Expected utility models

and optimal investments

Lecture III

Market uncertainty, risk preferences and investments

Portfolio choice and stochastic optimization

- Maximal expected utility models
- Preferences are given exogeneously
- Methods

  Primal problem (HJB eqn under stringent model assumptions)

  Dual problem (Linearity under market completeness)
- Optimal policies: consumption and portfolios

Maximal expected utility models

- Market uncertainty

$(\Omega, \mathcal{F}, \mathbb{P})$, $\quad W = (W^1, \dots, W^d)^*$ d-dim Brownian motion

Trading horizon: $[0, T]$, $(0, +\infty)$

Asset returns: $\quad dR_t = \mu_t \, dt + \sigma_t \, dW_t$

$\mu \in \mathcal{L}_1(R^m)$, $\sigma \in \mathcal{L}_2(\mathbb{R}^{d \times m})$

riskless asset

Wealth process: $dX_t = \pi_t \, dR_t - C_t \, dt$

Control processes: consumption rate C_t, asset allocation π_t

Maximal expected utility models

- Preferences: $U : \mathbb{R} \to \mathbb{R}$

  increasing, concave, asymptotically elastic....

  $$U(x) = \frac{1}{\gamma} x^{\gamma},\ \log x,\ -e^{-\gamma x}$$

- Objective: maximize intermediate utility of consumption and

  utility of terminal wealth

$$V(x,t) = \sup_{(C,\pi)} E_{\mathbb{P}} \left(\int_t^T U_1(C_s)\, ds + U_2(X_T) / X_t = x \right)$$

- Generalizations: infinite horizon, long-term average, ergodic criteria...

Recall that U_1, U_2 are not related
to the investment opportunities

Primal maximal expected utility problem

- V solves the Hamilton-Jacobi-Bellman eqn

$$\begin{cases} V_t + F(x, V_x, V_{xx}; U_1) = 0 \\ V(x, T) = U_2(x) \end{cases}$$

- Viscosity theory (Crandall-Lions)

  Z., Soner, Touzi, Duffie-Z., Elliott, Davis-Z., Bouchard

- Optimal policies in feedback form

$$C_s^* = \widetilde{C}((V_x^{-1})'(X_s^*, s)) \ , \quad \pi_s^* = \tilde{\pi}(V_x(X_s^*, s), V_{xx}(X_s^*, s))$$

- Degeneracies, discontinuities, state and control constraints

Dual maximal expected utility problem in complete markets

- Dual utility functional

$$U^*(y) = \max_x (U(x) - xy)$$

- Dual problem becomes linear – direct consequence of market completeness and representation, via risk neutrality, of replicable contingent claims

- Problem reduces to an optimal choice of measure – intuitive connection with the so-called state prices

  Cox-Huang, Karatzas, Shreve, Cvitanic, Schachermayer, Zitkovic, Kramkov, Delbaen et al, Kabanov, Kallsen, ...

Extensions

- Recursive utilities and Backward Stochasticc Differential Equations (BSDEs)

  Kreps-Porteus, Duffie-Epstein, Duffie-Skiadas, Schroder-Skiadas, Skiadas, El Karoui-Peng-Quenez, Lazrak and Quenez, Hamadene, Ma-Yong, Kobylanski

- Ambiguity and robust optimization

  Ellsberg, Chen-Epstein, Epstein-Schneider, Anderson et al., Hansen et al, Maenhout, Uppal-Wang, Skiadas

- Mental accounting and prospect theory

  Discontinuous risk curvature
  Huang-Barberis, Barberis et al., Thaler et al., Gneezy et al.

- Large trader models

  Feedback effects
  Kyle, Platen-Schweizer, Bank-Baum, Frey-Stremme, Back, Cuoco-Cvitanic

- Social interactions

  Continuous of agents – Propagation of fronts
  Malinvaud, Schelling, Glaesser-Scheinkman, Horst-Scheinkman, Foellmer

- Fund management and fee structure

  Non-zero sum stochastic differential games
  Huggonier-Kaniel

Optimal portfolios

- HJB equation yields the optimal policy in feedback form

$$\pi_s^* = \pi(X_s^*, s)$$

$$\pi(x,t) = \prod(x, V_x, V_{xx}, \ldots)$$

- Duality yields the optimmal policy via a martingale representation theorem or via replicating strategies of a dual "pseudo-claim"

