# Optimal investments under dynamic performance critria

Lecture IV

## Utility-based measurement of performance

# Deterministic environment

## Utility traits

$$u(x,t) \quad : \quad x \text{ **"wealth"** and } t \text{ **"time"**}$$

- Monotonicity $\quad u_x(x,t) > 0$

- Risk aversion $\quad u_{xx}(x,t) < 0$

- Impatience $\quad u_t(x,t) < 0$

Fisher (1913, 1918), Koopmans (1951),

Koopmans-Diamond-Williamson (1964) ...

## Stochastic environment

### Important ingredients

- Time evolution concurrent with the one of the investment universe
- Consistency with up to date information
- Incorporation of available opportunities and constraints
- Meaningful optimal utility volume

## Dynamic utility

$U(x,t)$ is an $\mathcal{F}_t$-adapted process

- As a function of $x$, $U$ is increasing and concave
- For each self-financing strategy, represented by $\pi$, the associated (discounted) wealth $X_t$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi},t) \mid \mathcal{F}_s) \geq U(X_s^{\pi},s) \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by $\pi^*$, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*},t) \mid \mathcal{F}_s) = U(X_s^{\pi^*},s) \qquad 0 \leq s \leq t$$

## Traditional framework

A deterministic utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U(x,T) = u_T(x)$$

**Backwards in time generation of optimal utility volume**

$$V(x,t) = \sup_{\pi} E_{\mathbb{P}}(u(X_T^{\pi}, T)|\mathcal{F}_t; X_t^{\pi} = x)$$

$$V(x,t) = \sup_{\pi} E_{\mathbb{P}}(V(X_s^{\pi}, s)|\mathcal{F}_t; X_t^{\pi} = x) \qquad \text{(DPP)}$$

$$V(x,t) = E_{\mathbb{P}}(V(X_s^{\pi^*}, s)|\mathcal{F}_t; X_t^{\pi^*} = x)$$

$$\Downarrow$$

$$U(x,t) \equiv V(x,t) \qquad 0 \leq t < T$$

**The dynamic utility coincides with the traditional value function**

A deterministic utility datum $u_0(x)$ is assigned at the beginning of the trading horizon, $t = 0$

$$U(x, 0) = u_0(x)$$

**Forward in time generation of optimal utility volume**

$$U(X_s^{\pi^*}, s) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t)|\mathcal{F}_s) \qquad 0 \leq s \leq t$$

- Dynamic utility can be defined for all trading horizons
- Utility and allocations take a very intuitive form
- Difficulties due to the "inverse in time" nature of the problem

**Utility is not exogeneously given but is**

**implied/calibrated w.r.t. investment opportunities**

# Motivational examples

## An incomplete multiperiod binomial example

## Exponential utility datum

- Traded security: $S_t$, $t = 0, 1, ...$

  $\xi_{t+1} = \dfrac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi^d_{t+1}, \xi^u_{t+1}$ with $0 < \xi^d_{t+1} < 1 < \xi^u_{t+1}$

  Second traded asset is riskless yielding zero interest rate

- Stochastic factor: $Y_t$, $t = 0, 1, ...$

  $\eta_{t+1} = \dfrac{Y_{t+1}}{Y_t}$, $\eta_{t+1} = \eta^d_{t+1}, \eta^u_{t+1}$ with $\eta^d_t < \eta^u_t$

- Probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$

  $\{S_t, Y_t : t = 0, 1, ...\}$ : a two-dimensional stochastic process

- State wealth process: $X_t$, $t = s+1, s+2, \ldots, \ldots$

  $\alpha_i$ : the number of shares of the traded security held in this portfolio over the time period $[i-1, i]$

$$X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \triangle S_i$$

- Forward dynamic exponential utility

$$\begin{cases} U(X_s^{\alpha^*}, s) = E_{\mathbb{P}}(U(X_t^{\alpha^*}, t)|\mathcal{F}_s) \\ U(x, 0) = -e^{-\gamma x}\,, \qquad \gamma > 0 \end{cases}$$

- A forward dynamic utility

$$U(x,t) = \begin{cases} -e^{-\gamma x} & \text{if} \quad t = 0 \\ -e^{-\gamma x + \sum_{i=1}^{t} h_i} & \text{if} \quad t \geq 1 \end{cases}$$

- Auxiliary quantities : local entropies $h_i$

$$h_i = q_i \log \frac{q_i}{\mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)} + (1 - q_i) \log \frac{1 - q_i}{1 - \mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)}$$

with

$$A_i = \{\xi_i = \xi_i^u\} \quad \text{and} \quad q_i = \mathbb{Q}\left(A_i \,|\mathcal{F}_{i-1}\right)$$

for $i = 0, 1, ..$ and $\mathbb{Q}$ being the minimal relative entropy measure

## Important insights

The forward utility process

$$U(x,t) = -e^{-\gamma x + \sum_{i=1}^{t} h_i}$$

is of the form

$$U(x,t) = u(x, A_t)$$

where $u(x,t)$ is the deterministic utility function

$$u(x,t) = -e^{-\gamma x + \frac{1}{2}t}$$

and $A_t$ corresponds to a time change depending on the "market input"

$$A_t = 2\sum_{i=1}^{t} h_i$$

## Important insights (continued)

- The variational utility input

$$u(x,t) = -e^{-\gamma x + \frac{1}{2}t}$$

solves the partial differential equation

$$\begin{cases} u_t \; u_{xx} = \frac{1}{2} u_x^2 \\ u(x,0) = -e^{-\gamma x} \end{cases}$$

- The stochastic market input

$$A_t = 2 \sum_{i=1}^{t} h_i$$

plays now the role of "time". It depends exclusively on the market parameters.

## A continuous-time example

- Investment opportunities

  Riskless bond : $r = 0$

  Risky security : $dS_t = \sigma_t S_t(\lambda_t dt + dW_t)$

- Utility datum at $t = 0$ : $u_0(x)$

- Wealth process

$$\begin{cases} dX_t = \sigma_t \pi_t(\lambda_t dt + dW_t) \\ X_0 = x \end{cases}$$

- Market input : $\lambda_t$, $A_t$

$$\begin{cases} dA_t = \lambda_t^2 dt \\ A_0 = 0 \end{cases}$$

- Building the martingale $U(X_t^{\pi^*}, t)$

Assume that we can construct $U(x,t)$ via

$$\begin{cases} U(X_t^{\pi^*}, t) = u(X_t^{\pi^*}, A_t) \\ U(x,0) = u(x,0) = u_0(x) \end{cases}$$

where $u(x,t)$ is the variational utility input and $A_t$ the stochastic market input

$$dU(X_t^{\pi}, t) = u_x(X_t, A_t)\sigma_t \pi_t \, dW_t$$

$$+\underbrace{(u_t(X_t^{\pi}, A_t)\lambda_t^2 + u_x(X_t^{\pi}, A_t)\sigma_t\pi_t\lambda_t + \tfrac{1}{2}u_{xx}(X_t^{\pi}, A_t)\sigma_t^2\pi_t^2)}_{\leq 0}dt$$

- Variational utility input condition

$$\begin{cases} u_t \; u_{xx} = \frac{1}{2} u_x^2 \\ \\ u(x,0) = u_0(x) \end{cases}$$

- The optimal allocations in stock, $\pi_t^*$, and in bond, $\pi_t^{0,*}$,

$$\begin{cases} \pi_t^* = -\sigma_t^{-1} \lambda_t \dfrac{u_x(X_t^{\pi^*}, A_t)}{u_{xx}(X_t^{\pi^*}, A_t)} = \sigma_t^{-1} \lambda_t R_t \\ \\ \pi_t^{0,*} = X_t^{\pi^*} - \sigma_t^{-1} \lambda_t R_t \end{cases}$$

$$R_t = r(X_t^{\pi^*}, A_t) \quad ; \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

**The local risk tolerance $r(x,t)$ and the subordinated**

**risk tolerance process $R_t$ emerge as important quantities**

## Dynamic utility measurement

time $t_1$, information $\mathcal{F}_{t_1}$

asset returns

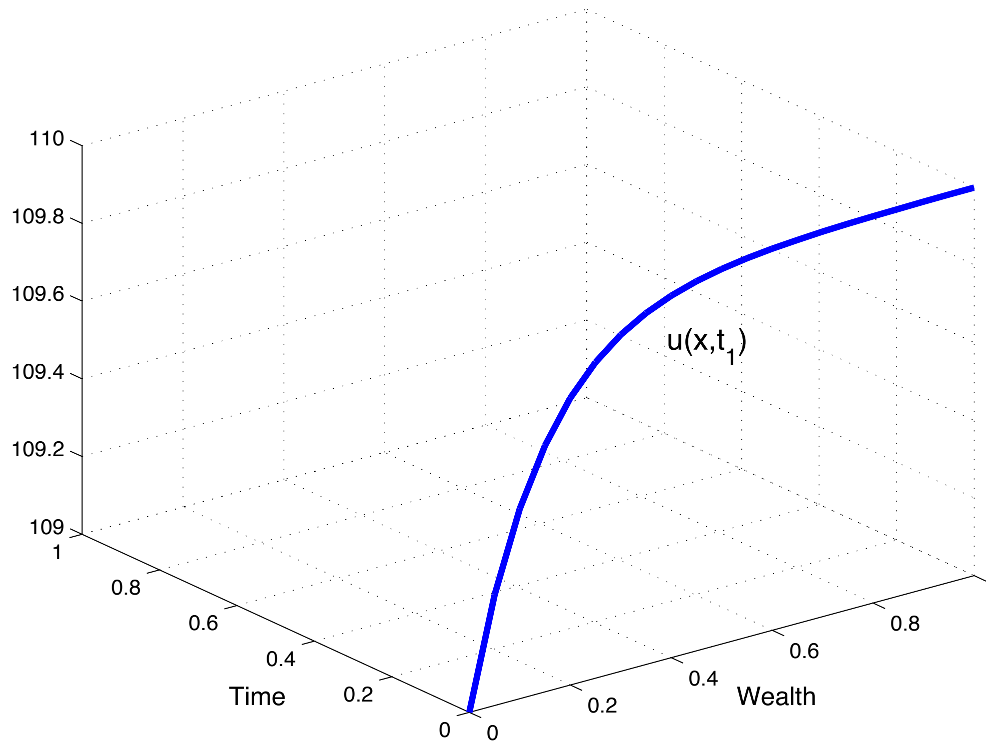
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$




$$MI(t_1) \dashrightarrow + \dashleftarrow u(x, t_1)$$

$$\Downarrow$$

$$\boxed{U(x, t_1; MI) \in \mathcal{F}_{t_1} \qquad \pi(x, t_1; MI) \in \mathcal{F}_{t_1}}$$

