Old and new examples of scale functions for spectrally negative Lévy processes

Andreas E. Kyprianou[1]

Department of Mathematical Sciences, University of Bath

7th Winter school on Mathematical Finance, January 21-23

[1]based on joint work with Friedrich Hubalek and Victor Rivero
"Old and new examples of scale functions for spectrally negative Lévy processes" (K. and Hubalek - preprint)
"Special, conjugate and complete scale functions for spectrally negative Lévy processes" (K. and Rivero - preprint).

Recollections

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator).

Recollections

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator).
- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

Recollections

- $X = \{X_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process (i.e. $\Pi(0,\infty) = 0$ and $-X$ is not a subordinator).
- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}).$$

- For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0,\infty)$ defined by $W^{(q)}(x) = 0$ for $x < 0$ and otherwise continuous satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large is fundamental to virtually **all** fluctuation identities concerning spectrally negative processes. For example the classical identity

$$\mathbb{E}_x(e^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

for $q \geq 0$, $0 \leq x \leq a$.

Papers which (implicitly) use scale functions

Asmussen, S., Avram, F. and Pistorius, M. **(2004)** Russian and American put options under exponential phase-type Lévy models. *Stochast. Process. Appl.* **109**, 79–111.

Avram, F., Chan, T. and Usabel, M. **(2002)** On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts. *Stochast. Process. Appl.* **100**, 75–107

Avram, F., Kyprianou, A.E. and Pistorius, M.R. **(2004)** Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, 215–235.

Avram, F., Palmowski, Z. and Pistorius, M.R. **(2006)** On the optimal dividend problem for a spectrally negative Lévy process, *Annals of Applied Probability, accepted.*

Baurdoux, E.J. **(2007)** Last exit before an exponential time for spectrally negative Lévy processes. *Preprint.*

Baurdoux, E.J. and Kyprianou A.E. **(2007)** The McKean stochastic game driven by a spectrally negative Lévy process. *Preprint.*

Baurdoux, E.J. and Kyprianou A.E. **(2007)** The Shepp-Shiryaev stochastic game driven by a spectrally negative Lévy process. *Preprint.*

Bertoin, J. (1996b) On the first exit time of a completely asymmetric stable process from a finite interval, *Bull. London Math. Soc.* **28**, 514–520.

Bernyk, V, Dalang, R.C. and Peskir, G. **(2007)** The law of the supremum of a stable Levy process with no negative jumps. *Research Report No. 29* **(2006)**, *Probab. Statist. Group Manchester*

Bertoin, J. (1997b) Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval, *Ann. Appl. Probab.* **7**, 156–169.

Bingham, N.H. (1975) Fluctuation theory in continuous time, *Adv. Appl. Probab.* **7**, 705–766.

- Bingham, N.H. (1976) Continuous branching processes and spectral positivity. *Stochast. Process. Appl.* **4**, 217–24
- Boxma, O. J. and Cohen, J.W.(1998) The $M/G/1$ queue with heavy-tailed service-time distribution. *IEEE Journal on Selected Areas in Communications*. **16**, 749–763.
- Chan, T. **(2004)** Some applications of Lévy processes in insurance and finance. *Finance* **25**, 71–94.
- Chaumont, L. (1994) Sur certains processus de Lévy conditionnés à rester positifs. *Stoch. Stoch. Rep.* **47**, 1–20.
- Chaumont, L. (1996) Conditionings and path decomposition for Lévy processes. *Stochast. Process. Appl.* **64**, 39–54.
- Chiu, S.K. and Yin, C. **(2005)** Passage times for a spectrally negative Lévy process with applications to risk theory. *Bernoulli* **11**, 511–522.
- Doney, R.A. (1991) Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.* **44**, 566–576.
- Doney, R.A. **(2005)** Some excursion calculations for spectrally one-sided Lévy processes. *Séminaire de Probabilités XXXVIII.* 5–15. Lecture Notes in Mathematics, Springer, Berlin Heidelberg New York.
- Doney, R.A. **(2007)** *Ecole d'été de Saint-Flour.* Lecture notes in Mathematics, Springer, Berlin Heidelberg New York.
- Doney, R.A. and Kyprianou, A.E. **(2005)** Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.* **16**, 91–106.
- Dube, P., Guillemin, F. and Mazumdar, R. **(2004)** Scale functions of Lévy processes and busy periods of finite capacity. *J. Appl. Probab.* **41**, 1145–1156.
- Hilberink, B. and Rogers. L.C.G. **(2002)** Optimal capital structure and endogenous default. *Finance and Stochastics*, **6** 237–263.
- Emery, D.J. (1973) Exit problems for a spectrally positive process, *Adv. Appl. Probab.* **5**, 498–520.
- Furrer, H. (1998) Risk processes perturbed by α-stable Lévy motion. *Scandinavian. Actuarial Journal.* **1**, 59–74