These representations, albeit general, offer very little intuition

and are of very low practical importance, if any

Incomplete markets

- Duality "breaks" down
- HJB equation too complex and stringent assumptions are needed
- Portfolios consist of the myopic and the non-myopic component
- Myopic portfolio is the investment as if the Sharpe ratio were constant
- Non-myopic component is the excess risky demand, known as the hedging component
- Notion of hedging opaque

An example with myopic and non-myopic portfolios

Optimal investments under CRRA preferences

Market environment

$$dS_s = M(Y_s, s)S_s\, ds + \Sigma(Y_s, s)S_s\, dW_s^1$$

$$dY_s = B(Y_s, s)\, ds + A(Y_s, s)\, dW_s$$

riskless bond of zero interest rate

Preferences

$$U(x) = \frac{x^\alpha}{\alpha} \qquad (\alpha < 0,\ 0 < \alpha < 1)$$

Value function

$$V(x,y,t) = \sup_{\pi} E\left(\frac{X_T^{\alpha}}{\alpha} \mid X_t = x,\ Y_t = y\right)$$

State controlled wealth process

$$dX_s = M(Y_s, s)\pi_s\, ds + \Sigma(Y_s, s)\pi_s\, dW_s^1$$

$$X_t = x, \quad x \geq 0$$

Objective

Characterize the optimal investment process π_s^*

Feedback controls $\pi_s^* = \pi^*(X_s^*, Y_s, s)$

(Wachter, Campell and Viciera, Liu, ...)

The Hamilton-Jacobi-Bellman equation

$$V_t + \max_{\pi}\left(\frac{1}{2}\Sigma^2(y,t)\pi^2 V_{xx} + \pi(R\Sigma(y,t)A(y,t)V_{xy} + M(y,t)V_x)\right)$$

$$+\frac{1}{2}A^2(y,t)V_{yy} + B(y,t)V_y = 0$$

$$V(x,y,T) = \frac{x^\alpha}{\alpha}\ ; \qquad (x,y,t) \in D = R^+ \times R \times [0,T]$$

Optimal policies

$$\pi_s^* = \pi^*(X_s^*, Y_s, s)$$

$$= -\left(\frac{M(Y_s, s)}{\Sigma^2(Y_s, s)}\right)\frac{V_x(X_s^*, Y_s, s)}{V_{xx}(X_s^*, Y_s, s)} - \left(R\frac{A(Y_s, s)}{\Sigma(Y_s, s)}\right)\frac{V_{xy}(X_s^*, Y_s, s)}{V_{xx}(X_s^*, Y_s, s)}$$

$$dX_s^* = M(Y_s, s)\pi_s^* \, ds + \Sigma(Y_s, s)\pi_s^* \, dW_s^1$$

- **Normalized HJB Equation** (Krylov, Lions)

Non-compact set of admissible controls

$$\max_\pi \left(\frac{1}{1+\pi^2}\left(V_t + \max_\pi \left(\frac{1}{2}\Sigma^2(y,t)\pi^2 V_{xx} + \pi(RA(y,t)\Sigma(y,t)V_{xy}\right.\right.\right.$$
$$\left.\left. + M(y,t)V_x)\right) + \frac{1}{2}A^2(y,t)V_{yy} + B(y,t)V_y\right) = 0$$

$$U(x,y,T) = \frac{x^\alpha}{\alpha}$$

V is the unique constrained viscosity solution of the normalized HJB equation

V is a constrained viscosity solution of the original HJB equation (Duffie-Z.)

V is unique in the appropriate class
(Ishii-Lions, Duffie-Z., Katsoulakis, Touzi, Z.)

Solution

$$V(x,y,t) = \frac{x^\alpha}{\alpha} v(y,t)^\varepsilon \qquad \varepsilon = \frac{1-\alpha}{1-\alpha+R^2\alpha}$$

$$v_t + \frac{1}{2}A^2(y,t)v_{yy} + \left(B(y,t) + R\frac{\alpha}{1-\alpha}L(y,t)A(y,t)\right)v_y$$

$$+\frac{1}{2\varepsilon}\frac{\alpha}{1-\alpha}L^2(y,t)v = 0$$

$$L(y,t) = \frac{M(y,t)}{\Sigma(y,t)}$$

$$\pi^*(x,y,t) = \frac{1}{1-\alpha}\,\frac{M(y,t)}{\Sigma^2(y,t)}x + R\frac{\varepsilon}{1-\alpha}\frac{A(y,t)}{\Sigma(y,t)}\,\frac{v_y(y,t)}{v(y,t)}x$$

