. Let S be another stopping time. What can we say about the time S price $(\tau, \mathcal{O}_\tau)_S$ of this option? When $S \leq \tau \leq T$, we simply have

$$(\tau, \mathcal{O}_\tau)_S = \mathcal{O}_S.$$

That is, pricing to time τ and then pricing to time S is the same as pricing directly to time S. This follows from the fact that if (M_t) is a martingale, then optional sampling theorem and the law of iterated expectations combine to yield $M_S = \mathbb{E}[M_T \mid \mathcal{F}_\tau \mid \mathcal{F}_S]$.

Practical operations on options

Let $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ be two options. Define

- *Sum:* $\mathcal{O}^1 + \mathcal{O}^2 := (T_1 \wedge T_2, \mathcal{O}^1_{T_1 \wedge T_2} + \mathcal{O}^2_{T_1 \wedge T_2})$.
  By price transitivity law, the sum operator is associative and the price operator is linear: if $T \leq T_1 \wedge T_2$, then
  $(\mathcal{O}^1 + \mathcal{O}^2)_T = \mathcal{O}^1_T + \mathcal{O}^2_T$.

- *Rollover:* $\mathcal{O}^1 \rightarrowtail \mathcal{O}^2 := (T_2, O^1 O^2 / \mathcal{O}^2_{T_1})$.
  "Rolling $\mathcal{O}^1$ over $\mathcal{O}^2$", i.e., investing at T_1 the first option payoff O^1 in the second option $\mathcal{O}^2$, assuming here
  $T_1 \leq T_2, \; O^2 > 0$.

- *T-expiry swaption:* $(T, (\mathcal{O}^1_T - \mathcal{O}^2_T)^+), \quad T \leq T_1 \wedge T_2$.
  Option to swap, i.e., exchange, $\mathcal{O}^2$ with $\mathcal{O}^1$ at T.

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Indistinguishable options

Let $\mathcal{O}^1 = (T_1, O^1)$ and $\mathcal{O}^2 = (T_2, O^2)$ be two options.

- $\mathcal{O}^1$ and $\mathcal{O}^2$ are *Equivalent* if $T_1 = T_2$ a.s. and $O^1 = O^2$ a.s.

- $\mathcal{O}^1$ and $\mathcal{O}^2$ are *Indistinguishable* if price processes $(\mathcal{O}_t^1)$ and $(\mathcal{O}_t^2)$ are indistinguishable, i.e., a.s. $\mathcal{O}_t^1 = \mathcal{O}_t^2$ all t.

- Easy to see two equivalent options are indistinguishable.

- Converse is not true, e.g., $\mathcal{O}^2$ is obtained from $\mathcal{O}^1$ by "*postponing zero payments*" of $\mathcal{O}^1$.

Theorem: Options $\mathcal{O}^1$ and $\mathcal{O}^2$ are indistinguishable if and only if $O^1 = O^2$ a.s. and $\{T_1 \neq T_2\} \subset \{O^1 = 0\}$ a.s.

So, T_1 and T_2 need not be the same, but they differ only at zero payments.

Nonnegative options and nonnegative arbitrage

Assume option $\mathcal{O} = (T, O)$ is *Nonnegative*, i.e., $O \geq 0$ a.s. Then

(In practice, most options are nonnegative, but are not positive either, for they can have zero payoff in some states (i.e., $\mathbb{P}[O = 0] > 0$) , e.g., call and put options and swaptions.)

- Obviously, $\mathcal{O}_t \geq 0$ a.s. all t, and $\mathcal{O}_0 > 0$ if $\mathbb{P}[O > 0] > 0$.

- Right continuity implies: a.s. $\mathcal{O}_t \geq 0$ all t.

- *Nonnegative Arbitrage*: Once the option price becomes zero, it stays zero thereafter. More precisely, for almost all paths ω, if $\mathcal{O}_t(\omega) = 0$, then $\mathcal{O}_s(\omega) = 0$ for all $s \geq t$.

- In particular, for almost all paths ω, if $O(\omega) > 0$, then $\mathcal{O}_t(\omega) > 0$ for all $t \leq T(\omega)$.

- In particular, if β is a numeraire, then a.s. $\beta_t > 0$ all t.

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Semipositive options

A nonnegative option $\mathcal{O}$ is *Semipositive* if its price is positive before expiry T, i.e., at each time t, $\mathcal{O}_t > 0$ a.s. on $\{t < T\}$.