Huzak, M., Perman, M., Šikić, H. and Vondracek, Z. **(2004)** Ruin probabilities and decompositions for general perturbed risk processes. *Ann. Appl. Probab.* **14**, 1378–397.

Huzak, M., Perman, M., Šikić, H. and Vondracek, Z. **(2004)** Ruin probabilities for competing claim processes. *J. Appl. Probab.* **41**, 679–90.

Klüppelberg, C., and Kyprianou, A.E. **(2006)** On extreme ruinous behaviour of Lévy insurance risk processes. *J. Appl. Probab.* **43**, 594-598.

Korolyuk, V.S. (1974) Boundary problems for a compound Poisson process. *Theory Probab. Appl.* **19**, 1–14.

Korolyuk V.S. (1975a) *Boundary Problems for Compound Poisson Processes*. Naukova Dumka, Kiev (in Russian).

Korolyuk, V.S. (1975b) On ruin problem for compound Poisson process. *Theory Probab. Appl.* **20**, 374–376.

Korolyuk, V.S. and Borovskich, Ju. V. (1981) *Analytic Problems of the Asymptotic Behaviour of Probability Distributions*. Naukova Dumka, Kiev (in Russian).

Korolyuk, V.S., Suprun, V.N. and Shurenkov, V.M. (1976) Method of potential in boundary problems for processes with independent increments and jumps of the same sign. *Theory Probab. Appl.* **21**, 243–249.

Krell, N. **(2006)** Multifractal spectra and precise rates of decay in homogeneous fragmentations. *Preprint.*

Kyprianou, A.E. and Palmowski, Z. **(2005)** A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Séminaire de Probabilités XXXVIII.* 16–29. Lecture Notes in Mathematics, Springer, Berlin Heidelberg New York.

Kyprianou, A.E. and Surya, B. **(2005)** On the Novikov–Shiryaev optimal stopping problem in continuous time. *Electron. Commun. Probab.* **10**, 146–154.

Kyprianou, A.E. and Surya, B. **(2007)** Principles of smooth and continuous fit in the determination of endogenous bankruptcy levels. *inance and Stochastics* **11**, 131-152.

Lambert, A. **(2000)** Completely asymmetric Lévy processes confined in a finite interval. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, **36**, 251–274.

Lambert, A. **(2007)** Quasi-stationary distributions and the continuous-state branching process conditioned never to be extinct. *Elect. J. Probab.* 420-446.

Loeffen, R. (2007) On optimality of the barrier strategy in de Finettiï■¡ dividend problem for spectrally negative Lévy processes. *Preprint.*

Nguyen-Ngoc, L. and Yor, M. **(2005)** Some martingales associated to reflected Lévy processes. *Séminaire de Probabilités XXXVII.* 42–69. Lecture Notes in Mathematics, Springer, Berlin Heidelberg New York.

Obloj, J. and Pistorius, M.R. **(2007)** On an explicit Skorokhod embedding for spectrally negative Levy processes. http://arxiv.org/abs/math/0703597

Pistorius, M.R. **(2003)** On doubly reflected completely asymmetric Lévy processes, *Stoch. Proc. Appl.*, **107**,131-143.

Pistorius, M.R. **(2004)** On exit and ergodicity of the completely asymmetric Lévy process reflected at its infimum. *J. Theor. Probab.* **17**, 183–220.

Pistorius, M.R. **(2006)** On maxima and ladder processes for a dense class of Lévy processes. *J. Appl. Probab.* **43** no. 1, 208-220.

