TU Delft

Double-Sided Parisian Options

joint work with J.A.M. van der Weide

Winter School Mathematical Finance - January 22th 2007 – Jasper Anderluh

TU Delft

Overview

- The Parisian Option Contract : Introduction and notation

  Contract Pay-off, Parisian Stopping time,

  Contract types, Applications.

- Fourier Transform

  Transform of damped probability,

  Transform of the Parisian stopping time.

- Numerical Examples

  Price behavior of double-sided Parisian in call,

  Greek behavior, various contract types.

Opties: AEX

AEX | -- Optietype -- | maart | 2007 | Zoeken | Alle series | Refresh

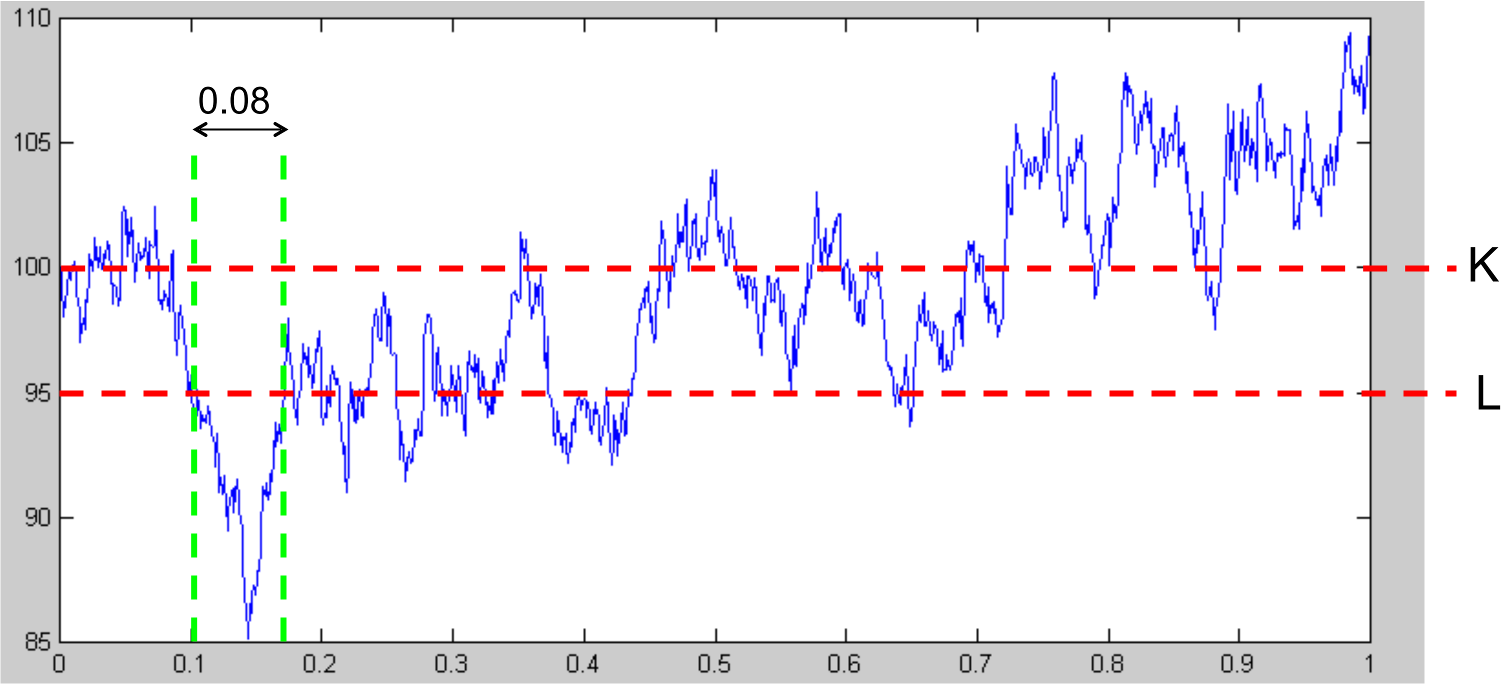
AEX Index	AMX Mid Cap	AScX Small Cap	Dow Jones	Nasdaq	BEL 20	CAC40	Xetra Dax [D]	Euro Stoxx 50 [D]
482,37 -1,21%	636,48 -0,65%	656,03 -0,59%	12326,95 0,04%	2465,98 0,45%	4149,88 -1,07%	5360,31 -1,19%	6403,43 -1,11%	4049,17 -0,90%

Fonds	Laatste	Bied	Laat	+/-	%	Tijd	Vol	Slot	Open	Hoog	Laag	Val
AEX Index	482,74			-5,90	-1,13%	10:52		488,27	487,08	487,70	482,37	EUR