## Dynamic utility measurement

time $t_2$, information $\mathcal{F}_{t_2}$

asset returns

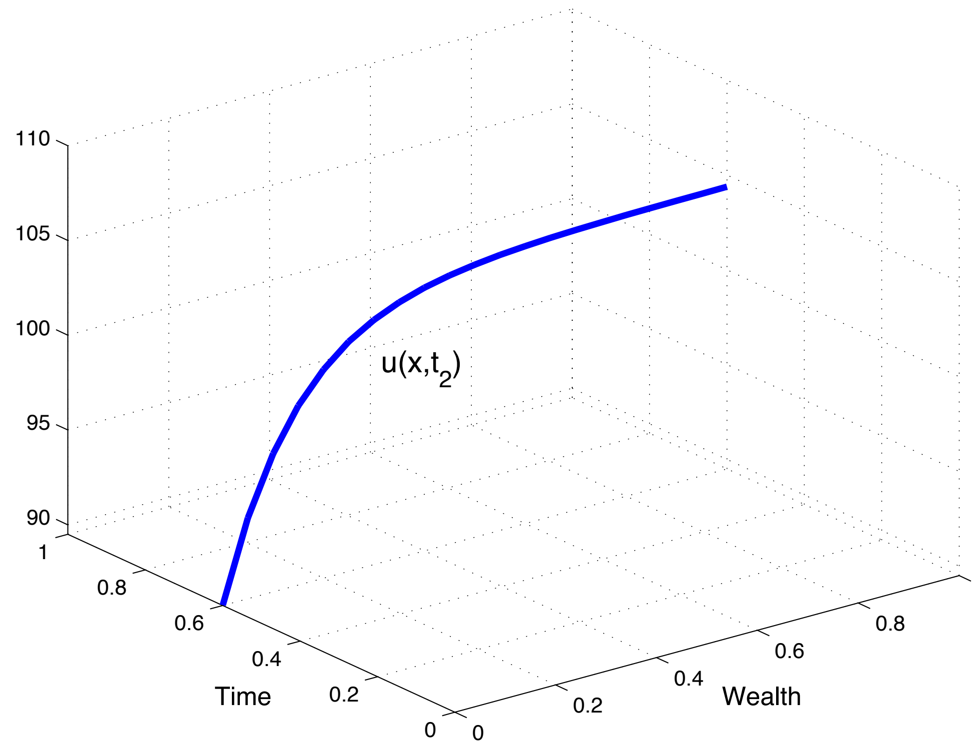
constraints

market view
away from equilibrium

benchmark   numeraire

calendar time   subordination

$\longrightarrow$




$MI(t_2)$ $\dashrightarrow$ $+$ $\dashleftarrow$ $u(x, t_2)$

$\Downarrow$

$$U(x, t_2; MI) \in \mathcal{F}_{t_2} \qquad \pi(x, t_2; MI) \in \mathcal{F}_{t_2}$$

## Dynamic utility measurement

time $t_3$, information $\mathcal{F}_{t_3}$

asset returns

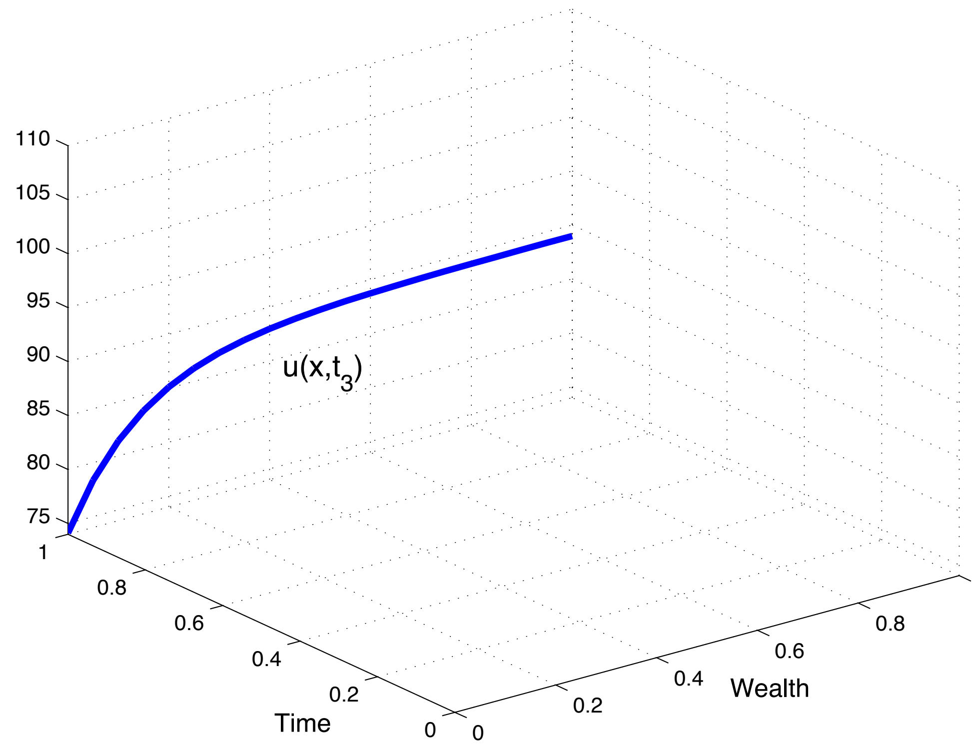
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$




$$MI(t_3) \dashrightarrow + \dashleftarrow u(x, t_3)$$

$$\Downarrow$$

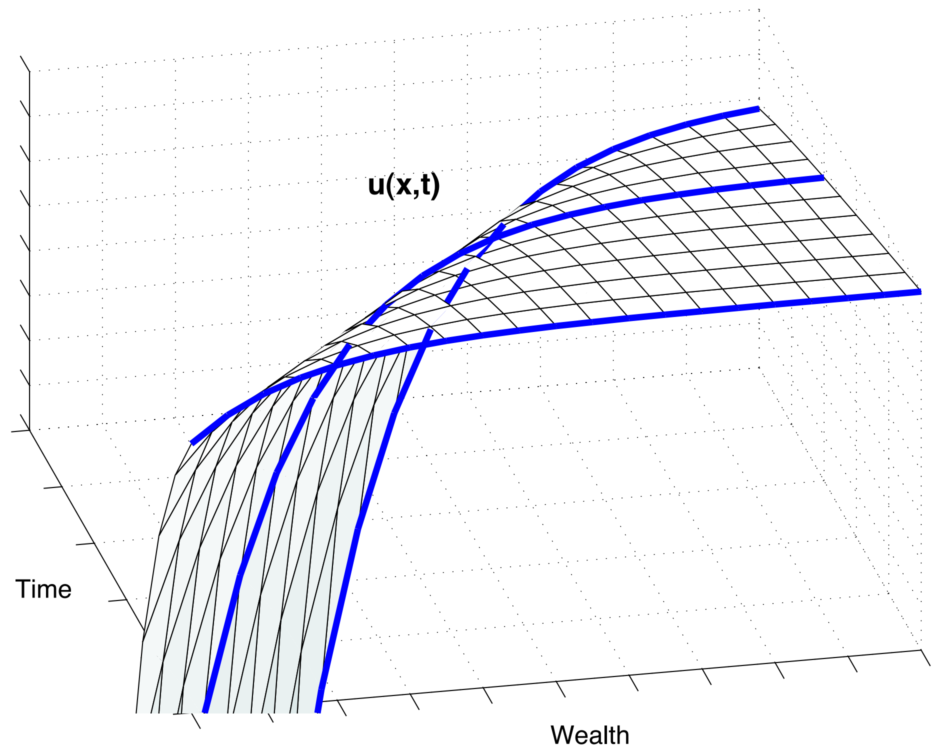
$$\boxed{U(x, t_3; MI) \in \mathcal{F}_{t_3} \qquad \pi(x, t_3; MI) \in \mathcal{F}_{t_3}}$$

## Dynamic utility measurement

time $t$, information $\mathcal{F}_t$

| |
|---|
| asset returns<br>additional<br>market input |

$\longrightarrow$




$$MI(t) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x,t)$$

$$\Downarrow$$

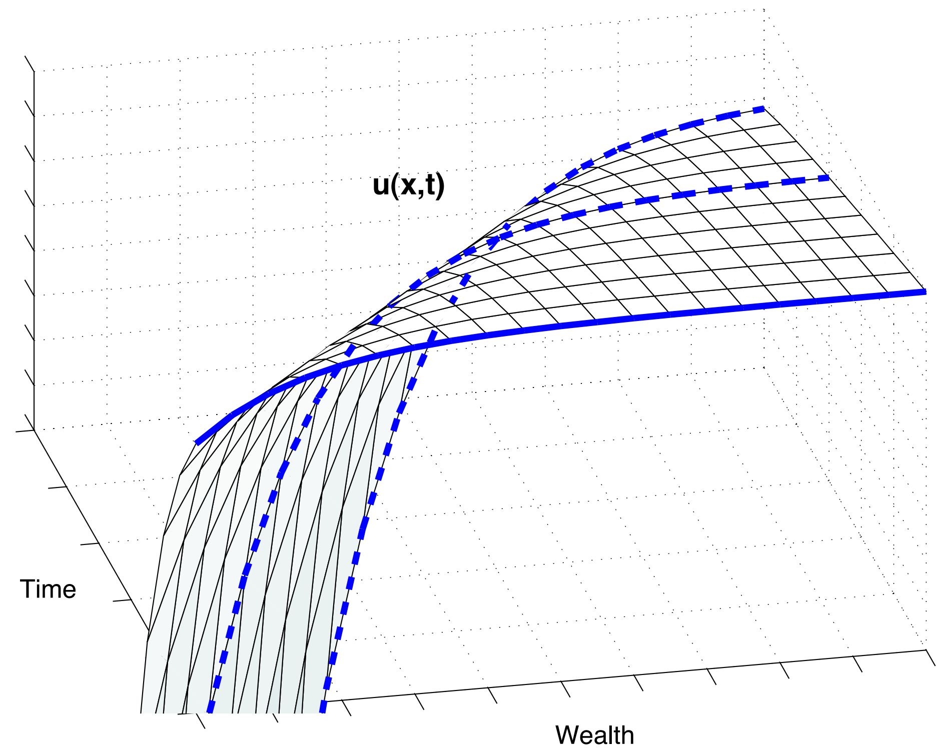
$$U(X_t^*, t) \in \mathcal{F}_t \qquad \pi^*(X_t^*, t) \in \mathcal{F}_t$$