Pistorius, M.R. **(2006)** An excursion theoretical approach to some boundary crossing problems and the Skorokhod embedding for reflected Lévy processes. *Seminaire de Probabilites, to appear.*

Pistorius, M.R. **(2005)** A potential theoretical review of some exit problems of spectrally negative Lévy processes. *Seminaire de Probabilites* **38**. 30-41.

Renaud, J. and Zhou, X. **(2007)** Distribution of the dividend payments in a general Lévy risk model. *J. Appl. Probab.* **44**, 420-427.

Rogers, L.C.G. (1990) The two-sided exit problem for spectrally positive Lévy processes, *Adv. Appl. Probab.* **22**, 486–48

Rogers, L. C. G. **(2000)** Evaluating first-passage probabilities for processes. *J. Appl. Probab.* **37**, 1173-1180.

Suprun, V.N. (1976) Problem of ruin and resolvent of terminating processes with independent increments. *Ukrainian Math. J.* **28**, 39–45.

Surya, B. A. **(2006)** Evaluating scale functions of spectrally negative Lévy processes. *Preprint.*

Takács, L. (1966) *Combinatorial Methods in the Theory of Stochastic Processes.* Wiley, New York.

Zolotarev, V.M. (1964) The first passage time of a level and the behaviour at infinity for a class of processes with independent increments, *Theory Prob. Appl.* **9**, 653–661.

The problem with scale functions.....

The problem with scale functions.....

Very few tractable examples. (Concentrating henceforth on the case $q = 0$ in which case we shall write W instead of $W^{(q)}$) Examples include:

The problem with scale functions.....

Very few tractable examples. (Concentrating henceforth on the case $q = 0$ in which case we shall write W instead of $W^{(q)}$) Examples include:

- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

The problem with scale functions.....

Very few tractable examples. (Concentrating henceforth on the case $q = 0$ in which case we shall write W instead of $W^{(q)}$) Examples include:

- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

- Variants on this theme can be dealt with when the exponentially distributed jumps are replaced by jump distributions having rational transforms.

The problem with scale functions.....

Very few tractable examples. (Concentrating henceforth on the case $q = 0$ in which case we shall write W instead of $W^{(q)}$) Examples include:

- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

- Variants on this theme can be dealt with when the exponentially distributed jumps are replaced by jump distributions having rational transforms.
- Brownian motion with drift μ.

$$W(x) = \frac{1}{\mu}(1 - e^{-2\mu x})$$

The problem with scale functions.....

Very few tractable examples. (Concentrating henceforth on the case $q = 0$ in which case we shall write W instead of $W^{(q)}$) Examples include:

- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

- Variants on this theme can be dealt with when the exponentially distributed jumps are replaced by jump distributions having rational transforms.
- Brownian motion with drift μ.

$$W(x) = \frac{1}{\mu}(1 - e^{-2\mu x})$$

- α-stable process with $\alpha \in (1, 2)$.

$$W(x) = x^{\alpha-1}/\Gamma(\alpha)$$

Dig a little deeper ....

- Furrer (1998) studies ruin of an α-stable process with $\alpha \in (1,2)$ plus a drift ct and deduces that

$$W(x) = \frac{1}{c}(1 - E_{\alpha-1,1}(-cx^{\alpha-1}))$$

where

$$E_{\alpha-1,1}(z) = \sum_{k\geq 0} z^k/\Gamma(1+(\alpha-1)k)$$

is the two-parameter Mittag-Leffler function with indices $\alpha - 1$ and 1.

Dig a little deeper ....

- Furrer (1998) studies ruin of an α-stable process with $\alpha \in (1,2)$ plus a drift ct and deduces that

$$W(x) = \frac{1}{c}(1 - E_{\alpha-1,1}(-cx^{\alpha-1}))$$

where

$$E_{\alpha-1,1}(z) = \sum_{k\geq 0} z^k/\Gamma(1+(\alpha-1)k)$$

is the two-parameter Mittag-Leffler function with indices $\alpha-1$ and 1.