Optieserie	Laatste	Bied	Laat	+/-	%	Tijd	Vol	Slot	Open	Hoog	Laag	Val
AEX CALL mrt '07 360,00		125,80	126,40					130,95				EUR
AEX CALL mrt '07 380,00		106,20	106,80					110,85				EUR
AEX CALL mrt '07 400,00		86,85	87,40					97,50				EUR
AEX CALL mrt '07 420,00		68,00	68,40					72,80				EUR
AEX CALL mrt '07 440,00	52,00	49,95	50,30	-2,05	-3,79%	10:37	10	54,05	52,00	52,00	52,00	EUR
AEX CALL mrt '07 460,00		33,45	33,75					37,35				EUR
AEX CALL mrt '07 480,00	20,00	19,50	19,60	-2,85	-12,47%	10:50	66	22,85	20,65	20,95	20,00	EUR
AEX CALL mrt '07 500,00	9,60	9,00	9,20	-1,60	-14,29%	10:47	121	11,20	10,55	10,55	9,60	EUR
AEX CALL mrt '07 520,00	3,40	3,15	3,30	-0,75	-18,07%	10:50	109	4,15	3,95	3,95	3,40	EUR
AEX CALL mrt '07 540,00	1,00	0,85	0,95	-0,15	-13,04%	10:35	47	1,15	1,10	1,10	1,00	EUR
AEX CALL mrt '07 560,00	0,25	0,15	0,25			10:33	40	0,25	0,30	0,30	0,25	EUR
AEX PUT mrt '07 360,00		0,20	0,25					0,20				EUR
AEX PUT mrt '07 380,00	0,45	0,40	0,55	0,10	28,57%	10:47	32	0,35	0,45	0,45	0,45	EUR
AEX PUT mrt '07 400,00	0,85	0,85	1,00	0,20	30,77%	10:46	12	0,65	0,80	0,85	0,80	EUR
AEX PUT mrt '07 420,00	1,75	1,75	1,95	0,25	16,67%	10:48	19	1,50	1,45	1,75	1,45	EUR
AEX PUT mrt '07 440,00	3,50	3,55	3,65	0,90	34,62%	10:51	252	2,60	2,70	3,50	2,70	EUR
AEX PUT mrt '07 460,00	6,50	6,65	6,80	1,40	27,45%	10:50	90	5,10	5,55	6,50	5,55	EUR
AEX PUT mrt '07 480,00	11,95	12,25	12,50	2,25	23,20%	10:50	289	9,70	10,05	11,95	10,05	EUR
AEX PUT mrt '07 500,00	20,10	21,65	21,90	2,35	13,24%	10:33	73	17,75	18,50	20,40	18,45	EUR
AEX PUT mrt '07 520,00	34,30	35,50	35,90	3,20	10,29%	10:47	23	31,10	32,80	34,30	32,80	EUR
AEX PUT mrt '07 540,00		52,90	53,30					47,85				EUR
AEX PUT mrt '07 560,00	70,65	72,00	72,40	11,95	20,36%	10:18	21	58,70	69,00	70,65	69,00	EUR

Introduction and Notation (2)

- In general, the pay-off of a path-dependent option depends on the whole stock price path.



- A Parisian option knocks in or out as soon as the stock price *S* makes an excursion below or above some barrier *L* for time *D*.

Introduction and Notation (3)

- Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a filtered probability space with $\{W_t, \mathcal{F}_t; t \geq 0\}$ a Brownian motion. We use the Black-Scholes economy, given by the following dynamics :

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ S_0 = s_0 \end{cases} \qquad \begin{cases} dB_t = B_t r dt \\ B_0 = 1 \end{cases}$$

- Using classical results we can compute the value $V_\Phi(t)$ at time *t* of a claim Φ, the payoff at expiry time *T* by :

$$V_\Phi(t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [\Phi]$$

- The pay-off Φ should represent the Parisian option pay-off, so we introduce the Parisian stopping time.

Introduction and Notation (4)

- Introduce the following notation for the last time before *t* we hit level *L*,