Structural and characterization results on optimal policies

- Long-term horizon problems
  Logarithmic utilities, approximations for other utilities (Campbell)

- Finite horizon and exponential utilities
  The excess hedging demand (non-myopic is identified with the indifference delta of a pseudo-claim with payoff depending on risk aversion and aggregate Sharpe ratio

Other limitations

Time horizon

- How do we know our utility say 30 years from now?
- How do we manage our liabilities beyond the time the utility is prespecified?
- Are our portfolios consistent across different units?

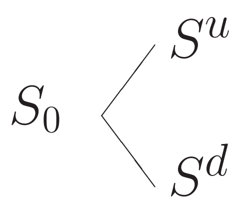
Units, numeraires and expected utility

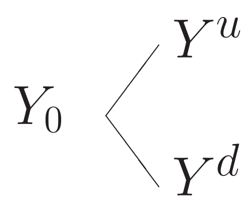
A toy incomplete model

- Probability space

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad \mathbb{P}\{\omega_i\} = p_i, \quad i = 1, ..., 4$$

- Two risks





- Random variables S_T and Y_T

$$S_T(\omega_1) = S^u, Y_T(\omega_1) = Y^u \qquad S_T(\omega_3) = S^d, Y_T(\omega_3) = Y^u$$

$$S_T(\omega_2) = S^u, Y_T(\omega_2) = Y^d \qquad S_T(\omega_4) = S^d, Y_T(\omega_4) = Y^d$$

Investment opportunities

- We invest the amount β in bond $(r = 0)$ and the amount α in stock
- Wealth variable

$$X_0 = x, \quad X_T = \beta + \alpha S_T = x + \alpha(S_T - S_0)$$

Indifference price

- For a general claim C_T , we define the value function

$$V^{C_T}(x) = \max_{\alpha} E(-e^{-\gamma(X_T - C_T)})$$

- The indifference price is the amount $\nu(C_T)$ for which,

$$\boxed{V^0(x) = V^{C_T}(x + \nu(C_T))}$$

The indifference price (MZ 2004)

$$\nu(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)} \mid S_T)\right) = \mathcal{E}_{\mathbb{Q}}(C_T)$$

$$\mathbb{Q}(Y_T \mid S_T) = \mathbb{P}(Y_T \mid S_T)$$

Static arbitrage

Indifference prices in spot and forward units

Spot units

Wealth: $X_T^s = x + \alpha \left(\frac{S_T}{1+r} - S_0 \right)$

Value function: $V^{C_T}(x) = \sup_{\alpha} E_{\mathbb{P}} \left(-e^{-\gamma (X_T^s - \frac{C_T}{1+r})} \right)$

Pricing condition: $V^0(x) = V^{s,C_T}(x + \nu^s(C_T))$

Pricing measure: $E_{\mathbb{Q}^s} \left(\frac{S_T}{1+r} \right) = S_0$ and $\mathbb{Q}^s(Y_T | S_T) = \mathbb{P}(Y_T | S_T)$

Indifference price: $\nu^s(C_T) = \mathcal{E}_{\mathbb{Q}^s} \left(\frac{C_T}{1+r} \right) = E_{\mathbb{Q}^s} \left(\frac{1}{\gamma} \log E_{\mathbb{Q}^s} \left(e^{\gamma \frac{C_T}{1+r}} | S_T \right) \right)$

Forward units

Wealth: $X_T^f = X_T^s(1+r) = f + \alpha(F_T - F_0)\ ; \quad f = x(1+r)$

Value function: $V^{C_T}(f) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma(X_T^f - C_T)}\right)$

Pricing condition: $V^0(f) = V^{C_T}(f + \nu^f(C_T))$

Pricing measure: $E_{\mathbb{Q}^f}(F_T) = F_0$ and $\mathbb{Q}^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T)$

Indifference price: $\nu^f(C_T) = \mathcal{E}_{\mathbb{Q}^f}(C_T) = E_{\mathbb{Q}^f}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}|F_T\right)\right)$

Inconsistency across prices expressed in spot and forward units

Pricing measures: $\mathbb{Q}^s = \mathbb{Q}^f$

Spot price: $\nu^s(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\frac{C_T}{1+r}}|S_T\right)\right)$

Forward price: $\nu^f(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}|S_T\right)\right)$

$$\boxed{\nu^f(C_T) \neq (1+r)\nu^s(C_T)}$$

(WWW) What went wrong?

