Part I: Unconditional and Conditional Risk Functionals

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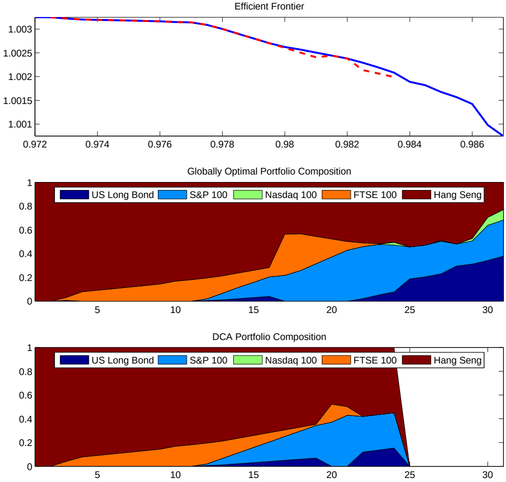
Why to measure risk/acceptability

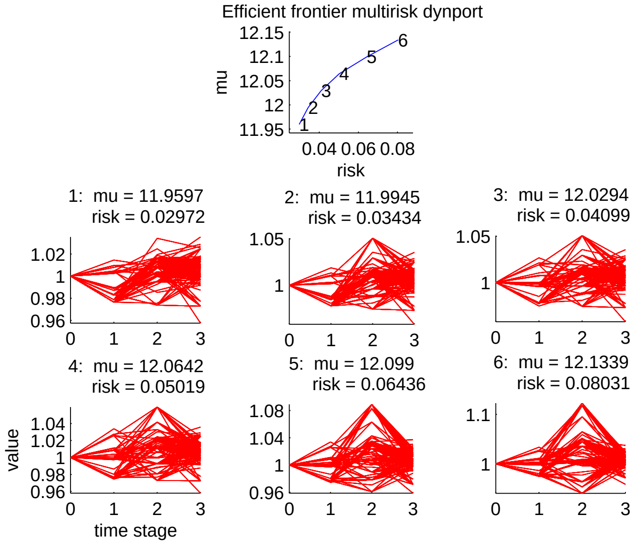
We aim at assigning a numerical value to the "risk" of a random variable Y, or -more generally- to a stochastic process $(Y_1, \ldots, Y_T)$ for the following purposes

- to define acceptable profit&loss distributions or profit&loss processes (from the standpoint of regulators or supervisors);
- to make possible a comparison between alternatives w.r.t. riskiness (for decision makers);
- to define objectives or constraints in financial optimization problems, which are meaningful and lead to low complexity algorithms.

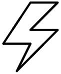
Example: static portfolio management



Example: dynamic portfolio management



Notations

- Let $(Y_\gamma)_{\gamma\in\Gamma}$ be a family of functions in $L_p(\Omega, \mathcal{F}, P), 1 \le p \le \infty$, which is bounded from below. Since the L_p spaces are order complete Banach lattices, there exists a function $\underline{Y} = \mathbf{inf}\{Y_\gamma : \gamma \in \Gamma\}$, called the infimum (or sometimes the essential infimum), with the properties
  - $\underline{Y} \le Y_\gamma$ a.s. for all $\gamma \in \Gamma$
  - if $Z \le Y_\gamma$ a.s. for all $\gamma \in \Gamma$, then $Z \le \underline{Y}$ a.s.
- $Y \lhd \mathcal{F}$ stands for: Y is measurable w.r.t. the σ-algebra $\mathcal{F}$, $(Y_1, \dots, Y_T) \lhd \boldsymbol{\mathcal{F}}$ stands for: The process $(Y_1, \dots, Y_T)$ is adapted to the filtration $\boldsymbol{\mathcal{F}} = (\mathcal{F}_1, \dots, \mathcal{F}_T)$.
-  denotes a counterintuitive fact or a counterexample

What is a risk/acceptability measure?

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A mapping $\mathcal{A} : \mathcal{Y} = L_p(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}}$ is called *acceptability functional* (negative risk functional) if it satisfies the following conditions for all $Y,\ \tilde{Y} \in \mathcal{Y}$, $c \in \mathbb{R}$, $\lambda \in [0,1]$:

(A1) $\mathcal{A}(Y + c) = \mathcal{A}(Y) + c$ (*translation-equivariance*, cash-invariance), or more generally

(A1') There is a linear subspace $\mathcal{W} \subseteq L_p$ and a function $Z^* \in L_q(\mathcal{F})$ $(1/p + 1/q = 1)$ such that for $W \in \mathcal{W}$

$$\mathcal{A}(Y+W) = \mathcal{A}(Y) + \mathbb{E}(W\,Z^*), \quad \textit{(the } (\mathcal{W}, Z^*) \textit{ translation property)}.$$

(A2) $\mathcal{A}(\lambda Y + (1-\lambda)\tilde{Y}) \geq \lambda \mathcal{A}(Y) + (1-\lambda)\mathcal{A}(\tilde{Y})$ (*concavity*),

(A3) $Y \leq \tilde{Y}$ implies $\mathcal{A}(Y) \leq \mathcal{A}(\tilde{Y})$ (*monotonicity*).

