Pricing and trading credit default swaps

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Begin at the beginning, and go on till you come to the end. Then, ...

L. Carroll, Alice's Adventures in Wonderland

A probability space $(\Omega, \mathbf{G}, \mathbb{P})$ is given. All the processes are assumed to be $\mathbf{G}$-adapted and càdlàg.

We denote

$$B_t = \exp(\int_0^t r(s)ds)$$

the savings account, where r is deterministic.

Self-Financing Trading Strategies and Dividend-paying Assets

Let S^i, $i = 1, \ldots, k$ denote the price processes of securities that pay dividends according to a process of finite variation D^i, with $D^i_0 = 0$, and S^j, $j = k+1, \ldots, m$ non-dividend-paying assets.

The **wealth process** associated to the strategy $\phi = (\phi^1, \ldots, \phi^m)$ is

$$V_t(\phi) = \sum_{\ell=1}^{m} \phi^\ell_t S^\ell_t \,.$$

A strategy ϕ is said to be **self-financing** if $V_t(\phi) = V_0(\phi) + G_t(\phi)$ where the *gains process* $G(\phi)$ is

$$G_t(\phi) = \sum_{i=1}^{k} \int_{]0,t]} \phi^i_u \, dD^i_u + \sum_{\ell=1}^{m} \int_{]0,t]} \phi^\ell_u \, dS^\ell_u .$$

We say that $\mathbb{Q}$, equivalent to $\mathbb{P}$, is a **martingale measure** if

• the discounted price $B_t^{-1} S_t^j$ of any non-dividend paying traded security is a $\mathbb{Q}$-martingale with respect to $\mathbf{G}$

• the ex-dividend price process S^i associated with the dividend process D^i satisfies:

$$S_t^i = B_t \, \mathbb{E}_Q \Big(S_T^i B_T^{-1} + \int_{]t,T]} B_u^{-1} \, dD_u^i \,\Big|\, \mathcal{G}_t \Big).$$

The processes $S_t^i B_t^{-1} + \int_{]0,t]} B_u^{-1} \, dD_u^i$ are $\mathbb{Q}$-martingales.

For any self-financing trading strategy ϕ, the discounted wealth process $B_t^{-1} V_t(\phi)$ is a $\mathbb{Q}$-martingale.

Defaultable Market

The probability space is endowed with a reference filtration $\mathbf{F}$.

The **default time** τ is a **non-negative random variable.**

$H_t = \mathbb{1}_{\{\tau \le t\}}$ is the **default process**, with natural filtration $\mathbf{H}$. Note that $\mathcal{H}_t = \sigma(t \wedge \tau)$ and that τ is a $\mathbf{H}$-stopping time.

We set $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$.

Some examples

- τ is a stopping time in a Brownian filtration
- λ is a given non-negative $\mathbf{F}$-adapted process and

$$\tau = \inf\{t \ : \ \int_0^t \lambda_u du \geq U\}$$

where U is a non-negative r.v. independent of $\mathbf{F}$.

Defaultable claim

A **defaultable claim maturing at** T is a quadruple (X, A, Z, τ), where

- X is an $\mathcal{F}_T$-measurable random variable,
- A is an $\mathbf{F}$-adapted continuous process of finite variation
- Z is an $\mathbf{F}$-predictable process.

The payoff X is done at time T if $\tau > T$

The payoff Z_τ is done at default time τ if $\tau \leq T$

The process A corresponds to a cumulative continuous payment till default time.

The **dividend process** D of a defaultable claim $(0, A, Z, \tau)$ equals, on $t \le T$,

$$\begin{aligned} D_t &= A_{\tau\wedge t} + \mathbb{1}_{\tau\le t} Z_\tau \\ &= \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u \end{aligned}$$

Credit Default Swap

A **credit default swap** with a constant rate κ and *recovery at default* is a defaultable claim $(0, A, Z, \tau)$, where

- $Z_t \equiv \delta(t)$
- $A_t = -\kappa\, t$ for every $t \in [0, T]$.

The function (or process) $\delta : [0, T] \to \mathbb{R}$ represents the **default protection**, and the constant $\kappa \in \mathbb{R}$ represents the CDS rate (also termed the **spread, premium** or *annuity* of a CDS).