- Risk preferences were not correctly specified!
- Risk preferences need to be consistent across units
- Risk aversion is not a constant

Indifference prices in spot and forward units

Spot units

Wealth: $X_T^s = x + \alpha\left(\frac{S_T}{1+r} - S_0\right)$

Value function: $V^{s,C_T}(x) = \sup_\alpha E_{\mathbb{P}}\left(-e^{-\gamma^s\left(X_T^s - \frac{C_T}{1+r}\right)}\right)$

Pricing condition: $V^{s,0}(x) = V^{s,C_T}(x + \nu^s(C_T))$

Pricing measure: $E_{\mathbb{Q}^s}\left(\frac{S_T}{1+r}\right) = S_0$ and $\mathbb{Q}^s(Y_T|S_T) = \mathbb{P}(Y_T|S_T)$

Indifference price: $\nu^s(C_T) = \mathcal{E}_{\mathbb{Q}^s}\left(\frac{C_T}{1+r}\right) = E_{\mathbb{Q}^s}\left(\frac{1}{\gamma^s}\log E_{\mathbb{Q}^s}\left(e^{\gamma^s\frac{C_T}{1+r}}|S_T\right)\right)$

Forward units

Wealth: $X_T^f = X_T^s(1+r) = f + \alpha(F_T - F_0)\ ; \quad f = x(1+r)$

Value function: $V^{f,C_T}(f) = \sup_\alpha E_{\mathbb{P}}\left(-e^{-\gamma^f(X_T^f - C_T)}\right)$

Pricing condition: $V^{f,0}(f) = V^{f,C_T}(f + \nu^f(C_T))$

Pricing measure: $E_{\mathbb{Q}^f}(F_T) = F_0$ and $\mathbb{Q}^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T)$

Indifference price: $\nu^f(C_T) = \mathcal{E}_{\mathbb{Q}^f}(C_T) = E_{\mathbb{Q}^f}\left(\frac{1}{\gamma^f}\log E_{\mathbb{Q}^f}\left(e^{\gamma^f C_T}|F_T\right)\right)$

Consistency across spot and forward units

$$\nu^f(C_T) = (1+r)\nu^s(C_T) \Longleftrightarrow \delta^s = \tfrac{1}{1+r}\delta^f$$

$$\delta^s = \frac{1}{\gamma^s}\,, \quad \delta^f = \frac{1}{\gamma^f} \quad : \quad \text{spot and forward risk tolerance}$$

Risk tolerance is not a number. It is expressed in wealth units.

- Utility functions

$$U^s(x) = -e^{-\gamma^s x} \quad ; \quad x \text{ in spot units}$$

$$U^f(x) = -e^{-\gamma^f x} \quad ; \quad x \text{ in forward units}$$

- Value function representations

$$V^{s,C_T}(x) = -e^{-\gamma^s(x-\nu^s(C_T))-H(\mathbb{Q}|\mathbb{P})} = U^s\left(x - \nu^s(C_T) + \delta^s H(\mathbb{Q}|\mathbb{P})\right)$$

$$V^{f,C_T}(x) = -e^{-\gamma^f(x-\nu^f(C_T))-H(\mathbb{Q}|\mathbb{P})} = U^f\left(x - \nu^f(C_T) + \delta^f H(\mathbb{Q}|\mathbb{P})\right)$$

$$\mathbb{Q} = \mathbb{Q}^s = \mathbb{Q}^f$$

Static no arbitrage constraint

$$\Updownarrow$$

Appropriate dependence across units needs to be

built into the risk preference structure

The stock as the numeraire

- Indifference price is a unitless quantity (number of stock shares)

- The "utility argument" $\gamma_T^s \frac{X_T}{S_T}$ needs to be unitless as well

- Static no arbitrage constraint strongly suggests that risk aversion needs to be stochastic

Stochastic risk preferences

Indifference prices and state dependent risk tolerance

- $\gamma_T = \gamma(S_T)$ $\mathcal{F}_T^S$-measurable random variable (in reciprocal to wealth units)

- Risk tolerance (in units of wealth)

- Risk tolerance (in units of wealth)

$$\delta_T = \frac{1}{\gamma_T}$$

- Should γ_T be allowed to be $\mathcal{F}_T^{(S,Y)}$-measurable?