Theorem: If $\mathcal{O}$ is a semipositive , then a.s. $\mathcal{O}_t > 0$ on $\{t < T\}$ all t, i.e., for almost all paths ω, $\mathcal{O}_t(\omega) > 0$ at all times $t < T(\omega)$.

A basic result on nonnegative options:
Let $\mathcal{O} = (T, O)$ be a nonnegative option. Then, there exists up to equivalence a unique semipositive option that is indistinguishable from $\mathcal{O}$, namely the option (T^0, O), where

$$T^0 := \inf\{t > 0 : \mathcal{O}_t = 0\}.$$

(It turns out that O is automatically $\mathcal{F}_{T^0}$-measurable - in fact, $O = \mathcal{O}_{T^0}$.)

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Trigger options

A progressively measurable process (Z_t) is a *Payoff Process* if a.s. $|Z_t| \leq \beta_t$ all t for some numeraire β.

Example: $Z_t = (\beta_t - K)^+$, $K > 0$.

Proposition: Let (Z_t) be a payoff process. Then for any numeraire β, the β-deflated process (Z_t/β_t) is $\mathbb{P}^\beta$-class D. In particular, for any stopping time T, the pair (T, Z_T) is an option.

If (Z_t) is a payoff process and T a stopping time, we refer to the T-expiry option (T, Z_T) as a $Z_.$-*Trigger Option*.

Examples: American and Bermudan options, barrier options, credit derivatives with recovery.

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Trigger option convergence theorem

Right-continuous version: Let (Z_t) be a right continuous payoff process and $(T_n)_{n=1}^{\infty}$ be a *decreasing* sequence of stopping times converging to a stopping time T, i.e., $T_n \searrow T$. Then, a.s., $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, all t.

Continuous version: Let (Z_t) be a continuous adapted process that is dominated by a continuous numeraire, and $(T_n)_{n=1}^{\infty}$ be a sequence of stopping times converging to a stopping time T. Then, a.s., $(T_n, Z_{T_n})_t \to (T, Z_T)_t$, all t.

These follow from a similar statement about convergence of prices of a sequence of options, which in turn is a simple consequence of the dominated convergence theorem.

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The Snell envelope

The *Snell Envelope* process of a right continuous payoff process (Z_t) is defined by

$$V_t := \sup_{T \geq t} (T, Z_T)_t,$$

where supremum is taken over the set all stopping times T satisfying $t \leq T \leq m$.

Theorem: (V_t) is a right continuous payoff process, and for all $t \leq s$,

$$(s, V_s)_t = \sup_{T \geq s} (T, Z_T)_t.$$

Corollary: (V_t) is a "superclaim", i.e., for $t \leq s$, we have

$$V_t := \sup_{T \geq t} (T, Z_T)_t \geq \sup_{T \geq s} (T, Z_T)_t = (s, V_s)_t.$$

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Definition of American option

Fix $t \in [0, m]$. In the definition of Snell envelop, the supremum $\sup_{T \geq t}(T, Z_T)_t$ is not necessarily attained at any stopping time T. If not, the American option is not defined. However, if the supreme is attained at some stopping time, the "first" stopping time at which it is attained is the "*post t-optimal stopping time*" T_t, defined as the first time $s \geq t$ such that Z_s reaches V_s.

A payoff process (Z_t) is *American* if for each t, the supremum $\sup_{T \geq t}(T, Z_T)_t$ is attained at some stopping time. The supremum will then actually be attained T_t, i.e., $V_t = (T_t, Z_{T_t})_t$, where

$$T_t := \inf\{s \in [t, m] : Z_s = V_s\}.$$

For each s, we then refer to the trigger option $\mathcal{A}^s := (T_s, Z_{T_s})$ as the *Post-s American Option*. Note, $\mathcal{A}_t^t = V_t$.

The American option $\mathcal{A}^s$ can be exercised only at or after s. So, $A^0 := (T_0, Z_{T_0})$ is the American option that can be exercised at any time. If $\mathcal{A}^0$ is not exercised by time s, then it becomes the same as $\mathcal{A}^s$. The *American Stream* (A^s) is "regenerative" in this sense.

American option stream pricing formula

Theorem: Let (Z_t) be an American payoff process. Then for $t \leq s$, we have a.s.

$$\mathcal{A}_t^s = \sup_{T \geq s}(T, Z_T)_t, \quad t \leq s;$$

while for $s \leq t$, we have a.s.

$$\mathcal{A}_t^s = 1_{t \leq T_s} V_t, \quad s \leq t.$$

In particular, $\mathcal{A}_t^0 = 1_{t \leq T_0} V_t$, all t.