Toy Model

We assume here that **F is the trivial filtration**. Let

$$G(t) = \mathbb{Q}(\tau > t) = \int_t^\infty f(u)du$$

be the $\mathbb{Q}$-survival probability. In that case, for any function h,

$$\begin{aligned} \mathbb{E}(h(\tau)|\mathcal{H}_t)\mathbb{1}_{\{t<\tau\}} &= \mathbb{1}_{\{t<\tau\}} \frac{1}{\mathbb{P}(t<\tau)} \mathbb{E}(h(\tau)\mathbb{1}_{\{t<\tau\}}) \\ &= \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \mathbb{E}\left(\int_t^\infty h(u) f(u) du \right) \end{aligned}$$

We assume that $r = 0$.

The ex-dividend price of a CDS maturing at T with rate κ is

$$S_t(\kappa) = \mathbb{E}_Q\left(\mathbb{1}_{\{t<\tau\leq T\}}\delta(\tau) \,\middle|\, \mathcal{H}_t\right) - \mathbb{E}_Q\left(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big) \,\middle|\, \mathcal{H}_t\right)$$
$$= \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(\int_t^T \delta(u)\, f(u)\, du - \kappa \int_t^T G(u)\, du \right).$$

For a CDS initiated at time 0, the value κ is determined so that $S_0(\kappa) = 0$, hence

$$\int_0^T \delta(u)\, f(u)\, du = \kappa \int_0^T G(u)\, du$$

Note that the price S_t can take negative values.

The process

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du = H_t - \int_0^{t\wedge\tau} \gamma(u)\, du,$$

where $\gamma(u) = \frac{f(u)}{G(u)}$ is a $(\mathbb{Q}, \mathbf{H})$-martingale.

The process

$$L_t = 1\!\!1_{\{t<\tau\}} \frac{1}{G(t)}$$

is a $(\mathbb{Q}, \mathbf{H})$-martingale which satisfies $dL_t = -L_{t-} dM_t$.

Using

$$S_t(\kappa) = L_t \left(\int_t^T \delta(u)\, f(u) du - \kappa \int_t^T G(u)\, du \right)$$

and IP formula, one proves that the dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt\,.$$

The dividend process associated with the CDS is

$$dD_t = -\kappa(1 - H_t)dt + \delta(t)dH_t$$

hence,

$$\begin{aligned} d(S_t(\kappa) + D_t) &= -S_{t-}(\kappa)\, dM_t + (1 - H_t)\,(\kappa - \delta(t)\gamma(t))\, dt \\ &\quad -\kappa(1 - H_t)\, dt + \delta(t)\, dH_t \\ &= (\delta(t) - S_{t-}(\kappa))\, dM_t \end{aligned}$$

The function $\widetilde{S}_t(\kappa)$ such that $1\!\!1_{\{t<\tau\}}\widetilde{S}_t(\kappa) = 1\!\!1_{\{t<\tau\}}S_t(\kappa)$ is the **predefault-price**, it satisfies

$$d\widetilde{S}_t(\kappa) = \left(\widetilde{S}_t(\kappa)\gamma(t) + (\kappa - \delta(t)\gamma(t))\right) dt,$$

We assume that $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0, T]$.

Hedging with CDS

Our aim is to find **a replicating strategy for the defaultable claim** $(X, 0, Z, \tau)$, where X is a constant and $Z_t = z(t)$.

Let $\widehat{g}$ and ϕ^1 be two functions defined as

$$\widehat{g}(t) = \frac{1}{G(t)} \left(\int_0^t z(s)dG(s) + XG(T) \right)$$

$$\phi^1(t) = \frac{h(t) - \widehat{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)},$$

Let $\phi_t^0 = V_t(\phi) - \phi^1(t)S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_Q(Y|\mathcal{H}_t)$ and

$$Y = \mathbb{1}_{\{T \geq \tau\}} z(\tau) + \mathbb{1}_{\{T < \tau\}} X$$

Then the self-financing strategy $\phi = (\phi^0, \phi^1)$ based on the savings account and the CDS is a replicating strategy.