Random utility and its value function

- Value function without the claim

$$V^0(x;\gamma_T) = -\exp\left(-\frac{x}{E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\right)} - H(\mathbb{Q}^*|\mathbb{P})\right)$$

- Value function and utility

$$V(x,0;T) = -e^{-\frac{x}{E_{\mathbb{Q}}(\frac{1}{\gamma_T})}-H(\mathbb{Q}^*|\mathbb{P})} \qquad U(X_T;T) = -e^{-\gamma_T X_T}$$

- Two minimal entropy measures

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{\delta_T}{E_{\mathbb{Q}}(\delta_T)}$$

$$E_{\mathbb{Q}}\left(S_T - (1+r)\, S_0\right) = 0$$

$$E_{\mathbb{Q}^*}\left(\gamma_T\left(S_T - (1+r)\, S_0\right)\right) = 0$$

Structural constraints between the market environment

and the risk preferences

Indifference price and value function

- The indifference price of C_T is given by

$$\nu\left(C_T;\gamma_T\right)=E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T\frac{C_T}{1+r}}\left|S_T\right.\right)\right)$$

- The utility

$$U(X_T;T)=-e^{-\gamma_T X_T}$$

- Value function with the claim

$$V^{C_T}\left(x;\gamma_T\right)=-\exp\left(-\left(\frac{x-\nu\left(C_T;\gamma_T\right)}{E_{\mathbb{Q}}\left(\delta_T\right)}\right)-H\left(\mathbb{Q}^*\left|\mathbb{P}\right.\right)\right)$$

Optimal policies for stochastic risk preferences

(in the presence of the claim)

$$\alpha^{C_T,*} = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*}$$

- Optimal demand due to market incompleteness: $\alpha^{0,*}$

$$\alpha^{0,*} = -\frac{\partial H\left(\mathbb{Q}^* \mid \mathbb{P}\right)}{\partial S_0} E_{\mathbb{Q}}\left(\delta_T\right)$$

- Optimal demand due to changes in risk tolerance: $\alpha^{1,*}$

$$\alpha^{1,*} = \frac{\partial \log E_{\mathbb{Q}}\left(\delta_T\right)}{\partial S_0} x$$

- Optimal demand due to liability: $\alpha^{2,*}$

$$\alpha^{2,*} = E_{\mathbb{Q}}\left(\delta_T\right) \frac{\partial}{\partial S_0} \left(\frac{\nu\left(C_T; \gamma_T\right)}{E_{\mathbb{Q}}\left(\delta_T\right)} \right)$$

Numeraire independence

Indifference prices and general numeraires

- The stock as the numeraire

Wealth: $X_T^S = \frac{x}{S_T} + \alpha\left(1 - \frac{S_0}{S_T}\right)$

Value function: $V^{S,C_T}(x^S) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma^S(S_T)(X_T^S - \frac{C_T}{S_T})}\right)$

Pricing condition: $V^{S,0}(x^S) = V^{S,C_T}(x^S + \nu^S(C_T))$

Pricing measure: $\mathbb{Q}^S(Y_T|S_T) = \mathbb{P}(Y_T|S_T)$; $\frac{B_t}{S_t}$ martingale w.r.t. $\mathbb{Q}^S$

Indifference price

$$\nu^S(C_T) = E_{\mathbb{Q}^S}\left(\frac{1}{\gamma^S(S_T)} \log E_{\mathbb{Q}^S}\left(e^{\gamma^S(S_T)\frac{C_T}{S_T}} \mid S_T\right)\right)$$

Numeraire consistency

$$\frac{\nu(C_T;\gamma_T)}{S_0} = \nu^S(C_T;\gamma_T^S) \quad \Longleftrightarrow \quad \delta_T = \delta_T^S\, S_T$$

The term structure of risk preferences

Fundamental questions

- What is the proper specification of the investors' risk preferences?
- Are risk preferences static or dynamic?
- Are they affected by the market environment and the trading horizon?
- Are there endogenous structural conditions on risk preferences?
- How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?

Requirements for a consistent indifference pricing system

(work in progress MZ)

Risk preferences need to be consistent across units and trading horizons

$$\Downarrow$$

Dynamic utilities

Martingality of risk tolerance process