The first formula follows from the formula $(s, V_s)_t = \sup_{T \geq s}(T, Z_T)_t$ valid for $t \leq s$. The second formula follows from the easily verified fact that $1_{t \leq T_s} = 1_{t \leq T_s = T_t}$ for $s \leq t$.

Doob-Meyer decomposition of superclaims

A *Superclaim* is a right-continuous payoff process (V_t) such that $V_t \geq (s, V_s)_t$ a.s. for all $t \leq s$. A *Supernumeraire* is a positive superclaim.

- As we saw, the Snell envelope is a superclaim.

Proposition: A right-continuous payoff process (V_t) is a superclaim if and only if the process (V_t/β_t) is a right-continuous $\mathbb{P}^\beta$-supermartingale for some (*hence all*) numeraire β.

Theorem: Let (V_t) be a supernumeraire and β be a numeraire. Then there exist a unique numeraire B with $B_0 = V_0$ and a decreasing predictable process (A_t) such that a.s. $V_t = \beta_t A_t + B_t$ all t.

Multiplicative minimax duality

Let (Z_t) be a *positive* right-continuous payoff process.
We know the Snell envelope (V_t) is then a supernumeraire.
We call a numeraire B a *Domineering Numeraire* if $B_0 = V_0$ and
$B_t \geq V_t$ for all t. The Doob-Meyer decomposition above implies
that there are many domineering numeraires.

Lemma: Let B be a domineering numeraire. Then

$$\sup_{t \geq 0} \left(\frac{Z_t}{B_t} \right) = 1.$$

Theorem: We have a.s. all t,

$$V_t = \inf_{\beta \in \mathcal{C}^+} \beta_t \mathbb{E}^\beta \left[\sup_{s \geq t} \left(\frac{Z_s}{\beta_s} \right) \mid \mathcal{F}_t \right].$$

Multiplicative minimax duality $\cdots$

- In particular, $V_0 = \inf_\beta \beta_0 \mathbb{E}^\beta[\sup_{t \geq 0}(Z_t/\beta_t)]$.

- The infimum is attained at any domineering numeraire.

- So, $\beta_0 \mathbb{E}^\beta[\sup_{t \geq 0}(Z_t/\beta_t)]$ is an *upper bound* for the American option price V_0, for any numeraire β.

- Such an upper bound can be computed by Monte-Carlo simulation. An suitable numeraire must first be chosen.

- These results extend to nonnegative payoff processes (Z_t).

- An "additive version" of minimax duality was previously derived by Rogers (2001) and Haugh & Kogan (2001), and further studied in Andersen & Broadie (2001), Joshi & Theis (2002), and Kolodko & Schoenmakers (2003).

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Multiplicative Doob-Meyer decomposition

A *Local Numeraire* is an adapted, right continuous, positive process (B_t) such that (B_t/β_t) is a $\mathbb{P}^\beta$-local martingale for some (*hence all*) numeraire β.

Theorem: Let (V_t) be a supernumeraire. Then there exists a unique decomposition $V_t = D_t B_t$, where (D_t) is a decreasing predictable process and (B_t) is a local numeraire with $B_0 = V_0$.

A supernumeraire (V_t) *Multiplicative*, if the local numeraire (B_t) in the multiplicative decomposition $V_t = D_t B_t$ is actually the price process of a numeraire B. We then refer to this numeraire B as the *Rollover Numeraire* associated to (V_t). It is clearly a domineering numeraire, and as such relevant to minimax duality.

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An application of multiplicative decomposition

Corollary: Assume that discount factors satisfy $(s, 1)_t \leq 1$ for all $t \leq s$. Then there exists a unique increasing, predictable, local numeraire (B_t) with $B_0 = 1$.

The assumption is equivalent to the identically one process being a supernumeraire. Applying the multiplicative Doob-Meyer decomposition, we obtain a decomposition $1 = D_t B_t$. The resulting local numeraire (B_t) is increasing, as it equals $(1/D_t)$.

This increasing local numeraire (B_t) can be interpreted as the "*continuous money market account*". If it is further assumed to be absolutely continuous, then, taking its logarithmic derivative, we obtain the "*instantaneous interest-rate process*" r_t, which is nonnegative and satisfies $B_t = \exp(\int_0^t r_s ds)$.