$$\gamma_t^L(S) = \sup_{u \leq t} \{S_u = L\}$$

- For the single-sided Parisian stopping time,

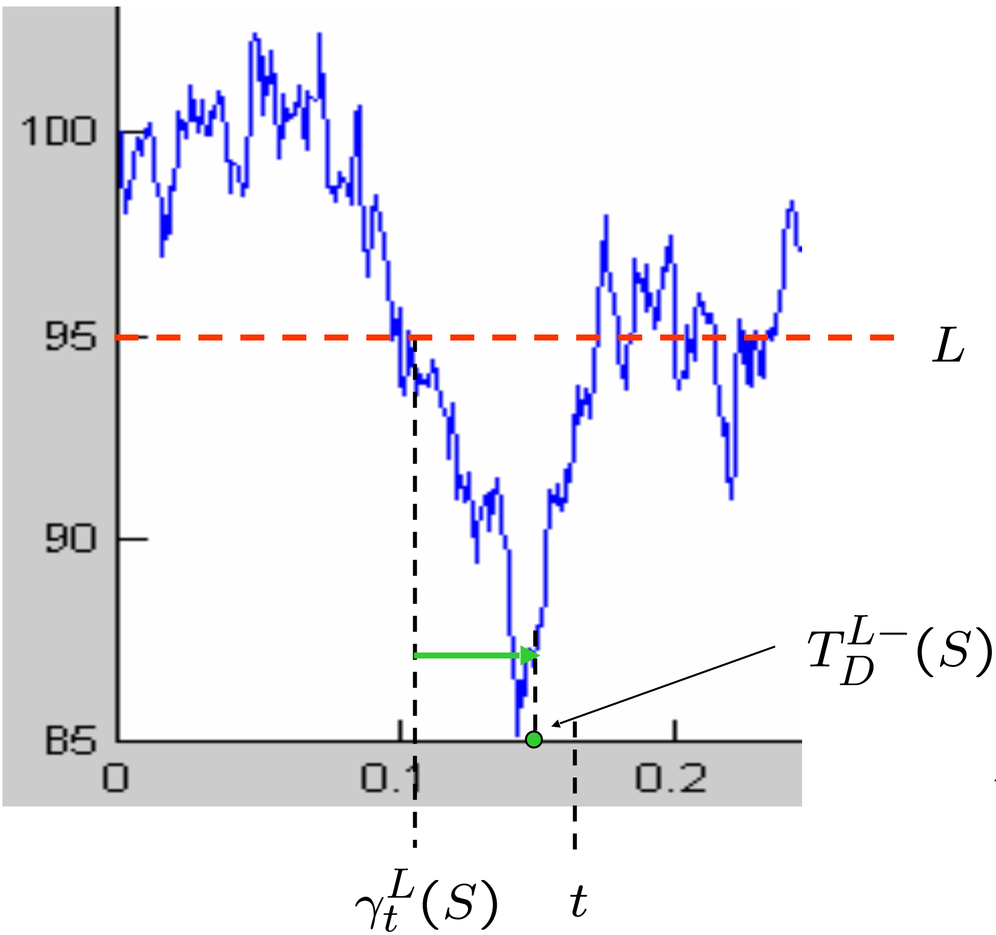
$$T_D^{L\pm}(S) = \inf \left\{ t > 0 | (t - \gamma_t^L(S)) 1_{\{S_t \gtrless L_1\}} > D \right\}.$$

- And for the double-sided Parisian stopping time,

$$T_{D_1,D_2}^{L_1-,L_2+}(S) := \min \left(T_{D_1}^{L_1-}(S), T_{D_2}^{L_2+}(S) \right)$$

Introduction and Notation (5)

- Consider the following example *D* is 10 days,



$$D = \frac{10}{250} = 0.04 = \text{———}$$

Introduction and Notation (6)

- For the double-sided Parisian in call the pay-off is given by,

$$\Phi = (S_T - K)^+ 1_{\left\{T_{D_1,D_2}^{L_1+,L_2-} \leq T\right\}}$$

- For the single-sided Parisian options, the following contract types can be constructed,

	Call	
	down-and	up-and
in	$(S_T - K)^+ 1_{\{T_D^{L-} \leq T\}}$	$(S_T - K)^+ 1_{\{T_D^{L+} \leq T\}}$
out	$(S_T - K)^+ 1_{\{T_D^{L-} \geq T\}}$	$(S_T - K)^+ 1_{\{T_D^{L+} \geq T\}}$
in	$(K - S_T)^+ 1_{\{T_D^{L-} \leq T\}}$	$(K - S_T)^+ 1_{\{T_D^{L+} \leq T\}}$
out	$(K - S_T)^+ 1_{\{T_D^{L-} \geq T\}}$	$(K - S_T)^+ 1_{\{T_D^{L+} \geq T\}}$
	down-and	up-and
	Put	

Introduction and Notation (7)

- Laplace Transform

  (1997) Chesney, Jeanblanc and Yor,

  “Brownian Excursions and Parisian Barrier Options”

- PDE approach

  (1999) Haber, Schonbucher and Wilmott,

  “Pricing Parisian Options”

- Even in the Black-Scholes world,

  obtaining accurate prices is not trivial.

Introduction and Notation (8)

- At present time not exchange traded, so are there applications?
- Building block of convertible bonds with soft-call constraint [Kwok].
- Appear in investment problems when considered from the point of view of real options [Gauthier].
- Used in modeling credit risk [Moraux].
- Application in life-insurance [Chen and Suchanecki].