- An unusual example from queuing theory due to Boxma and Cohen (1998). Let $\eta(x) = e^x \mathrm{erfc}(\sqrt{x})$ and consider a compound Poisson with rate λ satisfying $1-\lambda > 0$, negative jumps with d.f. $F(x,\infty) = (2x+1)\eta(x) - 2\sqrt{x/\pi}$ and unit positive drift. Then

$$W(x) = \frac{1}{1-\lambda}\left(1 - \frac{\lambda}{\nu_1 - \nu_2}(\nu_1\eta(x\nu_2^2) - \nu_2\eta(x\nu_1^2))\right).$$

where $\nu_{1,2} = 1 \pm \sqrt{\lambda}$.

- Asmussen in his book 'Ruin Probabilities' studies a compound Poisson with rate λ, negative jump of fixed size α and positive drift c. Then

$$W(x) = \frac{1}{c} \sum_{n=1}^{\lfloor x/\alpha \rfloor} e^{-\lambda(\alpha n - x)/c} \frac{1}{n!} \left(\frac{\lambda}{c} \right)^n (\alpha n - x)^n$$

- Asmussen in his book 'Ruin Probabilities' studies a compound Poisson with rate λ, negative jump of fixed size α and positive drift c. Then

$$W(x) = \frac{1}{c} \sum_{n=1}^{\lfloor x/\alpha \rfloor} e^{-\lambda(\alpha n - x)/c} \frac{1}{n!} \left(\frac{\lambda}{c}\right)^n (\alpha n - x)^n$$

- Two new scale function occurring in study of Lévy-Lamperti processes [Chaumont, K. and Pardo (2007)]. The Lévy processes in question have unbounded variation processes with no Gaussian component and jump measure which is stable like (with stability parameter $\alpha \in (1,2)$) near the origin and has exponentially decaying tails. Their Laplace exponents are $\Gamma(\theta+\alpha)/[\Gamma(\theta)\Gamma(\alpha)]$ and $\Gamma(\theta-1+\alpha)/[\Gamma(\theta-1)\Gamma(\alpha)]$ and the respective scale functions are

$$W(x) = (1-e^{-x})^{\alpha-1} \text{ and } W(x) = (1-e^{-x})^{\alpha-1} e^{x}.$$

New examples: preliminaries

Henceforth we shall restrict ourselves to discussing the case of 0-scale functions for processes which do not drift to $-\infty$. [But with mild adaptation the the forthcoming methodology works for generating q-scale functions of oscillating LPs or LPs that drift to $-\infty$.]
We make use of two simple facts.

New examples: preliminaries

Henceforth we shall restrict ourselves to discussing the case of 0-scale functions for processes which do not drift to $-\infty$. [But with mild adaptation the the forthcoming methodology works for generating q-scale functions of oscillating LPs or LPs that drift to $-\infty$.]
We make use of two simple facts.

- The definition of the scale function through its LT, integration by parts and the Wiener-Hopf factorization for spec. neg. processes imply that

$$\int_0^\infty e^{-\beta x} W'(x)\, dx = \frac{1}{\phi(\theta)}$$

where ϕ is the Laplace exponent of the descending ladder height process $H = \{H_t : t \geq 0\}$.

New examples: preliminaries

Henceforth we shall restrict ourselves to discussing the case of 0-scale functions for processes which do not drift to $-\infty$. [But with mild adaptation the the forthcoming methodology works for generating q-scale functions of oscillating LPs or LPs that drift to $-\infty$.]
We make use of two simple facts.

- The definition of the scale function through its LT, integration by parts and the Wiener-Hopf factorization for spec. neg. processes imply that
$$\int_0^\infty e^{-\beta x} W'(x)\,dx = \frac{1}{\phi(\theta)}$$
where ϕ is the Laplace exponent of the descending ladder height process $H = \{H_t : t \geq 0\}$.
- $$\int_0^\infty dt \cdot \mathbb{P}(H_t \in dx) = W'(x)dx.$$

A simple idea for generating scale functions

- Pick your favouite subordinator H or equivalently Laplace exponent ϕ for which one knows its potential density OR can invert the Laplace transform $1/\phi(\theta)$.