Proof: The terminal value of the wealth is

$$V_T = z(\tau)1\!\!1_{\tau<T} + X1\!\!1_{T<\tau}$$

On the one hand

$$\begin{aligned} E(V_T|\mathcal{H}_t) = V_t &= z(\tau)1\!\!1_{\tau\leq t} + 1\!\!1_{\tau<t}\frac{1}{G(t)}\left(XG(T) + \int_0^t z(s)dG(s)\right) \\ &= \int_0^t z(s)dH_s + (1-H_t)\frac{1}{G(t)}\left(XG(T) + \int_0^t z(s)dG(s)\right) \end{aligned}$$

hence $dV_t = (z(t) - \hat{g}(t))\,dM_t$ with $\hat{g}(t) = \frac{1}{G(t)}(\int_0^t z(s)dG(s) + XG(T))$.

On the other hand, $dV_t = \phi_t^1\,dS_t(\kappa) = \phi_t^1(\delta(t) - S_{t-}(\kappa))\,dM_t$.

First to default

We assume again that $\mathbf{F}$ is the trivial filtration.

We now assume that two CDS's with default times τ_1 and τ_2 are given.
Let G be the survival probability of the pair (τ_1, τ_2)

$$G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)\,.$$

We assume that the pair (τ_1, τ_2) admits a density f. Some easy computation lead to $\mathbb{P}(t < \tau_1 | \tau_2) = h(t, \tau_2)$ where:

$$h(t, s) = \frac{\partial_2 G(t, s)}{\partial_2 G(0, s)}$$

Martingales

- **Filtration $\mathbf{H}^i = \sigma(\tau_i \wedge t)$** The processes

$$M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \frac{f_i(s)}{1 - F_i(s)} ds$$

where

$$F_i(s) = \mathbb{P}(\tau_i \le s) = \int_0^s f_i(u) du$$

are **$\mathcal{H}_t^i$-martingales**. In terms of G:

$$1 - F_1(t) = G(t, 0), \; f_1(t) = -\partial_1 G(t, 0)$$

• **Filtration $\mathbf{H} = \mathbf{H}^1 \vee \mathbf{H}^2$** Let $F^{(1)}$ be the $\mathbf{H}^2$-submartingale

$$F_t^{(1)} = \mathbb{P}(\tau_1 \leq t | \mathcal{H}_t^2)$$

with decomposition $F_t^{(1)} = Z_t^{(1)} + \int_0^t a_s^{(1)} ds$ where $Z^{(1)}$ is an $\mathbf{H}^2$-martingale.

The process

$$M_t^{(1)} = H_t^1 - \int_0^{t \wedge \tau_1} \frac{a_s^{(1)}}{1 - F_s^{(1)}} ds$$

is a **H-martingale**

In a closed form, the process

$$M_t^{(1)} = H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{1 - F_s^{(1)}} ds$$

is a **H**-martingale, where

- $a_t^{(1)} = -H_t^2 \partial_1 h^{(1)}(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)}$
- $h^{(1)}(t, s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$
- $F_t^{(1)} = \mathbb{P}(\tau_1 \leq t | \mathcal{H}_t^2)$

Indeed, some easy computation enables us to write

$$\begin{aligned} F_t^{(1)} &= H_t^2 \mathbb{P}(\tau_1 \leq t | \tau_2) + (1 - H_t^2) \frac{\mathbb{P}(\tau_1 \leq t < \tau_2)}{\mathbb{P}(\tau_2 > t)} \\ &= H_t^2 (1 - h^{(1)}(t, \tau_2)) + (1 - H_t^2) \frac{G(0,t) - G(t,t)}{G(0,t)} \end{aligned}$$

where

$$h^{(1)}(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)} \, .$$

$$M_t^{(1)} = H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \gamma_1(s)ds - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \gamma^{1|2}(s,\tau_2)ds$$

with

$$\begin{aligned} \gamma_1(s) &= -\frac{\partial_1 G(s,s)}{G(s,s)} \\ \gamma^{1|2}(t,s) &= -\frac{f(t,s)}{\partial_2 G(t,s)} \end{aligned}$$

Note that γ_1 is the intensity of τ_1 before τ_2 and $\gamma^{1|2}(t,\tau_2)$ is the intensity of τ_1 after τ_2.