Fourier Transform (1)

- It is convenient to use the following short-hand notation,

$$\tau = T_{D_1,D_2}^{L_1-,L_2+}(S) = \min\left(T_{D_1}^{L_1-}(S), T_{D_2}^{L_2+}(S)\right) = \min\left(\tau^-, \tau^+\right)$$

- In the Black-Scholes world $\{S_t\}_{t\geq 0}$ is given by a GBM. Like Carr and Madan we find by using Girsanov / Change of Numeraire,

$$V_{PIC} = S_0 \mathbb{P}_{r+\frac{1}{2}\sigma^2}\left[S_T > K; \tau \leq T\right] - Ke^{-rT}\mathbb{P}_r\left[S_T > K; \tau \leq T\right]$$

- So, the quantity of interest is the following,

$$P_r(T) = \mathbb{P}\left[S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} > K; \tau \leq T\right]$$

Fourier Transform (2)

- Note, that in order to proceed, we restate everything in terms of the underlying standard Brownian motion *W*,

$$P_r(T) = e^{-\frac{1}{2}m^2 T}\mathbb{E}\left[e^{mW_T}1_{\{W_T>k\}}1_{\{\tau\le T\}}\right]$$

- Now we compute the following Fourier Transform,

$$\phi(v) = \int_0^\infty e^{ivT}e^{-aT}P_r(T)dT$$

- Substitute $P_r(T)$ and use $\alpha = a + \frac{1}{2}m^2$ we get

$$\begin{aligned}\phi(v) &= \int e^{(iv-\alpha)T}\mathbb{E}\left[e^{mW_T}1_{\{W_T>k\}}1_{\{\tau\le T\}}\left(1_{\{\tau^+<\tau^-\}} + 1_{\{\tau^-<\tau^+\}}\right)\right]dT\\ &= \phi_+(v) + \phi_-(v).\end{aligned}$$

Fourier Transform (3)

Now we have the following lemma,

For the Fourier transforms ϕ_+ and ϕ_- the following holds,

$$\phi_+(v) = \mathbb{E}\left[e^{(iv-\alpha)\tau} 1_{\{\tau^+ < \tau^-\}}\right] \mathbb{E}\left[\int_0^\infty e^{(iv-\alpha)\rho} h\left(\rho, W_{\tau^+}\right) d\rho\right],$$

$$\phi_-(v) = \mathbb{E}\left[e^{(iv-\alpha)\tau} 1_{\{\tau^- < \tau^+\}}\right] \mathbb{E}\left[\int_0^\infty e^{(iv-\alpha)\rho} h\left(\rho, W_{\tau^-}\right) d\rho\right].$$

where,

$$h(\rho, w) = \mathbb{E}\left[e^{m(W_\rho + w)} 1_{\{W_\rho + w > k\}}\right]$$

Note the independence between $\tau 1_{\{\tau^+ < \tau-\}}$ and W_τ. Where does it come from? Can we compute the left-hand side expectations?

Fourier Transform (4)

- The Brownian meander at time *t*>0 is defined by,

$$m_u^{(t)} = \frac{1}{\sqrt{t-\gamma_t}} |W_{\gamma_t + u(t-\gamma_t)}|, \qquad u \leq 1,$$

- We are only interested in its final value *(u*=1*)*, denoted by,

$$n_t = \frac{1}{\sqrt{t-\gamma_t}} |W_t|.$$

- By CJY this final value is for every *t*>0 independent of the pair and $(\gamma_t, \text{sgn}(W_t))$ e *N* ha $n_t \stackrel{d}{=} N$ llowing density,

$$\mathbb{P}[N \in dx] = x e^{-\frac{x^2}{2}} 1_{\{x \geq 0\}} dx$$

Fourier Transform (5)

For the double-sided case we introduce,

$$n_t^l = \frac{1_{\{T_l<t\}}}{\sqrt{t-\gamma_t^l}}|W_t - l| \qquad \mu_t^l = 1_{\{T_l<t\}}\text{sgn}(W_t - l)\sqrt{t-\gamma_t^l}$$

And we have the following lemma,

For any bounded measurable function *f* we have,

$$\mathbb{E}\left[1_{\{\tau^+<\tau^-\}} f(n_\tau^{l_2}) \middle| \mathcal{H}_\tau\right] = 1_{\{\tau^+<\tau^-\}}\mathbb{E}\left[f(N)\right] \quad \text{a.s.}$$

where,

$$\mathcal{H}_\tau = \sigma(\mu_\tau^{l_1}, \mu_\tau^{l_2}, \gamma_\tau^{l_1}, \gamma_\tau^{l_2})$$

Fourier Transform (6)

Now we have a martingale argument,

$$1 = \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}\right] = \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^+<\tau^-\}}\right] + \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^-<\tau^+\}}\right],$$

And by the previous lemma we can compute,

$$\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^+<\tau^-\}}\right] = \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda\left(\mu_\tau^{l_2}n_\tau^{l_2}+l_2\right)}1_{\{\tau^+<\tau^-\}}\right]$$
$$= e^{\lambda l_2}\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau}1_{\{\tau^+<\tau^-\}}\mathbb{E}\left[e^{\lambda\sqrt{D_2}n_\tau}\middle|\mathcal{H}_\tau\right]\right]$$
$$= e^{\lambda l_2}\mathbb{E}\left[e^{\lambda\sqrt{D_2}N}\right]\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau}1_{\{\tau^+<\tau^-\}}\right] = e^{\lambda l_2}\Psi(\lambda\sqrt{D_2})\mathbb{E}_+(\lambda),$$