## Dynamic utility measurement

time $t_1$, information $\mathcal{F}_{t_1}$

asset returns
additional
market input

$\longrightarrow$




$MI(t_1) \dashrightarrow + \dashleftarrow u(x, t_1)$

$\Downarrow$

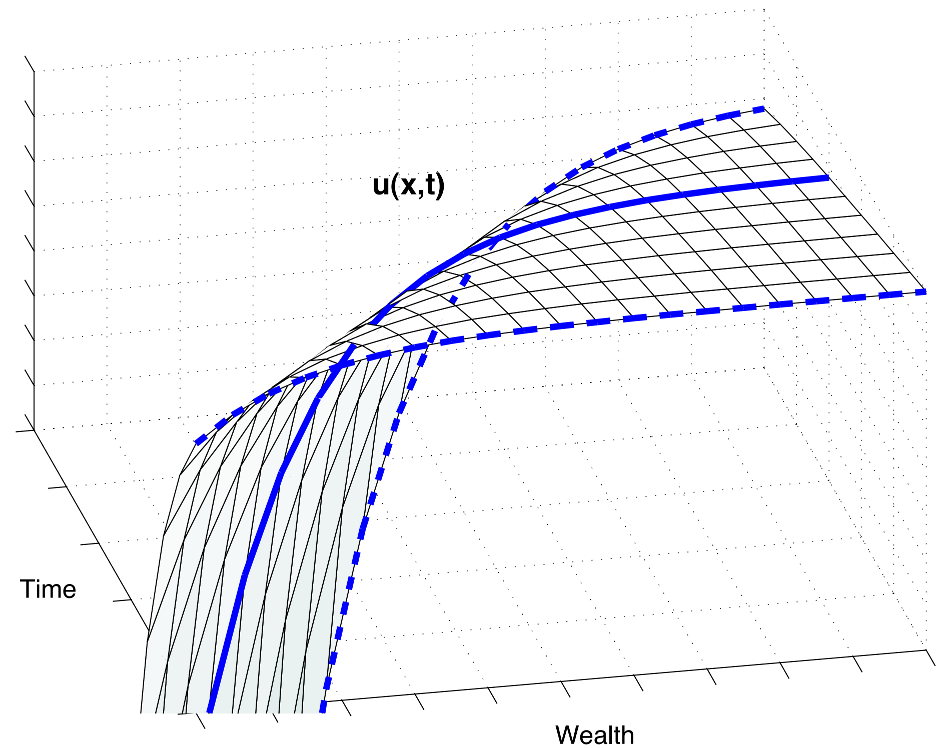
$$U(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1} \qquad \pi^*(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1}$$

## Dynamic utility measurement

time $t_2$, information $\mathcal{F}_{t_2}$

asset returns
additional
market input

$\longrightarrow$




$MI(t_2) \dashrightarrow + \dashleftarrow u(x, t_2)$

$\Downarrow$

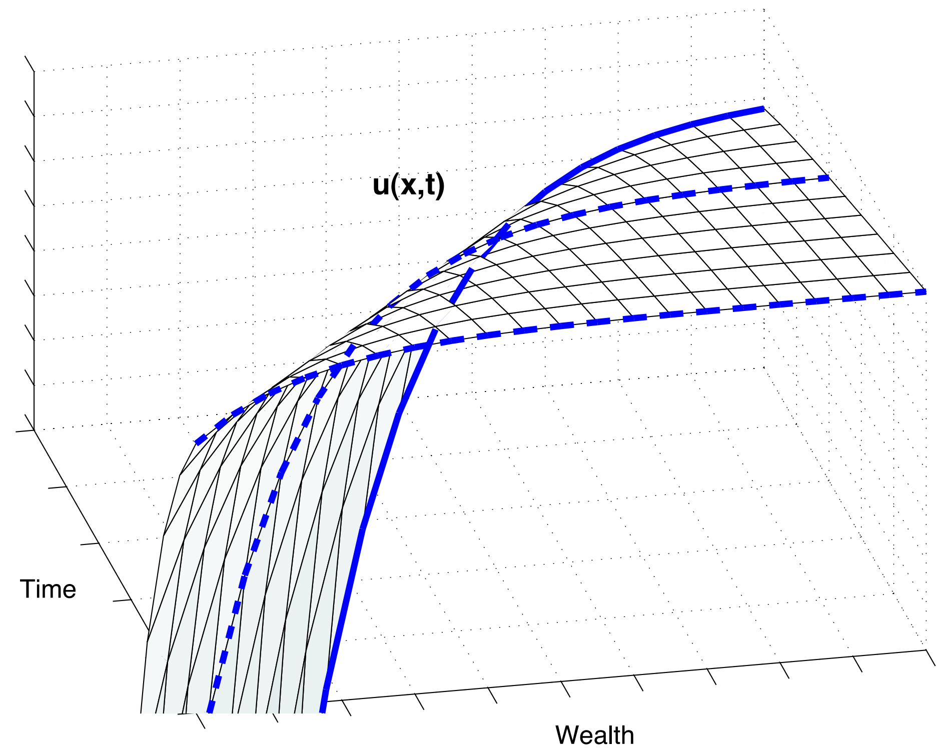
$$U(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2} \qquad \pi^*(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2}$$

## Dynamic utility measurement

time $t_3$, information $\mathcal{F}_{t_3}$

| |
|---|
| asset returns |
| additional |
| market input |

$\longrightarrow$




$MI(t_3)$ $\dashrightarrow$ $+$ $\dashleftarrow$ $u(x, t_3)$

$\Downarrow$

$$U(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3} \qquad \pi^*(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3}$$

## Construction of a class of forward dynamic utilities

## Creating the martingale that yields the optimal utility volume

Minimal model assumptions

Stochastic optimization problem "inverse" in time

**Key idea**

Stochastic input

**Market**

Variational input

**Individual**

Maximal utility — Optimal allocation

## Variational input – utility surfaces

## Utility surface

A model independent variational constraint on

impatience, risk aversion and monotonicity

- Initial utility datum

$$u_0(x) = u(x, 0)$$

- Fully non-linear pde

$$\begin{cases} u_t \; u_{xx} = \dfrac{1}{2} u_x^2 \\ u(x, 0) = u_0(x) \end{cases}$$

## Utility transport equation

The utility equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

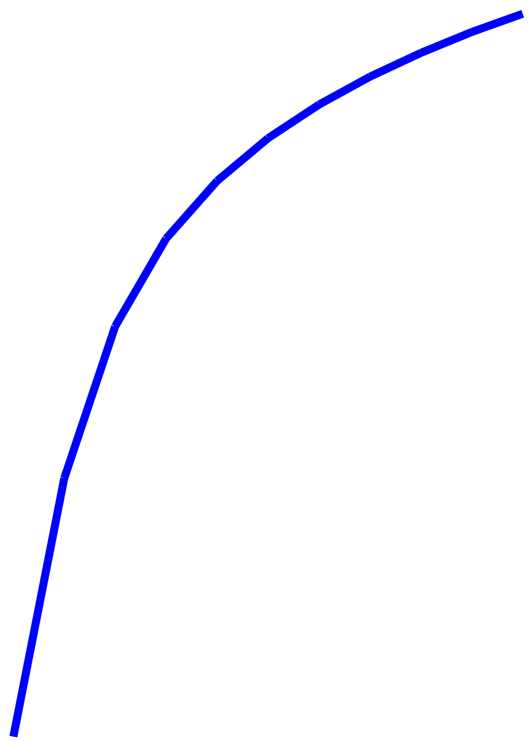
$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Characteristic curves: $$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t),t)$$

## Construction of utility surface $u(x,t)$ using characteristics

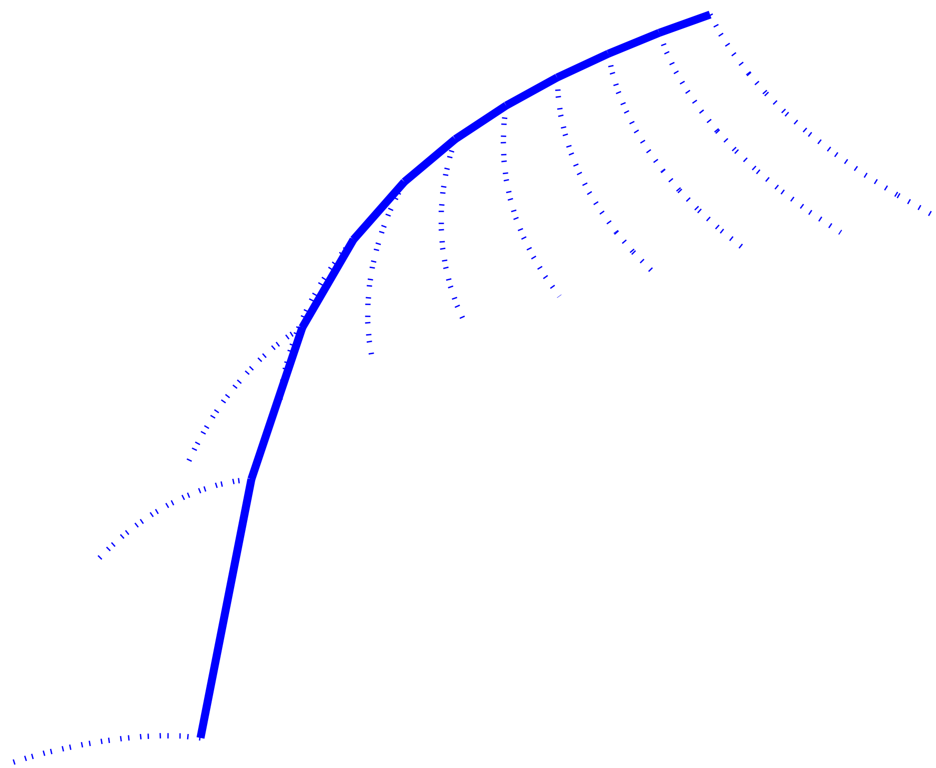
$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$