The process

$$M_t^2 = H_t^2 - \int_0^{t\wedge\tau_2} \frac{a_s^{(2)}}{1 - F_s^{(2)}} ds$$

where

- $a_t^{(2)} = -H_t^1 \partial_2 h^{(2)}(\tau_1, 1) - (1 - H_t^1)\frac{\partial_2 G(t,t)}{G(t,0)}$
- $h^{(2)}(t, s) = \frac{\partial_1 G(t,s)}{\partial_1 G(t,0)}$.
- $F_t^{(2)} = \mathbb{P}(\tau_2 \leq t | \mathcal{H}_t^1)$

is a **H**-martingale.

It is rather easy to find the dynamics of S^1. One starts from the fact that, on the set $\{\tau_1 > t, \tau_2 > t\}$

$$
\begin{aligned}
S_t^1 &= \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) G(du, t) - \kappa \int_t^T du \, G(u,t) \right) \\
&= V^1(t)
\end{aligned}
$$

and, on the set $\{\tau_1 > t > \tau_2\}$

$$
\begin{aligned}
S_t^1 &= \frac{1}{\partial_2 G(t,\tau_2)} \left(- \int_t^T du \, \delta(u) f(u, \tau_2) - \kappa \int_t^T du \, \partial_2 G(u,\tau_2) \right) \\
&= V^2(t, \tau_2)
\end{aligned}
$$

Hence

$$S^1_t = (1 - H^1_t)(1 - H^2_t)\, V^1(t) + (1 - H^1_t)\, H^2_t\, V^2(t, \tau_2)$$

and

$$\begin{aligned} dS^1_t &= (1 - H^1_t)(1 - H^2_t)\, dV^1(t) + (1 - H^1_t)\, H^2_t\, dV^2(t, \tau_2) \\ &\quad - S^1_{t-}\, dH^1_t - (1 - H^1_t)\{V^1(t) - V^2(t, \tau_2)\}\, dH^2_t \end{aligned}$$

where

$$\begin{aligned} dV^1(t) &= \Big((\gamma_1(t) + \gamma_2(t))\, V^1(t) + \kappa_1 - \delta_1(t)\gamma_1(t) - S^1_{t|2}(\kappa_1)\gamma_2(t)\Big)\, dt \\ dV^2(t, \tau_2) &= \Big(\gamma^{1|2}(t, \tau_2) V^2(t, \tau_2) - \gamma^{1|2}(t, \tau_2)\delta_1(t) + \kappa_1\Big)\, dt \end{aligned}$$

and the function $S^1_{t|2}(\kappa_1)$ equals

$$S^1_{t|2}(\kappa_1) = \frac{\int_t^T \delta_1(u) f(u,t) du}{\int_t^\infty f(u,t)\, du} - \kappa_1 \frac{\int_t^T du \int_u^\infty dz f(z,t)}{\int_t^\infty f(u,t)\, du}.$$

Note that $V^2(\tau_2, \tau_2) = S^1_{\tau_2|2}(\kappa_1)$

Let $S^1_t(\kappa_1)1\!\!1_{\{t<\tau_{(1)}\}} = \widetilde{S}^1_t(\kappa_1)1\!\!1_{\{t<\tau_{(1)}\}}$ where $\tau_{(1)} = \tau_1 \wedge \tau_2$.

The dynamics of the pre-default price $\widetilde{S}^1_t(\kappa_1)$ are

$$d\widetilde{S}^1_t(\kappa_1) = (\gamma_1(t) + \gamma_2(t))\widetilde{S}^1_t(\kappa_1)\,dt + \big(\kappa_1 - \delta_1(t)\gamma_1(t) - S^1_{t|2}(\kappa_1)\gamma_2(t)\big)\,dt,$$

The pre-default price of a FtD claim $(X, 0, Z, \tau_{(1)})$, where $Z = (Z_1, Z_2)$ and $X = c(T)$, equals

$$\begin{aligned} \widetilde{\pi}(t) &= \frac{\int_t^T du Z_1(u) \int_t^\infty dv\, f(u,v) + \int_t^T dv Z_2(v) \int_t^\infty du\, f(u,v)}{G(t,t)} \\ &\quad + c(T) \frac{G(T,T)}{G(t,t)}. \end{aligned}$$