Fourier Transform (7)

Finally resulting in the following theorem,

For the restricted Laplace transforms of τ the following holds,

$$\mathbb{E}_+(\lambda) = \frac{e^{\lambda l_1}\Psi(-\lambda_1) - e^{-\lambda l_1}\Psi(\lambda_1)}{e^{\lambda(l_1-l_2)}\Psi(-\lambda_1)\Psi(-\lambda_2) - e^{\lambda(l_2-l_1)}\Psi(\lambda_1)\Psi(\lambda_2)}$$

where,

$$\mathbb{E}_+(\lambda) = \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau}1_{\{\tau^+<\tau^-\}}\right]$$

By taking limits, we can show that the probability that a standard Brownian motion makes a positive excursion of length D_2 before making a negative excursion of length D_1 is given by,

$$\frac{\sqrt{D_1}}{\sqrt{D_1}+\sqrt{D_2}}$$

→ also follows from excursion theory

Fourier Transform (8)

- Combining these results gives,

$$\phi_+(v) = \mathbb{E}\left[e^{(iv-\alpha)\tau}1_{\{\tau^+<\tau^-\}}\right]\mathbb{E}\left[\int_0^\infty e^{(iv-\alpha)\rho}h\left(\rho, W_{\tau^+}\right)d\rho\right]$$

$$\downarrow$$

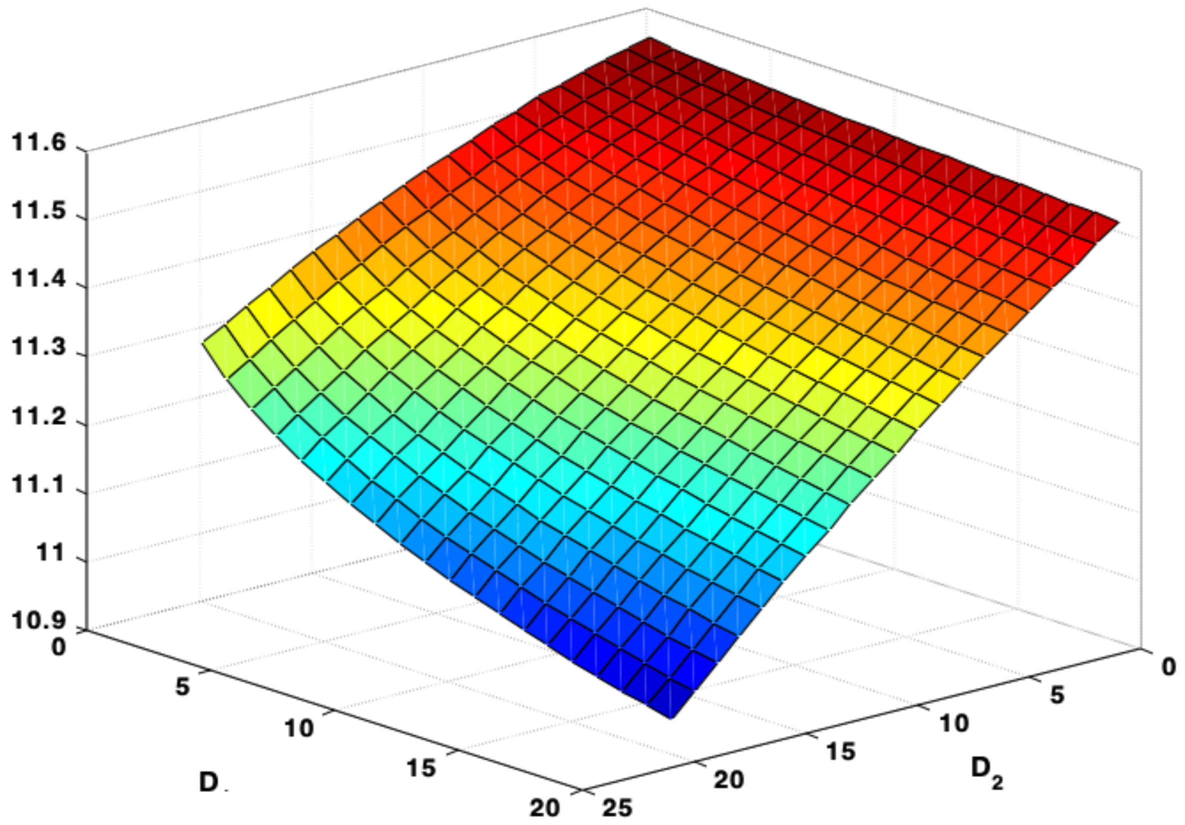
$$\phi_+(v) = \mathbb{E}_+(\tilde{v}_\alpha)\mathbb{E}\left[\int_0^\infty e^{(iv-\alpha)\rho}h\left(\rho, l_2+\sqrt{D_2}N\right)d\rho\right], \tilde{v}_\alpha = \sqrt{2(\alpha - iv)}$$