Utility datum $u_0(x)$

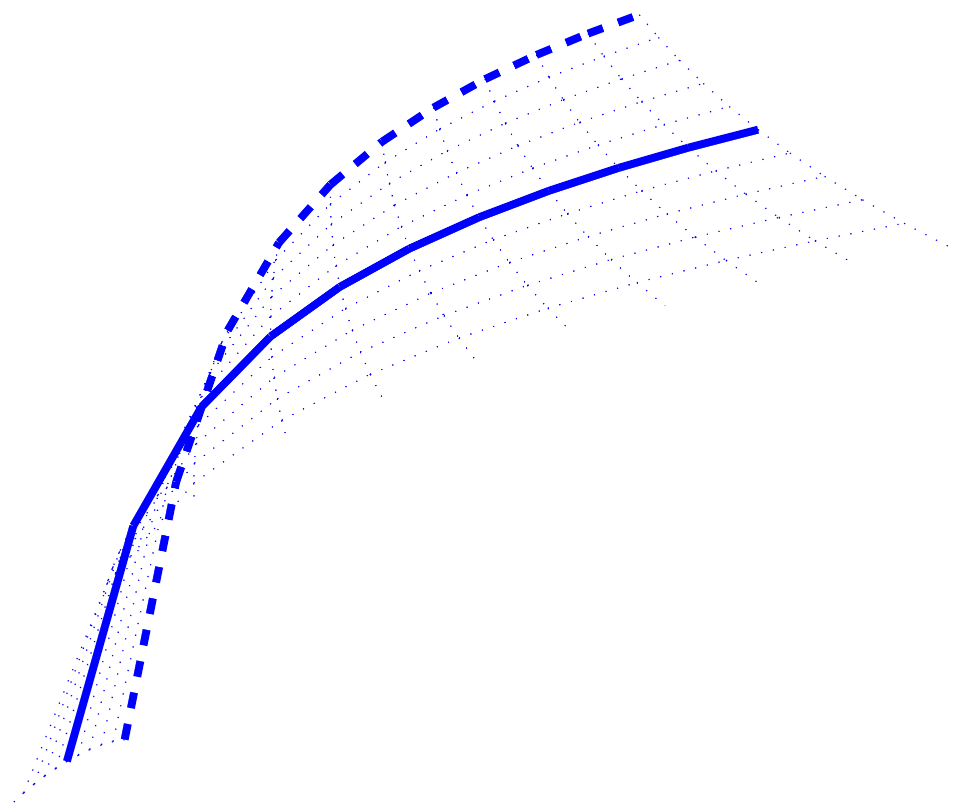
## Construction of characteristics

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$

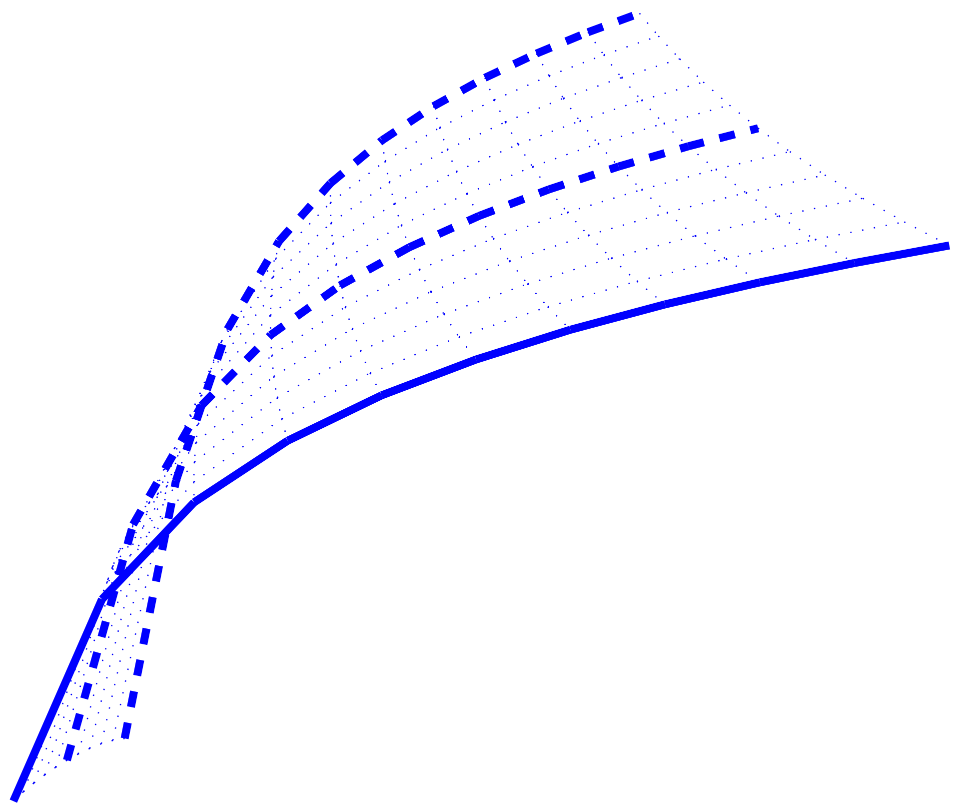


Utility datum $u(x, 0)$
Characteristic curves

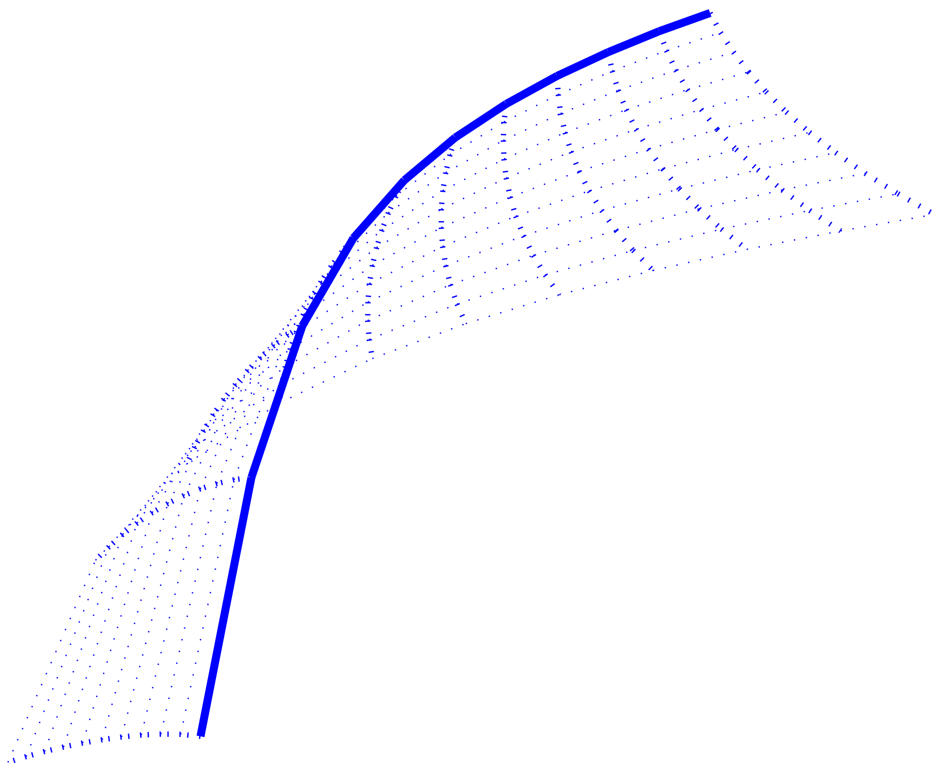
# Propagation of utility datum along characteristics



## Propagation of utility datum along characteristics



# Utility surface $u(x, t)$



## Two related pdes

- Fast diffusion equation for risk tolerance

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = r_0(x) \end{cases} \qquad \text{(FDE)}$$

Conductivity : $r^2$

- Porous medium equation for risk aversion

$$\gamma(x,t) = \frac{1}{r(x,t)}$$

$$\begin{cases} \gamma_t = \left( \frac{1}{\gamma} \right)_{xx} \\ \gamma(x,0) = \frac{1}{r_0(x)} \end{cases} \qquad \text{(PME)}$$

Pressure : $r^2$ and (PME) exponent: $m = -1$

## Difficulties

- Utility equation: $u_t \; u_{xx} = \frac{1}{2} u_x^2$

  Inverse problem and fully nonlinear

- Utility transport equation: $u_t + \frac{1}{2} r(x,t) u_x = 0$

  Shocks, solutions past singularities

- Fast diffusion equation: $r_t + \frac{1}{2} r^2 r_{xx} = 0$

  Inverse problem and backward parabolic, solutions might not exist, locally integrable data might not produce locally bounded slns in finite time

- Porous medium equation: $\gamma_t = (\frac{1}{\gamma})_{xx}$
  Majority of results for (PME), $\gamma_t = (\gamma^m)_{xx}$, are for $m > 1$, partial results for $-1 < m < 0$

# A rich class of risk tolerance inputs

- Addititively separable risk tolerance

$$r^2(x,t;\alpha,\beta) = m(x;\alpha,\beta) + n(t;\alpha,\beta)$$

Example

$$m(x;\alpha,\beta) = \alpha x^2 \qquad n(x;\alpha,\beta) = \beta e^{-\alpha t}$$

$$r(x,t;\alpha,\beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \qquad \alpha,\beta > 0$$

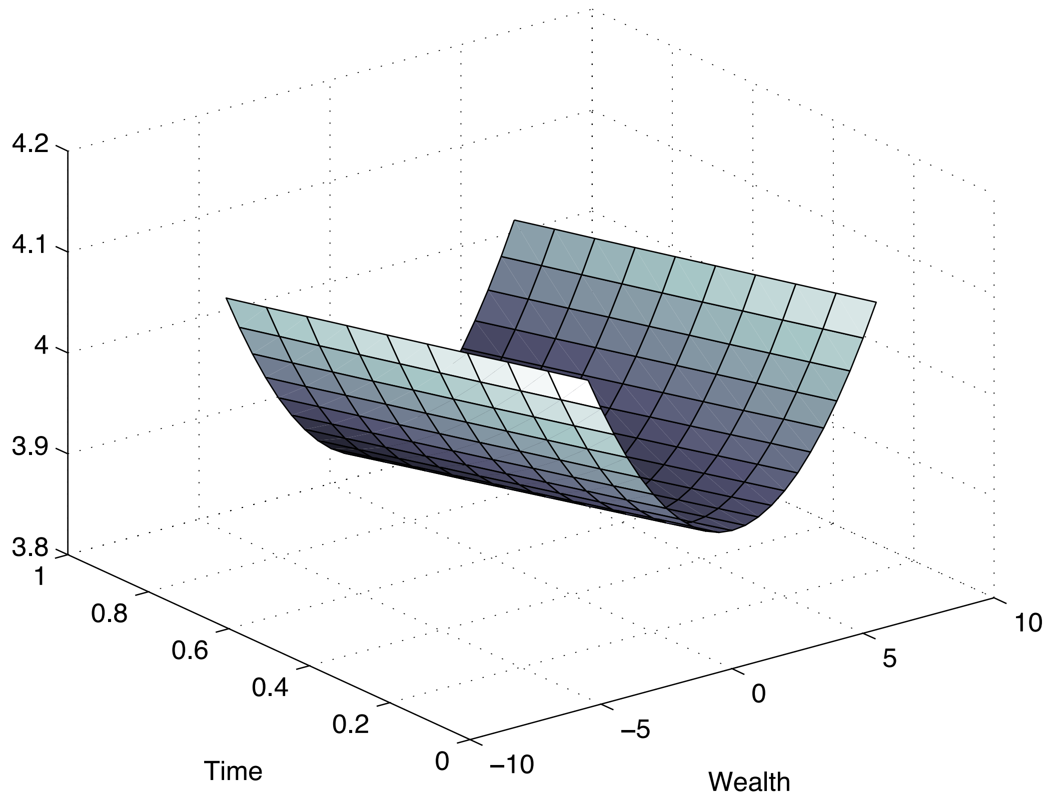
**(Very) special cases**

$$r(x,t;0,\beta) = \sqrt{\beta} \quad \longrightarrow \quad u(x,t) = -e^{-\frac{x}{\sqrt{\beta}}+\frac{t}{2}}$$