A simple idea for generating scale functions

- Pick your favouite subordinator H or equivalently Laplace exponent ϕ for which one knows its potential density OR can invert the Laplace transform $1/\phi(\theta)$.
- But then you need to know that a Lévy process exists for which your favourite H corresponds to its descending ladder height process.

A simple idea for generating scale functions

- Pick your favouite subordinator H or equivalently Laplace exponent ϕ for which one knows its potential density OR can invert the Laplace transform $1/\phi(\theta)$.
- But then you need to know that a Lévy process exists for which your favourite H corresponds to its descending ladder height process.
- No problem - thanks to the Wiener-Hopf factorisation!

Parent process for given H

Suppose that H is a (killed) subordinator with Laplace exponent

$$\phi(\theta) = \kappa + \zeta\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon(dx)$$

such that Υ is absolutely continuous with monotone non-increasing density. Then there exists a spectrally negative Lévy process X, henceforth referred to as the **parent process**, such height process is precisely the process H. The Lévy triple (a, σ, Π) of the parent process is uniquely identified as follows. The Gaussian coefficient is given by $\sigma = \sqrt{2\zeta}$. The Lévy measure is given by

$$\Pi(-\infty, -x) = \frac{d\Upsilon(x)}{dx}. \quad (1)$$

Finally

$$a = \int_{(-\infty,-1)} x\Pi(dx) - \kappa. \quad (2)$$

Bounded and unbounded variation

- **When** $\Upsilon(0,\infty) < \infty$. The parent process is given by

$$X_t = (\kappa + \Upsilon(0,\infty))t + \sqrt{2\zeta} B_t - S_t \qquad (3)$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion, $S = \{S_t : t \geq 0\}$ is an indpendent driftless subordinator with jump measure ν satisfying

$$\nu(x,\infty) = \frac{d\Upsilon}{dx}(x).$$

Bounded and unbounded variation

- **When** $\Upsilon(0,\infty) < \infty$. The parent process is given by

$$X_t = (\kappa + \Upsilon(0,\infty))t + \sqrt{2\zeta}B_t - S_t \tag{3}$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion, $S = \{S_t : t \geq 0\}$ is an indpendent driftless subordinator with jump measure ν satisfying

$$\nu(x,\infty) = \frac{d\Upsilon}{dx}(x).$$

- **When** $\Upsilon(0,\infty) = \infty$. The parent process X always has paths of unbounded variation.

Tempered stable descending ladder height

Introduce a new family of scale functions called **Gaussian-Tempered-Stable-Convolution class (GTSC)**. Choose a descending ladder height process H to have Laplace exponent:

$$\phi(\theta) = \kappa + \zeta\theta + c\Gamma(-\alpha)(\gamma^\alpha - (\gamma + \theta)^\alpha)$$

The associated Lévy measure is given by

$$\Upsilon(dx) = cx^{-\alpha-1}e^{-\gamma x}dx \qquad (x > 0).$$

Here the parameter regimes (partly to respect conditions on Υ) are:

$$\kappa \geq 0,$$
$$\zeta \geq 0$$
$$c > 0,$$
$$\alpha \in [-1, 1)$$
$$\gamma > 0.$$

GTSC parent processes

The parent process with the tempered stable choice of H is characterized as follows:

$$\Pi(dx) = c\frac{\gamma}{|x|^{\alpha+1}}e^{-\gamma|x|}dx + c\frac{(\alpha+1)}{|x|^{\alpha+2}}e^{-\gamma|x|}dx \text{ for } x<0$$

ie jump part is the result of the independent sum of a tempered stable subordinator and another spectrally negative Lévy process, also belonging to the class of (generalized) tempered stable processes, but whose jump component has unbounded variation.

$$\sigma = \sqrt{2\zeta}$$

Instead of giving a we give the Laplace exponent of parent process

$$\psi(\theta) = \kappa\theta + \zeta\theta^2 + c\theta\Gamma(-\alpha)(\gamma^\alpha - (\gamma+\theta)^\alpha).$$