- Now everything is known and the formulas for Fourier transforms follow by elaborate computation.

$$\phi_+(v) = \mathbb{E}_+(\tilde{v}_\alpha)\left(\frac{e^{(m-\tilde{v}_\alpha)k+l_2\tilde{v}_\alpha}\tilde{\Psi}_{u_2^*}(\tilde{v}_\alpha\sqrt{D_2})}{\tilde{v}_\alpha(\tilde{v}_\alpha - m)} + \frac{2e^{ml_2}\Psi_{u_2^*}(m\sqrt{D_2})}{\tilde{v}_\alpha^2 - m^2} - \frac{e^{(\tilde{v}_\alpha+m)k-l_2\tilde{v}_\alpha}\Psi_{u_2^*}(-\tilde{v}_\alpha\sqrt{D_2})}{\tilde{v}_\alpha(\tilde{v}_\alpha + m)}\right) \qquad k > l_2,$$

Numerical Examples (1)

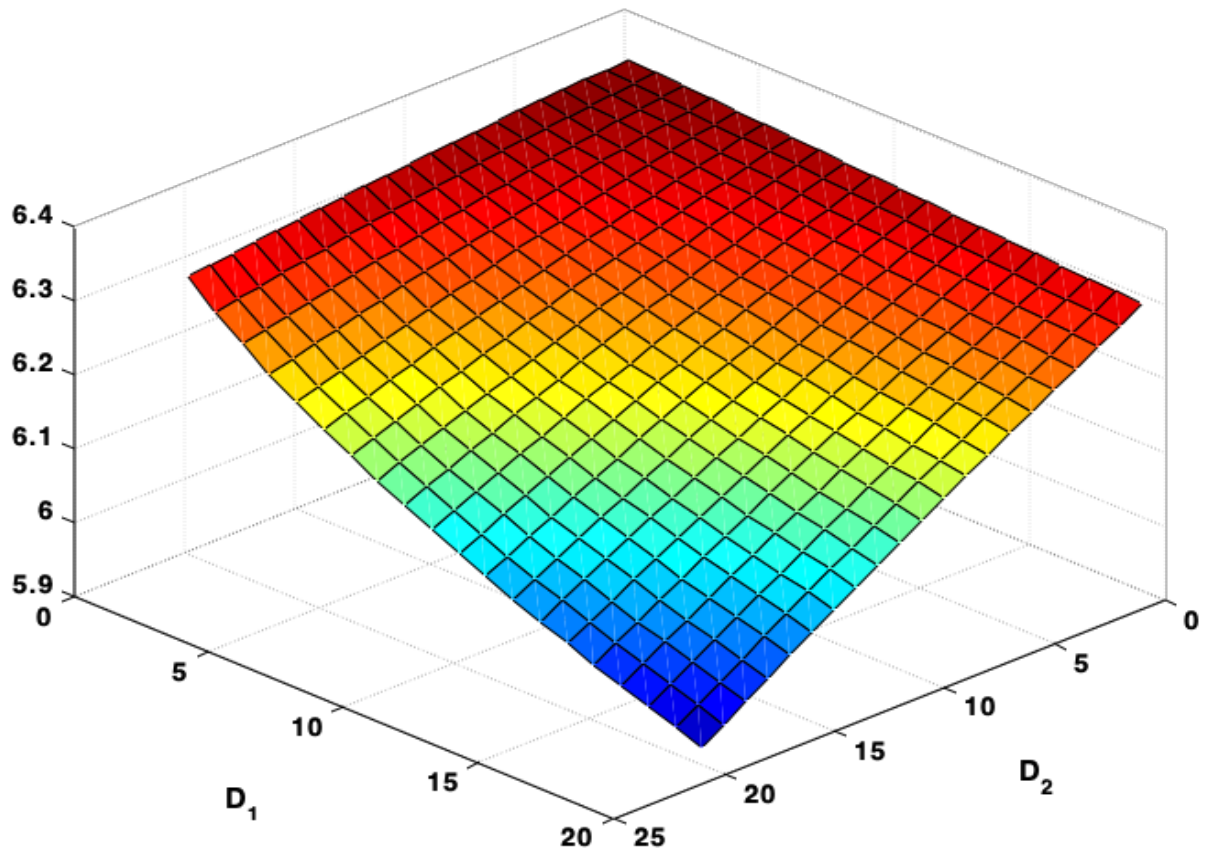
Double-sided Parisian in call prices.