$$r(x,t;1,0) = |x| \quad \longrightarrow \quad u(x,t) = \log x - \frac{t}{2}$$

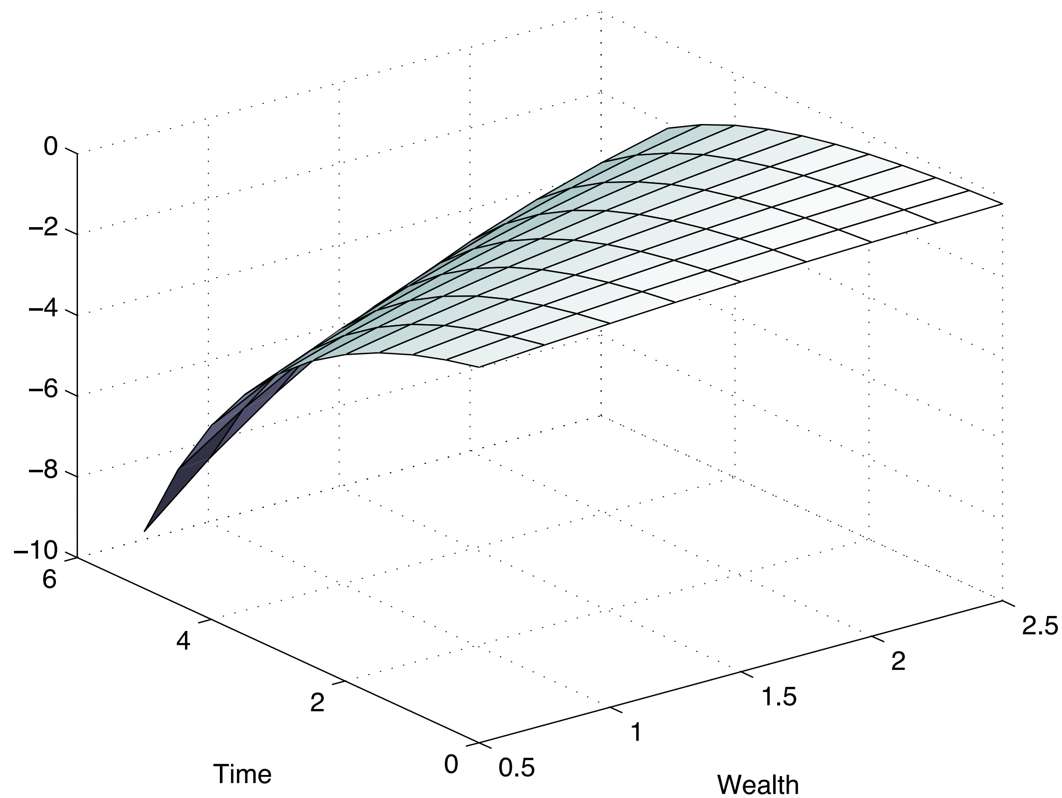
$$r(x,t;\alpha,0) = \sqrt{\alpha}\ |x| \ \longrightarrow \ u(x,t) = \frac{1}{\gamma}x^{\gamma}e^{-\frac{\gamma}{2(1-\gamma)}t}, \quad \gamma = \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}}$$

## Risk tolerance $r(x,t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}}$



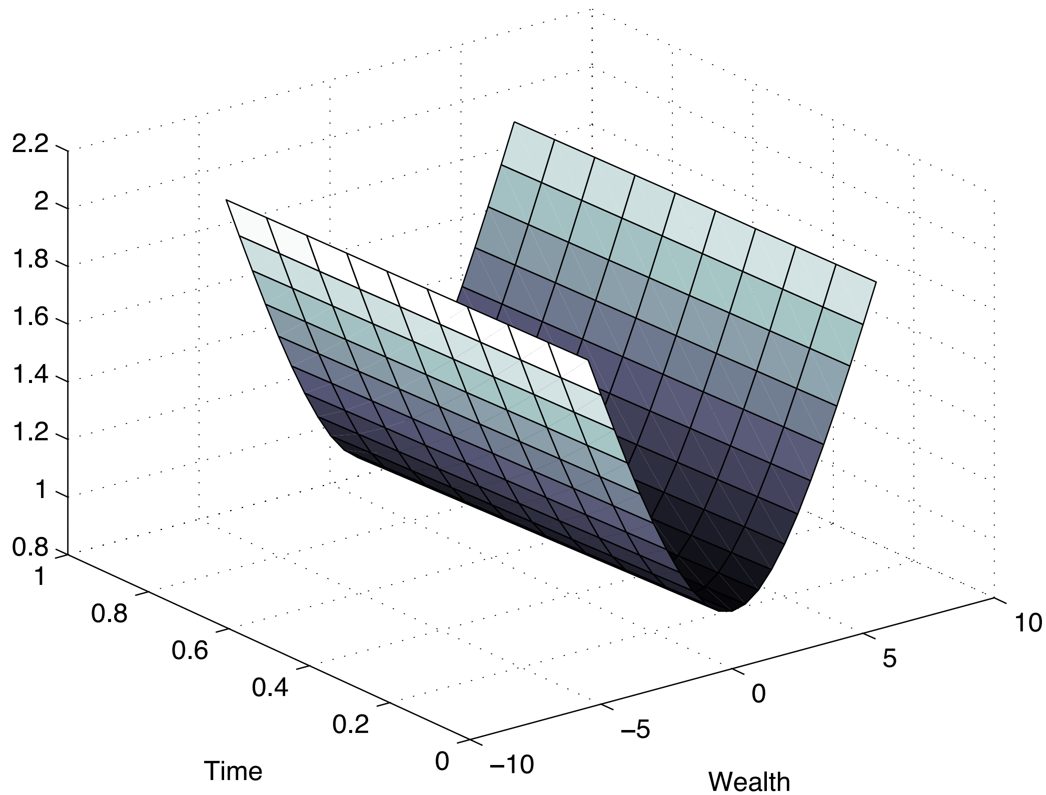
## Utility surface $u(x,t)$ generated by

## risk tolerance $r(x,t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}}$



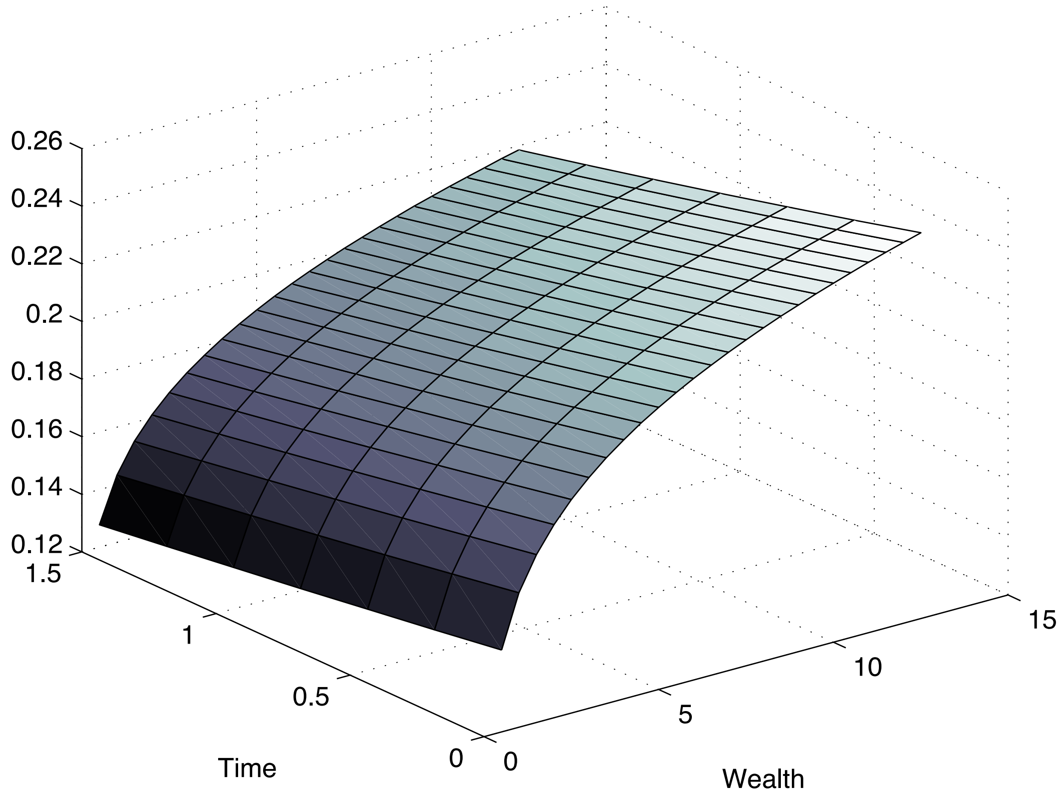

Characteristics: $$\frac{dx(t)}{dt} = \frac{1}{2}\sqrt{0.05x(t)^2 + 15.5e^{-0.05t}}$$