Moreover,

$$\begin{aligned} d\widetilde{\pi}(t) &= (\gamma_1(t) + \gamma_2(t))\widetilde{\pi}(t)\, dt - \sum_{i=1}^n Z_i(t)\gamma_i(t)\, dt, \\ &= \sum_{i=1}^n \gamma_i(t) \big(\widetilde{\pi}(t) - Z_i(t) \big)\, dt. \end{aligned}$$

Assume that the linear system

$$\begin{aligned} \phi_t^1\big(\widetilde{S}_t^1(\kappa_1) - \delta_1(t)\big) + \phi_t^2\big(\widetilde{S}_t^2(\kappa_2) - S_{t|1}^2(\kappa_2)\big) &= Z_1(t) - \widetilde{\pi}(t), \\ \phi_t^2\big(\widetilde{S}_t^2(\kappa_2) - \delta_2(t)\big) + \phi_t^1\big(\widetilde{S}_t^1(\kappa_1) - S_{t|2}^1(\kappa_1)\big) &= Z_2(t) - \widetilde{\pi}(t), \end{aligned}$$

admits a unique solution $\phi_t = (\phi_t^1, \phi_t^2)$ and let

$$\phi_t^0 = V_t(\phi) - \phi_t^1 S_t^1(\kappa_1) - \phi_t^2 S_t^2(\kappa_2)$$

where

$$dV_t(\phi) = \sum_{i=1}^{2} \phi_t^i\big(dS_t^i(\kappa_i) - \kappa_i dt\big)\,,\ V_0(\phi) = E_Q(Y)$$

Then the self-financing strategy ϕ replicates the first-to-default claim $(X, 0, Z, \tau_{(1)})$.

Stochastic intensity

We now assume that some **reference filtration** $\mathbf{F}$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ is given. We set $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbf{G}$ is referred to as to the **full filtration**.

We define the process

$$F_t = \mathbb{Q}\{\tau \le t \,|\, \mathcal{F}_t\},$$

and the **survival process** $G_t = 1 - F_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}$.

The process G

$$G_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}$$

is a supermartingale and admits a decomposition as

$$G_t = Z_t - A_t$$

where Z is an $\mathbf{F}$-martingale and A an $\mathbf{F}$ predictable increasing process. We assume that G is a continuous process with $G_0 = 1$ and $G_t > 0$.

From the remark that, if $(Y_t, t \geq 0)$ is a **G**-adapted process, there exists an **F** adapted process $(y_t, t \geq 0)$ such that

$$Y_t \mathbb{1}_{t<\tau} = y_t \mathbb{1}_{t<\tau}$$

we obtain the key formulae:

- For any integrable $\mathcal{G}_T$ measurable r.v. Y

$$\mathbb{E}(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}(G_T Y \,|\, \mathcal{F}_t)\,.$$

- Let y be an **F**-predictable process. Then,

$$\mathbb{E}(y_\tau \mathbb{1}_{\tau<T} | \mathcal{G}_t) = y_\tau \mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}(\int_t^T y_u dF_u | \mathcal{F}_t)$$

The ex-dividend price of a credit default swap, with a rate process κ and a protection payment δ_τ at default, equals, for every $t \in [s, T]$,

$$\begin{aligned} S_t(\kappa) &= \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\leq T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\Big) - \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\}} \int_t^{\tau\wedge T} \kappa_s ds \,\Big|\, \mathcal{G}_t\Big), \\ &= \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_Q\left(-\int_t^T \delta_u \, dG_u + \int_t^\infty dG_u \int_t^{u\wedge T} \kappa_v \, dv \,\Big|\, \mathcal{F}_t\right). \end{aligned}$$

We now assume that **(H) hypothesis holds** between **F** and **G**, that is **F**-martingales are **G**-martingales.

It is known that if the **F** market is complete and arbitrage free, and if using **G**-adapted strategies in the **F**-market does not induce arbitrage opportunities, then this hypothesis holds. It is well known that (H) hypothesis is equivalent to

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty)$$

hence the process F is increasing (G is decreasing). We assume that **F** is a Brownian filtration and that F is absolutely continuous wrt Lebesgue measure.