Take $\alpha = m/n \in \cap(0,1)$

Let $f(z) = \psi(z^n - \gamma)$. Because of the structure of the Laplace exponent a little algebra shows that

$$f(z) = c_0 + c_m z^m + c_n z^n + c_{m+n} z^{m+n} + c_{2n} z^{2n},$$

where

$$\begin{aligned} c_{2n} &= \zeta, \\ c_{n+m} &= -c\Gamma(-\alpha) \\ c_n &= \kappa - 2\zeta\gamma + c\Gamma(-\alpha)]\gamma^\alpha, \\ c_m &= c\Gamma(-\alpha)\gamma, \\ c_0 &= \zeta\gamma^2 - \gamma(\kappa - \zeta) - c\Gamma(-\alpha)\gamma^{\alpha+1}. \end{aligned}$$

Denote by $z_1, \ldots, z_\ell$ the distinct roots of $f(z)$ and by $m_1, \ldots, m_\ell$ their multiplicities and for $k = 1, 2, ..., \ell$ and $j = 1, ..., m_k$ let

$$A_{kj} = \frac{1}{(m_k - j)!} \frac{d^{m_k - j}}{dz^{m_k - j}} \left[\frac{(z - z_k)^{m_k}}{f(z)} \right]_{z = z_k}$$

Using Laplace inversion:

$$W(x) = \sum_{k=1}^{\ell} \sum_{j=0}^{m_k-1} A_{kj} \frac{1}{j!} e^{-\gamma x} x^{(j+1)/n-1} E^{(j)}_{\frac{1}{n},\frac{1}{n}}(z_k x^{\frac{1}{n}})$$

where

$$E^{(j)}_{\alpha,\beta}(x) = \frac{\partial^j}{\partial x^j} E_{\alpha,\beta}(x) = \sum_{n \geq k} \binom{n}{k} k! \frac{x^{n-k}}{\Gamma(n\alpha + \beta)}$$

and

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta + \alpha n)}$$

is the two-parameter Mittag-Leffler function.

Can perform similar analysis for the case $\alpha = -m/n \in [-1, 0)$

Special cases

- Take H as an inverse Gaussian subordinator: $\alpha = \frac{1}{2}, \kappa = \zeta = 0$, $\phi(\theta) = \delta((2\theta + \gamma^2)^{\frac{1}{2}} - \gamma)$ so that $\sigma = 0$ and

$$\Pi(dx) = \frac{3\delta}{2\sqrt{2\pi}}|x|^{-\frac{5}{2}} e^{-\frac{1}{2}\gamma^2|x|} + \frac{(\gamma^2/2 + \varphi)\delta}{\sqrt{2\pi}}|x|^{-\frac{3}{2}} e^{-\frac{1}{2}\gamma^2|x|} \text{ for } x < 0.$$

$$W(x) = \frac{1}{2\delta\gamma}\left[(1 + \gamma^2 x)\mathrm{erfc}(-\gamma\sqrt{x/2}) + \gamma\sqrt{2x/\pi}e^{-\gamma^2/2x} - 1\right]$$

Special cases

- Take H as an inverse Gaussian subordinator: $\alpha = \frac{1}{2}, \kappa = \zeta = 0$, $\phi(\theta) = \delta((2\theta + \gamma^2)^{\frac{1}{2}} - \gamma)$ so that $\sigma = 0$ and

$$\Pi(dx) = \frac{3\delta}{2\sqrt{2\pi}}|x|^{-\frac{5}{2}} e^{-\frac{1}{2}\gamma^2|x|} + \frac{(\gamma^2/2 + \varphi)\delta}{\sqrt{2\pi}}|x|^{-\frac{3}{2}} e^{-\frac{1}{2}\gamma^2|x|} \text{ for } x < 0.$$

$$W(x) = \frac{1}{2\delta\gamma}\left[(1+\gamma^2 x)\mathrm{erfc}(-\gamma\sqrt{x/2}) + \gamma\sqrt{2x/\pi}e^{-\gamma^2/2x} - 1\right]$$