$S_0 = 100, r = 0.035, \sigma = 0.25, L_1 = 90, L_2 = 110.$

Numerical Examples (2)

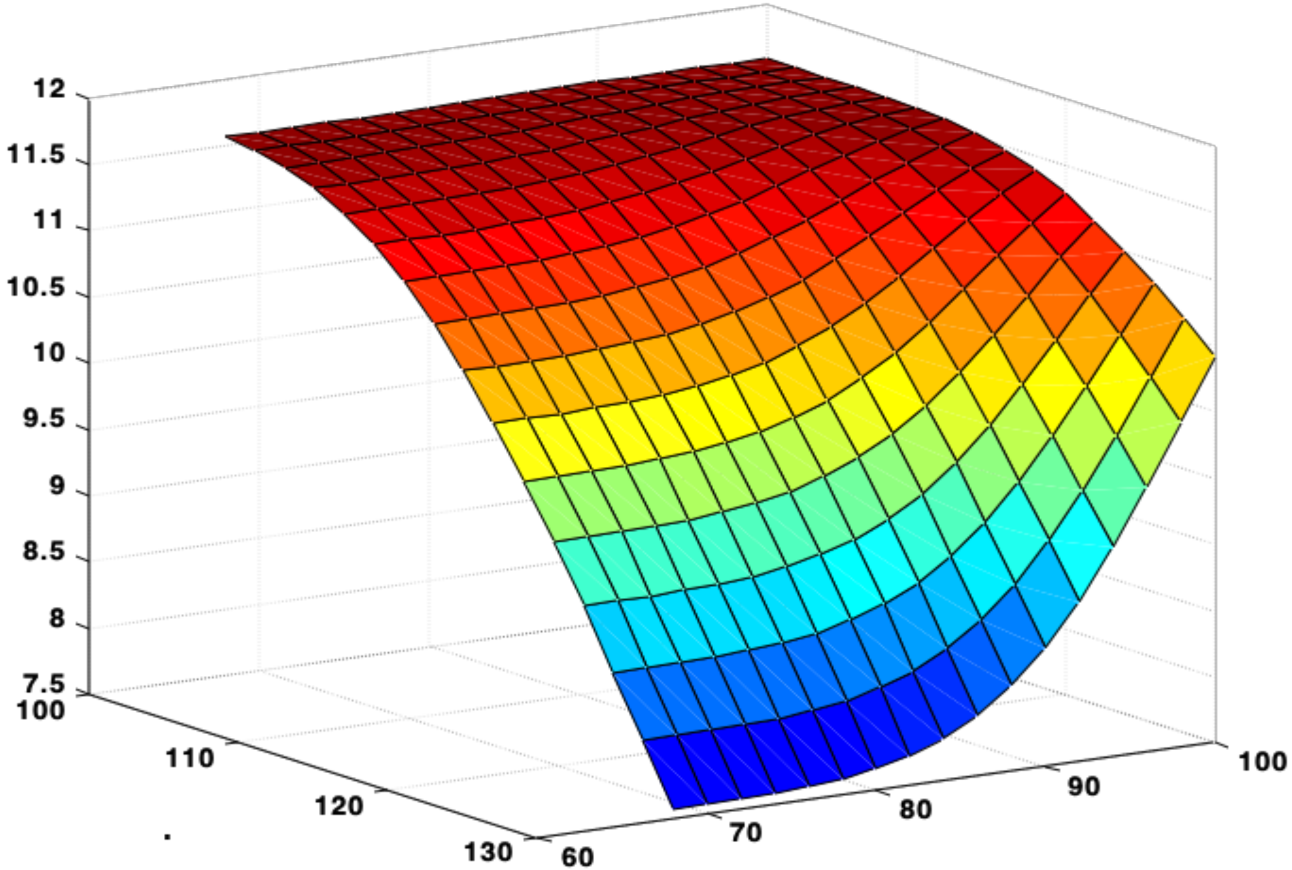
Double-sided Parisian in call prices.



$$S_0 = 90, r = 0.035, \sigma = 0.25, L_1 = 90, L_2 = 110.$$

Numerical Examples (3)