The process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du,$$

with $\gamma_t dt = \frac{dF_t}{G_t}$ is a **G**-martingale. The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t)B_t G_t^{-1}\, dm_t + (1-H_t)(r_t S_t(\kappa) + \kappa - \delta_t\gamma_t)\, dt,$$

where m is the $(\mathbb{Q}, \mathbf{F})$-martingale given by

$$m_t = \mathbb{E}_Q\left(\int_0^T B_u^{-1}\delta_u G_u \gamma_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t\right).$$

Hedging defaultable claims

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities T_i, rates κ_i and protection payment δ^i.
We assume $r = 0$. Let ζ_t^i defined as

$$m_t^i = \mathbb{E}_Q \left(\int_0^T \delta_u^i G_u \gamma_u \, du - \kappa_i \int_0^T G_u \, du \,\middle|\, \mathcal{F}_t \right) , \; dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q(-\int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t), \; dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist **F**-predictable processes ϕ^1, ϕ^2 such that

$$\sum_{i=1}^{2} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widehat{g}_t, \quad \sum_{i=1}^{2} \phi_t^i \zeta_t^i = \zeta_t,$$

where $\widehat{g}$ is given by

$$\widehat{g}_t = \frac{1}{G_t} \mathbb{E}_Q \left(- \int_t^T Z_u \, dG_u + G_T X \,\Big|\, \mathcal{F}_t \right).$$

Let $\phi_t^0 = V_t(\phi) - \sum_{i=1}^2 \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=1}^{2} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i\big)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

Pricing First to default claims

We now assume that some reference filtration $\mathbf{F}$ is given. Let the default times τ_i, $i = 1, 2$ be such that (H) hypothesis holds between $\mathbf{F}$ and $\mathbf{G} = \mathbf{F} \vee \mathbf{H}^1 \vee \mathbf{H}^2$ hence between $\mathbf{F}$ and $\mathbf{G}^1 = \mathbf{F} \vee \mathbf{H}^1$ (resp. $\mathbf{G}^2 = \mathbf{F} \vee \mathbf{H}^2$. We denote by

$$G(t, s; u) = P(\tau_1 > t, \tau_2 > s | \mathcal{F}_u)$$

Under H hypothesis, $G(t, t; t)$, $G(0, t; t)$ and $G(t, 0; t)$ are increasing processes, supposed to be continuous. Furthermore, for $t < u, s < u$

$$\mathbb{P}(\tau_1 \leq t, \tau_2 \leq s | \mathcal{F}_u) = \mathbb{P}(\tau_1 \leq t, \tau_2 \leq s | \mathcal{F}_\infty)$$

Then, one can generalize the previous results, established in the case of trivial filtration. In the case $r = 0$, the dynamics of the pre-default price $\widetilde{S}_t^1(\kappa_1)$ are

$$d\widetilde{S}_t^1(\kappa_1) = \Big((\gamma_1(t) + \gamma_2(t))\widetilde{S}_t^1(\kappa_1) + \kappa_1 - \delta_1(t)\gamma_1(t) - S_{t|2}^1(\kappa_1)\gamma_2(t) \Big)\, dt + G_t^{-1}\, dm_t,$$

with

$$\gamma_1(t) = -\frac{\partial_1 G(t,t;t)}{G(t,t;t)}$$

Assume that the recovery Z, paid at first default time, is a $\mathbf{F}$-predictable process. The first default time $\tau_{(1)}$ satisfies

$$\mathbb{P}(\tau_{(1)} > t|\mathcal{F}_t) = G(t,t;t) = G(t,t;\infty) = G^{(1)}(t)$$

and

$$\mathbb{E}(Z(\tau_{(1)})\mathbb{1}_{t<\tau_{(1)}<T}|\mathcal{G}_t) = \mathbb{1}_{\tau_{(1)}>t}\mathbb{E}(\int_t^T Z_u dG^{(1)}(u)|\mathcal{F}_t)$$

In the case where **τ_1 and τ_2 are conditionally independent with respect to $\mathcal{F}_t$**, then $G^{(1)}(u) = G^1(u)G^2(u)$ with $G^i(t) = \mathbb{P}(\tau_i > t|\mathcal{F}_t)$, hence

$$dG^{(1)}(u) = G^1(u)dG^2(u) + G^2(u)dG^2(u)$$

Begin at the beginning, and go on till you come to the end. ***Then, stop***.

L. Carroll, Alice's Adventures in Wonderland

Thank you for your attention