## Risk tolerance $r(x,t) = \sqrt{10x^2 + e^{-10t}}$



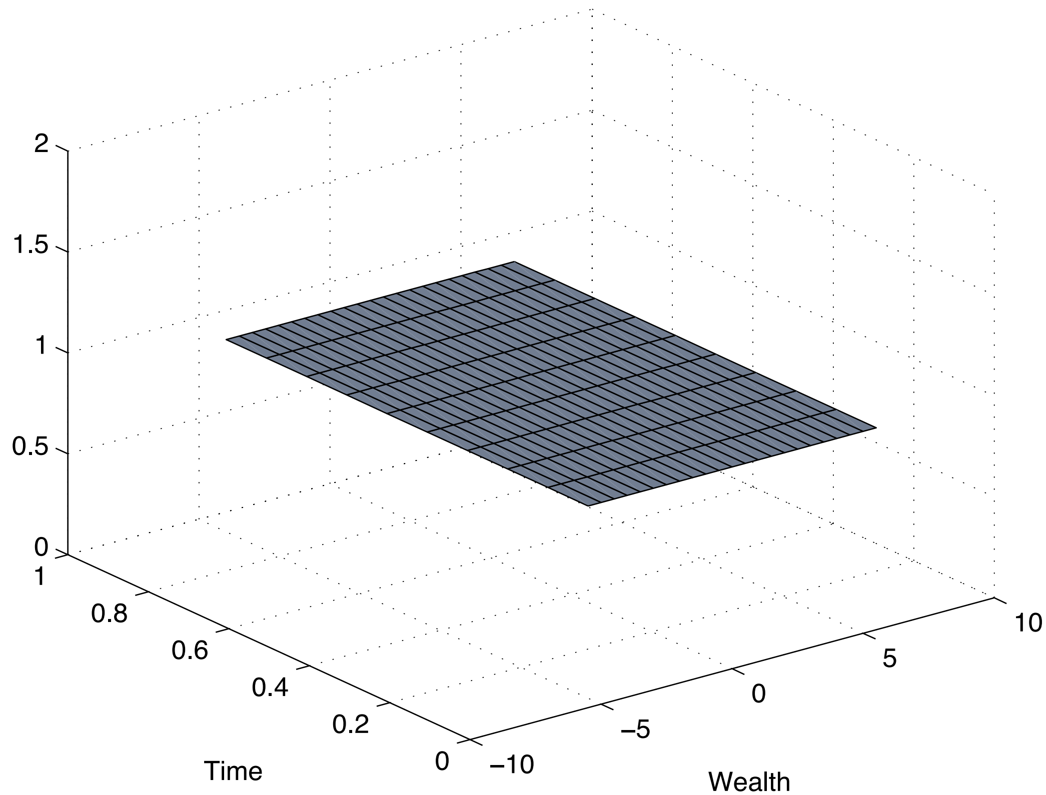
## Utility surface $u(x,t)$ generated by

## risk tolerance $\quad r(x,t) = \sqrt{10x^2 + e^{-10t}}$

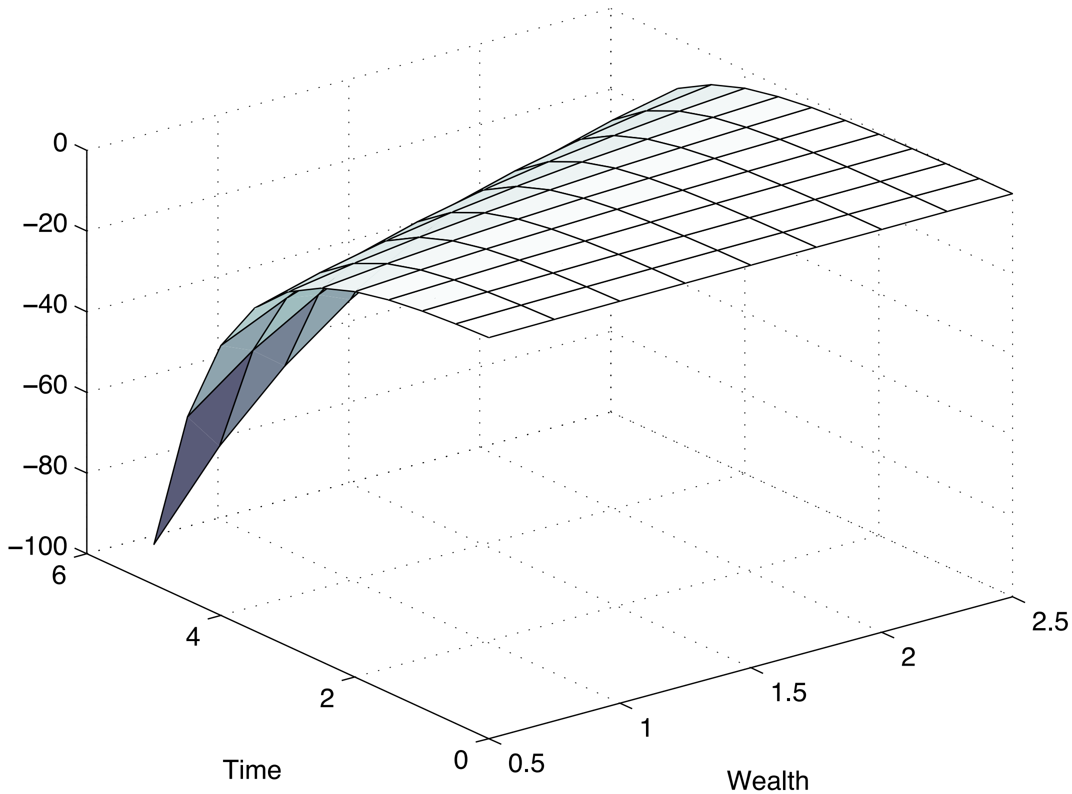



Characteristics: $\quad \dfrac{dx(t)}{dt} = \dfrac{1}{2}\sqrt{10x(t)^2 + e^{-10t}}$