- Take $\alpha \in (0,1), \kappa = \zeta = 0$: $\phi(\theta) = c\Gamma(-\alpha)(\gamma^\alpha - (\gamma+\theta)^\alpha)$ so that $\sigma = 0$ and

$$\Pi(dx) = c\frac{\gamma}{|x|^{\alpha+1}}e^{-\gamma|x|}dx + c\frac{(\alpha+1)}{|x|^{\alpha+2}}e^{-\gamma|x|}dx \text{ for } x < 0$$

$$W(x) = \int_0^x e^{-\gamma x} y^{\alpha-1} E_{\alpha,\alpha}(\gamma^\alpha y^\alpha)\,dy.$$

Computing potential density:

- Take H to be a Gamma process: $\alpha = \kappa = \zeta = 0$, $\phi(\theta) = -c\log(\gamma/(\gamma+\theta))$, so $\sigma = 0$

$$\Pi(dx) = c|x|^{-2}e^{-\gamma|x|}dx + c\gamma|x|^{-1}e^{-\gamma|x|}dx \text{ for } x < 0$$

$$W(x) = \int_0^{\gamma x} e^{-y} \int_0^{\infty} \frac{1}{\Gamma(ct)} y^{ct-1} dt dy$$

Computing potential density:

- Take H to be a Gamma process: $\alpha = \kappa = \zeta = 0$, $\phi(\theta) = -c\log(\gamma/(\gamma+\theta))$, so $\sigma = 0$

$$\Pi(dx) = c|x|^{-2}e^{-\gamma|x|}dx + c\gamma|x|^{-1}e^{-\gamma|x|}dx \text{ for } x < 0$$

$$W(x) = \int_0^{\gamma x} e^{-y} \int_0^{\infty} \frac{1}{\Gamma(ct)} y^{ct-1} dt dy$$

- Take H to be a compound Poisson with rate λ and gamma distributed jumps with parameters $\nu \in (0,1)$ and $\gamma > 0$, killed at rate $\kappa \geq 0$. Let $\rho = \lambda/(\lambda+\kappa)$

$$\Pi(dx) = \frac{\lambda(1-\nu)\gamma^{\nu}}{\Gamma(\nu)}|x|^{\nu-2}e^{-\gamma|x|}dx + \frac{\lambda\gamma^{\nu+1}}{\Gamma(\nu)}|x|^{\nu-1}e^{-\gamma|x|}dx \text{ for } x < 0$$

$$W(x) = \frac{1}{\lambda+\lambda} + \frac{\rho\gamma^{\nu}}{\lambda+\kappa}\int_0^x y^{\nu-1}e^{-\gamma y}E_{\nu,\nu}(\rho\gamma^{\nu}y^{\nu})dy.$$

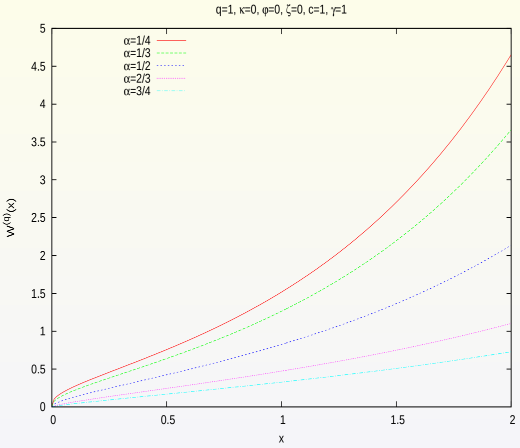

Figure: Scale functions $W^{(q)}(x)$ for a GTSC which oscillates: concavity/convexity proved by Ronnie Loeffen.

Special and conjugate scale functions

Instead of taking a tempered stable descending ladder height, take a **special Bernstein function**. That is to say, choose the Laplace exponent of the descending ladder height ϕ such that

$$\phi(\theta) = \kappa + \zeta\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon(dx) \text{ for } \theta \geq 0$$

with the assumption that Υ is absolutely continuous with a decreasing density and such that

$$\phi(\theta) = \frac{\theta}{\phi^*(\theta)} \text{ for } \theta \geq 0$$

where ϕ^* is also a Bernstein function (the conjugate to ϕ) which we shall write as

$$\phi^*(\theta) = \kappa^* + \zeta^*\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Upsilon^*(dx).$$

Special and conjugate scale functions ctd.