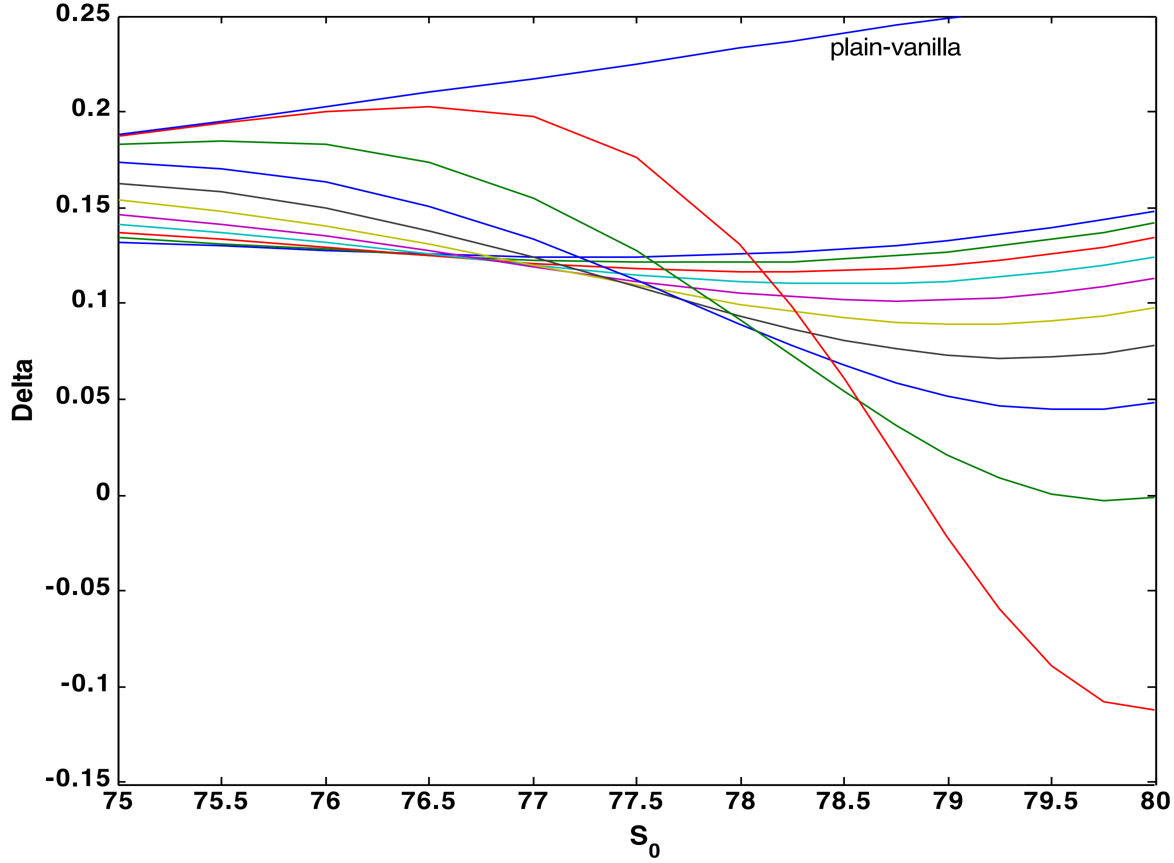
⇒ Double-sided Parisian in call prices.



$$S_0 = 100, r = 0.035, \sigma = 0.25, D_1 = D_2 = 10/250.$$

Numerical Examples (4)

Double-sided Parisian in call delta, NOTE : negative gamma.



$$L_1 = 80, L_2 = 120, r = 0.035, \sigma = 0.25, D_1 = D_2 = 10/250.$$

Numerical Examples (5)

Negative gamma? Then there should be negative theta.

	$S_0 = 74$		$S_0 = 76$		$S_0 = 78$		$S_0 = 80$	
d	Par	Plain	Par	Plain	Par	Plain	Par	Plain
10	1.474	1.547	1.737	1.923	1.987	2.358	2.257	2.856
9	1.474	1.536	1.740	1.911	1.984	2.344	2.240	2.841
8	1.475	1.525	1.744	1.898	1.982	2.330	2.222	2.825
7	1.476	1.514	1.750	1.885	1.984	2.316	2.205	2.809
6	1.476	1.503	1.759	1.873	1.988	2.302	2.187	2.794
5	1.475	1.492	1.770	1.860	1.997	2.288	2.170	2.778
4	1.472	1.481	1.783	1.848	2.014	2.274	2.152	2.762
3	1.467	1.470	1.797	1.835	2.041	2.260	2.135	2.747
2	1.459	1.459	1.809	1.823	2.087	2.245	2.117	2.731
1	1.448	1.448	1.809	1.810	2.164	2.231	2.100	2.715

$$L_1 = 80, L_2 = 120, r = 0.035, \sigma = 0.25, D_1 = D_2 = 10/250.$$

Numerical Examples (6)

Now you can construct the following contract types:

- $\phi = \phi_+ + \phi_-$. The double-sided Parisian in call. Most expensive.
- $\phi = \phi_+ + \phi_-$ with $l_1 \to \infty$ or $D_1 \to \infty$.

  The single-sided Parisian up-and-in call. Price inbetween.
- $\phi = \phi_-$ .The single-sided Parisian down-before-up-in call. Cheapest.

Contract type	$S_0 = 90$	$S_0 = 95$	$S_0 = 100$	$S_0 = 105$	S_0=110
Plain-vanilla	6.362	8.768	11.591	14.800	18.349
Double-sided knock-in	6.362	8.767	11.591	14.799	18.349
Double-sided P knock-in (1)	6.236	8.568	11.371	14.608	18.226
Single P up-in (2a)	5.792	8.218	11.113	14.435	18.129
P up-before-down-in (3a)	3.568	6.844	10.284	13.957	17.886
Single P down-in (2b)	2.676	1.742	1.123	0.719	0.457
P down-before-up-in (3b)	2.668	1.723	1.087	0.651	0.339