## Risk tolerance $r(x, t; 0, 1) = \sqrt{0x^2 + 1} = 1$

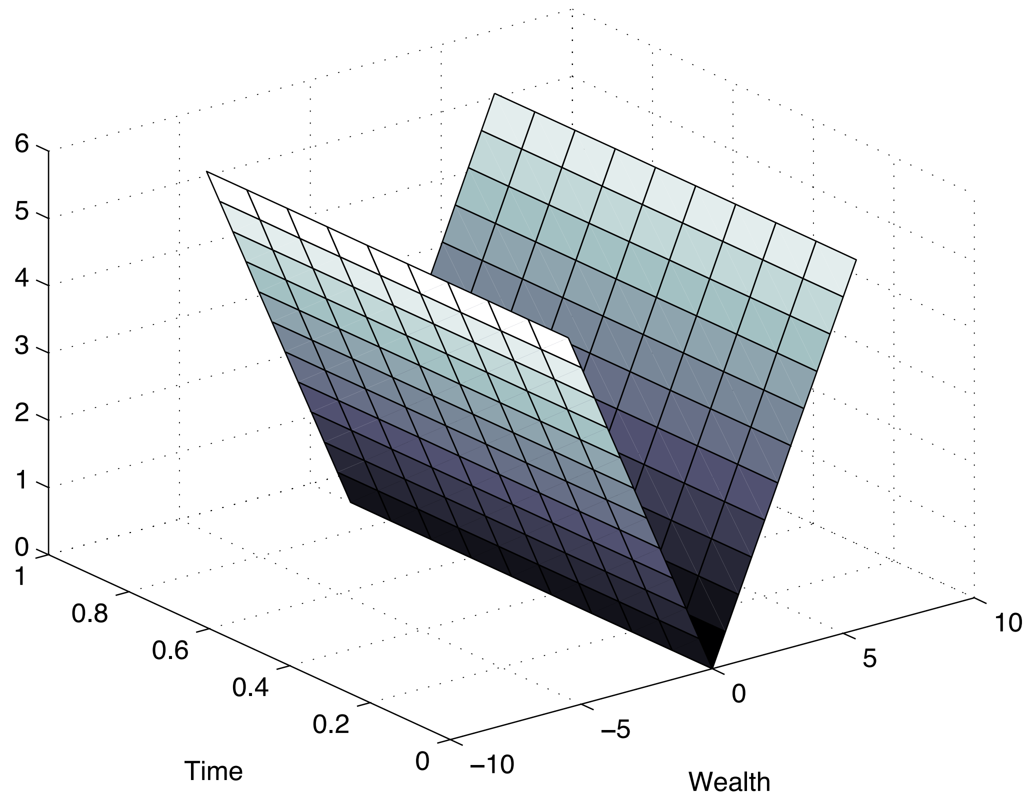


## Utility surface $u(x,t) = -e^{-x+\frac{t}{2}}$ generated by risk tolerance $r(x,t) = 1$

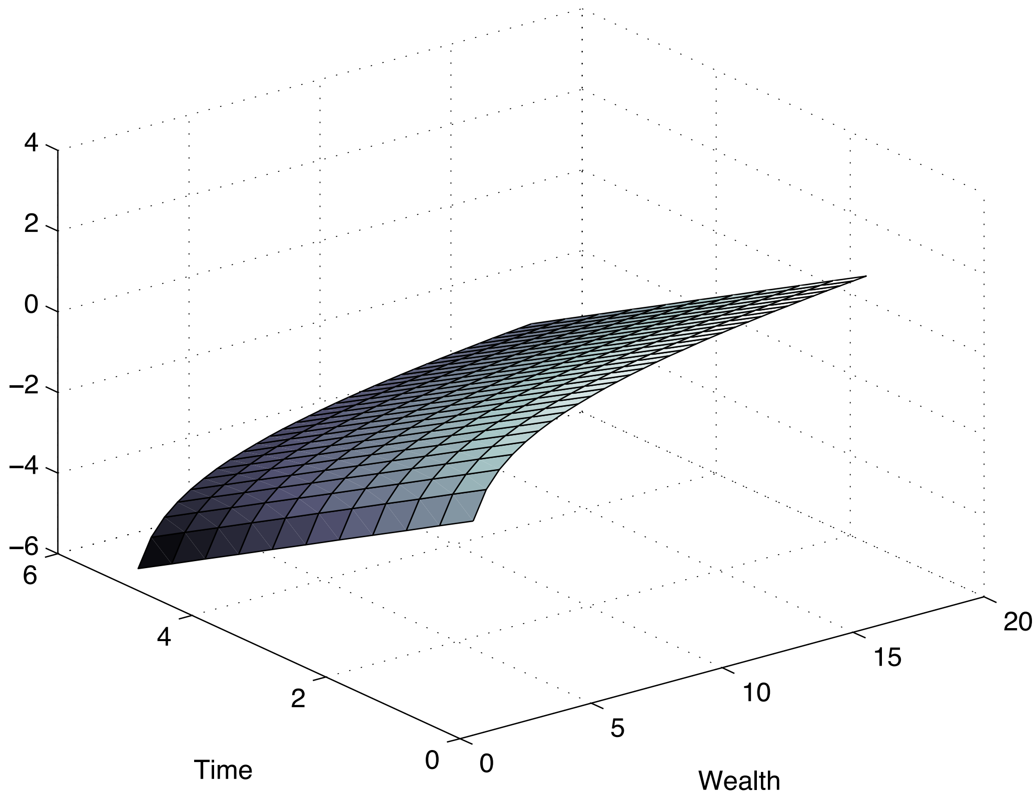



Characteristics: $$\frac{dx(t)}{dt} = \frac{1}{2}$$

**Risk tolerance** $r(x, t; 1, 0) = \sqrt{x^2 + 0e^{-t}} = |x|$

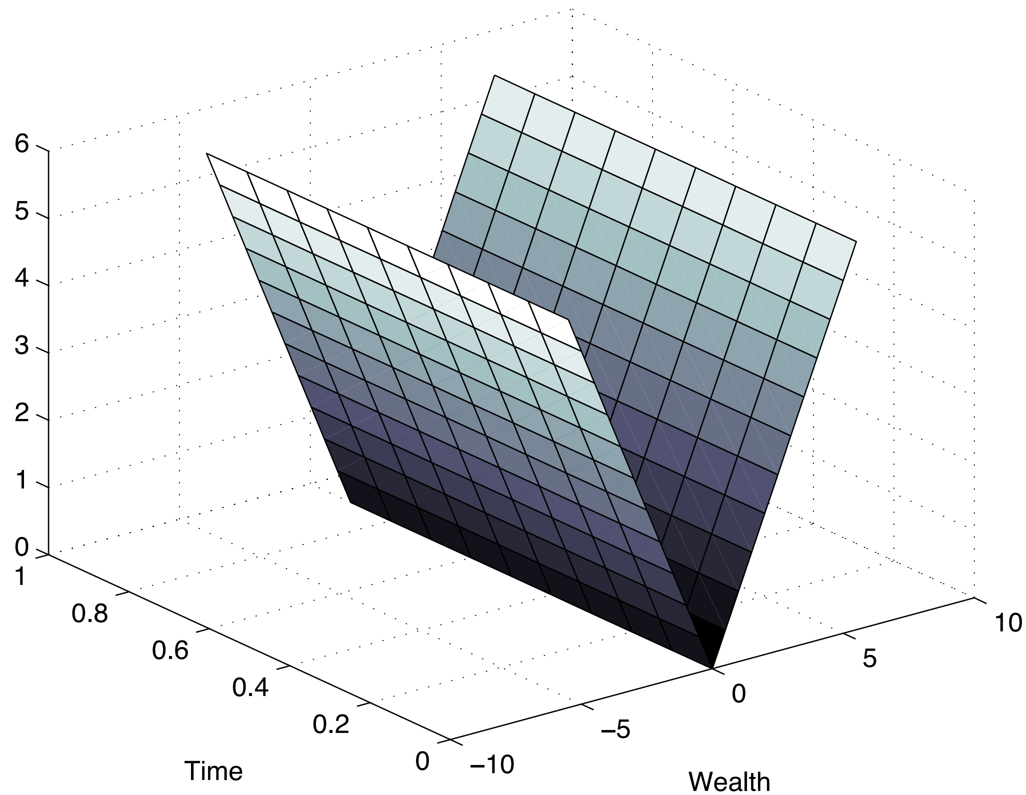


## Utility surface $u(x,t) = \log x - \frac{t}{2}$, $x > 0$ generated by risk tolerance $\quad r(x) = x$



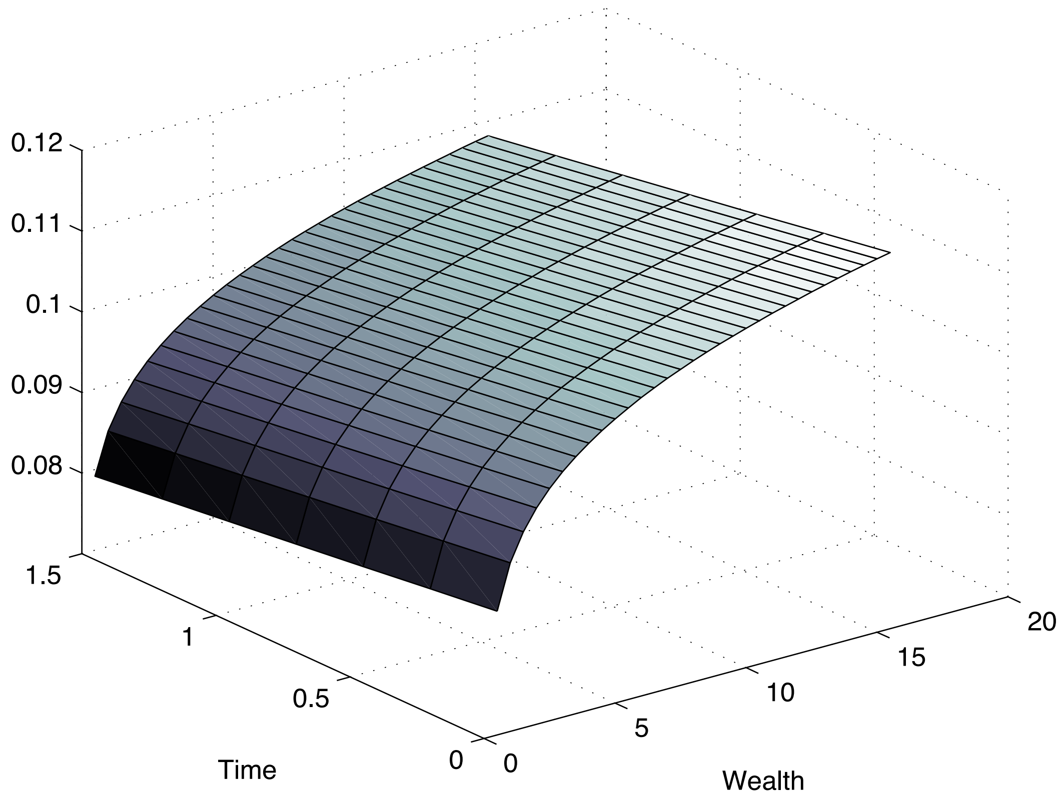

Characteristics: $\quad \dfrac{dx(t)}{dt} = \dfrac{1}{2}x(t)$

**Risk tolerance** $r(x,t;4,0)=\sqrt{4x^2+0e^{-4t}}=2|x|$



**Utility surface $u(x,t) = 2\sqrt{x}\, e^{-\frac{t}{2}}$, $x > 0$ generated by**

**risk tolerance** $r(x,t) = 2x$




Characteristics: $\dfrac{dx(t)}{dt} = x(t)$

## Multiplicatively separable risk tolerance

$$r(x,t;\alpha,\beta) = m(x;\alpha)n(t;\beta)$$

Example

$$m(x;\alpha) = \varphi(\Phi^{-1}(x;\alpha)) \qquad n(t;\beta) = \frac{1}{\sqrt{t+\beta}}\,, \qquad \beta > 0$$

$$\Phi(x;\alpha) = \int_{\alpha}^{x} e^{z^2/2}\,dz \qquad \varphi = \Phi'$$

$$r(x,t;\alpha,\beta) = \varphi(\Phi^{-1}(x;\alpha))$$

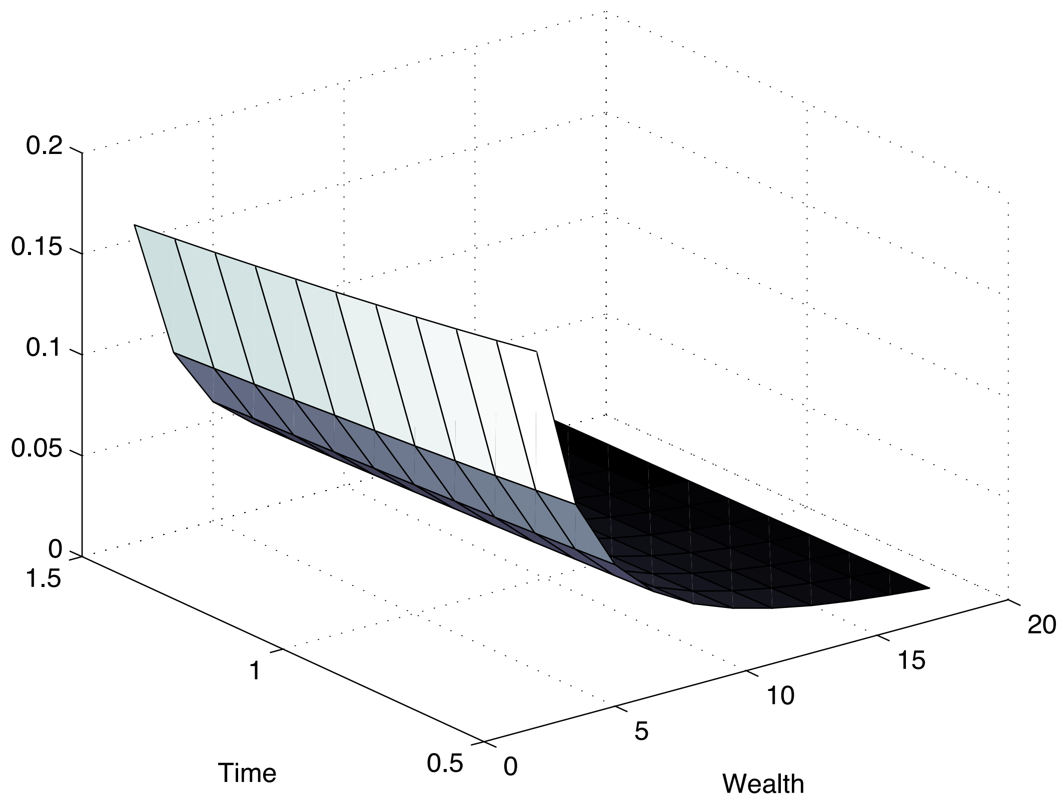
## (Very) special cases

$$m(x;\alpha) = \alpha,\ n(t;\beta) = 1 \quad \longrightarrow \quad u(x,t) = -e^{-\frac{x}{\alpha}+\frac{t}{2}}$$

$$m(x;\alpha) = x,\ n(t;\beta) = 1 \quad \longrightarrow \quad u(x,t) = \log x - \frac{t}{2}$$