- Potential analysis of special Bernstein functions gives us an expression for the potential function associated to ϕ and hence an expression for the the **special scale function** whose parent process has Laplace exponent $\psi(\theta) = \theta\phi(\theta)$:

$$W(x) = \zeta^* + \kappa^* x + \int_0^x \Upsilon^*(y,\infty)\,dy$$

and W is a concave function.

Special and conjugate scale functions ctd.

- Potential analysis of special Bernstein functions gives us an expression for the potential function associated to ϕ and hence an expression for the the **special scale function** whose parent process has Laplace exponent $\psi(\theta) = \theta\phi(\theta)$:

$$W(x) = \zeta^* + \kappa^* x + \int_0^x \Upsilon^*(y,\infty)dy$$

  and W is a concave function.

- If it so happens that Υ^* is absolutely continuous with non-increasing density, then we get the **conjugate scale function**

$$W^*(x) = \zeta + \kappa x + \int_0^x \Upsilon(y,\infty)dy.$$

  (also concave) whose parent process has Laplace exponent $\psi^*(\theta) = \theta\phi^*(\theta)$

Complete scale functions

- In particular if ϕ is a **complete Bernstein function** (ie it is a special Bernstein function such that Υ is absolutely continuous with a completely monotone density) then ϕ^* is completely monotone and both Υ and Υ^* have non-increasing densities.

Complete scale functions

- In particular if ϕ is a **complete Bernstein function** (ie it is a special Bernstein function such that Υ is absolutely continuous with a completely monotone density) then ϕ^* is completely monotone and both Υ and Υ^* have non-increasing densities.
- Since most known examples of special Bernstein functions are complete Bernstein functions, the latter remarks suggest that for each known example of pairs of conjugate complete Bernstein functions one gets for free examples of (conjugate) parent processes and their scale functions.

Example

- Take $\phi(\theta) = a\theta^{\beta-\alpha} + b\theta^{\beta}$ where $0 < \alpha < \beta \leq 1$ and so parent process has Laplace exponent

$$\psi(\theta) = a\theta^{\beta-\alpha+1} + b\theta^{\beta+1}, \ \theta \geq 0.$$

which is the sum of two independent spectrally negative stable processes.

Example

- Take $\phi(\theta) = a\theta^{\beta-\alpha} + b\theta^{\beta}$ where $0 < \alpha < \beta \leq 1$ and so parent process has Laplace exponent

$$\psi(\theta) = a\theta^{\beta-\alpha+1} + b\theta^{\beta+1},\ \theta \geq 0.$$

  which is the sum of two independent spectrally negative stable processes.

- The conjugate parent process has Laplace exponent

$$\psi^*(\theta) = \frac{\theta^2}{a\theta^{\beta-\alpha} + b\theta^{\beta}}$$

  which is an oscillating process, has no Gaussian component, and has Lévy measure given by

$$\Pi^*(-\infty, -x) = \frac{d^2}{dx^2}\left[\frac{1}{b}x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha}/b)\right]$$

- Finally the scale functions are given by

$$W(x) = \frac{1}{b}\int_0^x t^{\beta-1} E_{\alpha,\beta}(-at^{\alpha}/b)\,dt$$

and

$$W^*(x) = \frac{a}{\Gamma(2-\beta+\alpha)} x^{1-\beta+\alpha} + \frac{b}{\Gamma(2-\beta)} x^{1-\beta}, \ x \geq 0.$$

- Finally the scale functions are given by

$$W(x) = \frac{1}{b}\int_0^x t^{\beta-1}E_{\alpha,\beta}(-at^\alpha/b)\,dt$$

and

$$W^*(x) = \frac{a}{\Gamma(2-\beta+\alpha)}x^{1-\beta+\alpha} + \frac{b}{\Gamma(2-\beta)}x^{1-\beta}, \ \ x \geq 0.$$

- Note also that scale functions are continuous in the parameters of ψ and ψ^* and hence we may vary the parameters a, b, α, β to their extremes to get other examples. Eg

$$W^*(x) = b + \frac{a}{\Gamma(1-\alpha)}x^\alpha$$

is a scale function!

Moral of the story:

The Wiener-Hopf factorisation gives us a route for chanelling potential theory of subordinators into the theory of scale functions for spectrally negative Lévy processes.