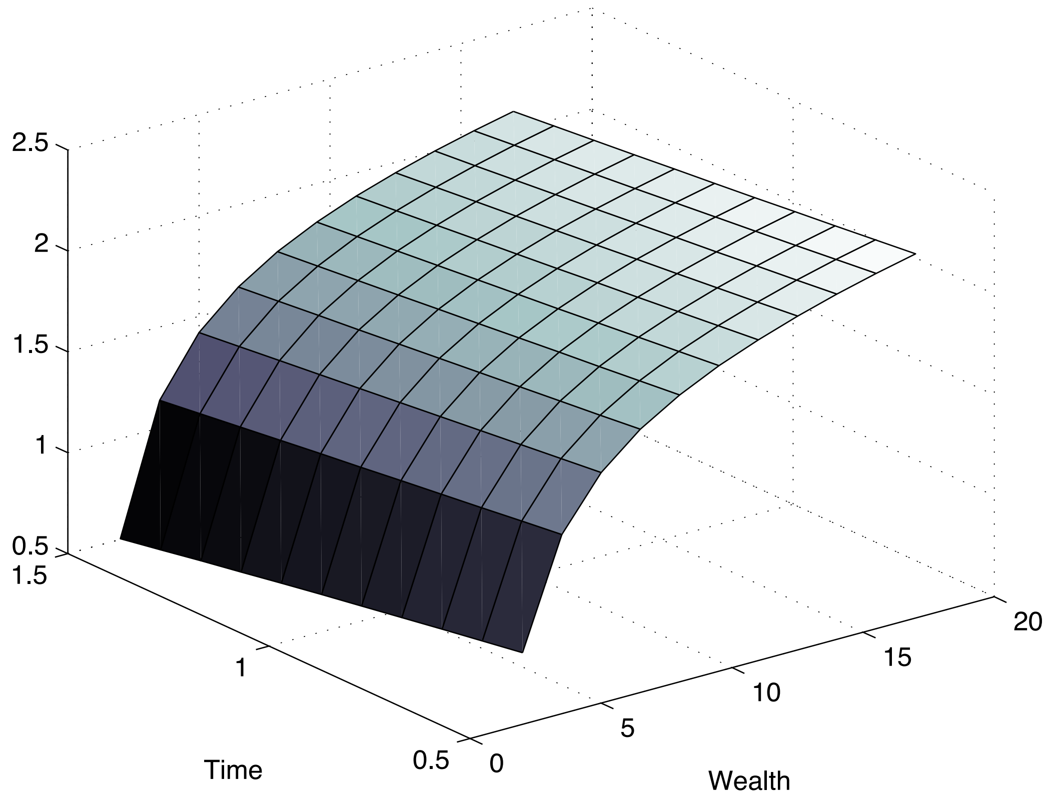
$$m(x;\alpha) = \alpha x,\ n(t;\beta) = 1 \longrightarrow u(x,t) = \frac{1}{\gamma}x^{\gamma}e^{-\frac{\gamma}{2(1-\gamma)}t}, \quad \gamma = \frac{\alpha-1}{\alpha}$$

**Risk tolerance** $r(x,t) = \dfrac{\varphi(\Phi^{-1}(x;0.5)}{\sqrt{t+5}}$



**Utility surface** $u(x,t) = \Phi(\Phi^{-1}(x;0.5) - \sqrt{t+5})$

**generated by risk tolerance** $r(x,t) = \dfrac{\varphi(\Phi^{-1}(x;0.5))}{\sqrt{t+5}}$

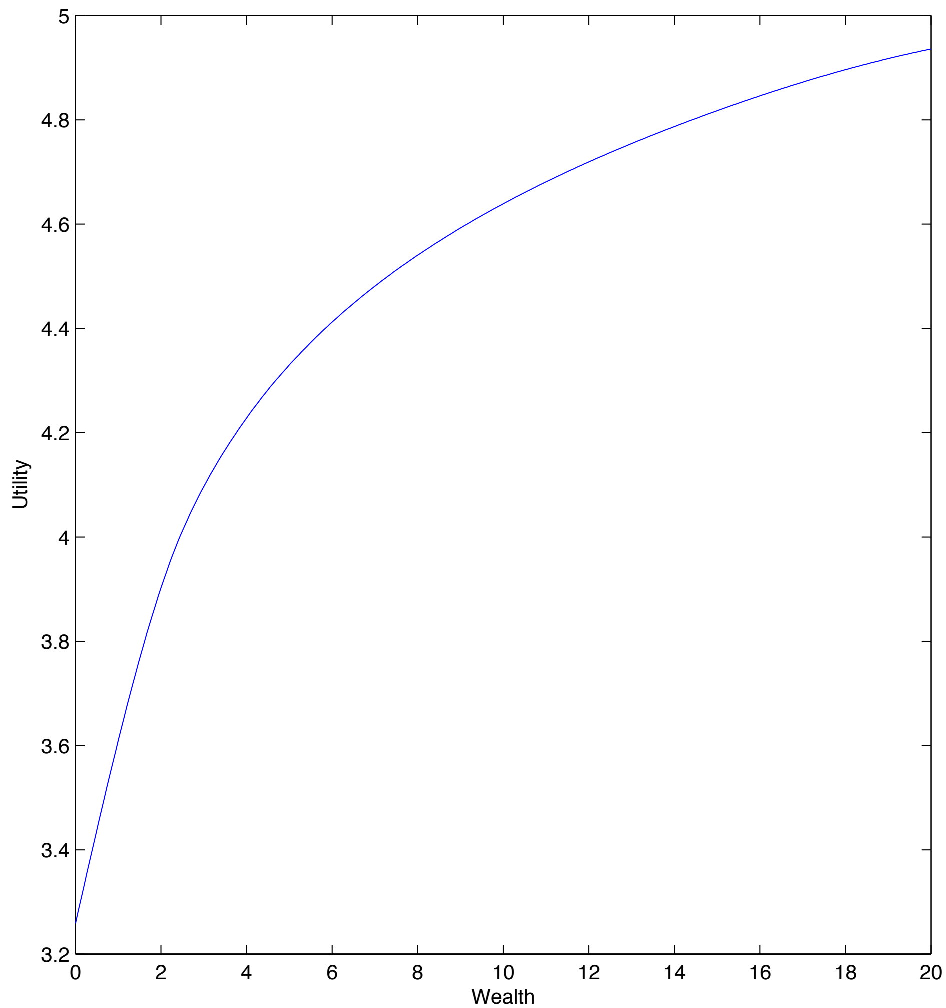



Characteristics: $$\frac{dx(t)}{dt} = \frac{\varphi(\Phi^{-1}(x(t);0.5))}{\sqrt{t+5}}$$

## **Utility function** $u(x, t_0)$

(fixed time)

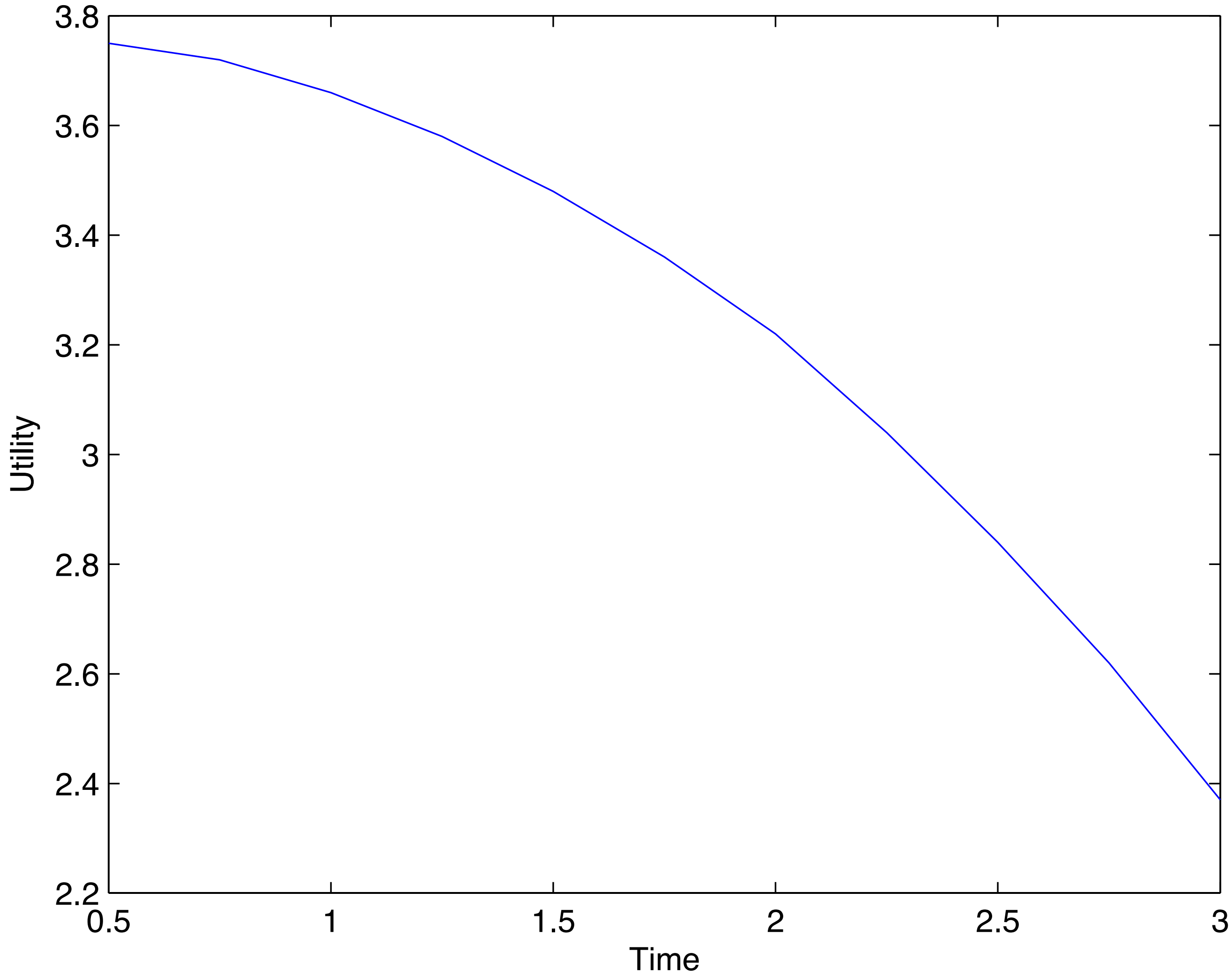
$t_0 = 2$



**Utility function** $u(x_0, t)$

(fixed wealth level)

$x_0 = 3.5$



## Summary on variational utility input

- Key state variables: wealth and risk tolerance

- Risk tolerance solves a fast diffusion equation posed inversely in time

$$\begin{cases} r_t + \frac{1}{2}r^2 r_{xx} = 0 \\ r(x,0) = -\frac{u_0'(x)}{u_0''(x)} \end{cases}$$

- Utility surface generated by a transport equation

$$\begin{cases} u_t + \frac{1}{2}r(x,t)u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

- Forward dynamic utility process constructed by compiling variational utility input and stochastic market input