An acceptability functional $\mathcal{A}$ is called

- *positively homogeneous* if

$$\mathcal{A}(\lambda Y) = \lambda \mathcal{A}(Y), \quad \forall \lambda \geq 0,\ Y \in \mathcal{Y}$$

- *version-independent* (law-invariant) if

  $\mathcal{A}(Y)$ depends only on the distribution function $G_Y(u) = \mathbb{P}\{Y \leq u\}$

$$\begin{array}{ccc} Y & \longrightarrow & G_Y \\ & \searrow & \downarrow \\ & & \mathcal{A}(Y) = \mathcal{A}\{G_Y\} \end{array}$$

Given an acceptability functional $\mathcal{A}$, the mappings

$$\rho := -\mathcal{A} \quad \text{and} \quad \mathcal{D} := \mathbb{E} - \mathcal{A}$$

are called *risk functional* and *deviation risk functional*, respectively. *Coherent* risk functionals are negative acceptability functionals which are positively homogeneous in addition. (Artzner et al., 1999).

By the Fenchel-Moreau Theorem, every concave upper semicontinuous (u.s.c.) functional $\mathcal{A}$ on $\mathcal{Y}$ has a representation of the form

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{A}^+(Z) : Z \in L_q\}, \qquad (1)$$

where $\mathcal{A}^+(Z) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{A}(Y) : Y \in \mathcal{Y}\}$ is the *conjugate* of $\mathcal{A}$. We call (1) a *dual representation* and $dom(\mathcal{A}^+) = \{Z : \mathcal{A}^+(Z) > -\infty\}$ the *set of supergradients*. Notice that if Z is a supergradient,

$$\mathcal{A}(Y) \le \mathbb{E}(Y\,Z) - \mathcal{A}^+(Z),$$

i.e. the affine-linear functional $Y \mapsto \mathbb{E}(Y\,Z) - \mathcal{A}^+(Z)$ is a majorant of $\mathcal{A}$. The Fenchel-Moreau inequality

$$\mathbb{E}(Y\,Z) \ge \mathcal{A}(Y) + \mathcal{A}^+(Z) \qquad (2)$$

follows.

The *supergradient of* $\mathcal{A}$ *at* $Y_0 \in dom(\mathcal{A})$ is

$$\partial\mathcal{A}(Y_0) = \{Z \in L_q : \mathbb{E}(Y_0 Z) = \mathcal{A}(Y_0) + \mathcal{A}^+(Z)\}$$

i.e. the Z, which fulfill the Fenchel-Moreau inequality (2) as equality. For $Z \in \partial\mathcal{A}(Y_0)$,

$$\mathcal{A}(Y) \le \mathbb{E}(Y Z) - \mathbb{E}(Y_0 Z) + \mathcal{A}(Y_0) = \mathcal{A}(Y_0) + \mathbb{E}[(Y - Y_0)Z].$$

Remark. If $Y \in L_\infty$, the dual representation looks like

$$\mathcal{A}(Y) = \inf\{\mathbb{E}_Q(Y) - f(Q) : Q \in \mathcal{Q}\}.$$

Properties of $\mathcal{A}$ follow from the dual representation

- $\mathcal{A}$ is monotonic, iff $dom(\mathcal{A}^+) \subseteq L_q^+$
- $\mathcal{A}$ is positively homogeneous, iff $\mathcal{A}^+$ takes only the values 0 and $-\infty$
- $\mathcal{A}$ has the $(\mathcal{W}, Z^*)$ translation property (A1'), iff $dom(\mathcal{A}^+) \subseteq \mathcal{W}^\perp + Z^*$
- In particular, $\mathcal{A}$ is translation-equivariant, iff all $Z \in dom(\mathcal{A}^+)$ satisfy $\mathbb{E}(Z) = 1$.

We prove here the last assertion, i.e. the fact that
the $(\mathcal{W}, Z^*)$-translation property is equivalent to the property, that for $W \in \mathcal{W}$, $\mathcal{A}(W) = \mathbb{E}(WZ^*)$.
Suppose that $Z \in dom(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the conjugate functional on L_q

$$\mathcal{A}^+(Z) = \inf\{\mathbb{E}(Z, Y) - \mathcal{A}(Y) : Y \in L_p\}.$$

This means that Z is a supergradient of $\mathcal{A}$

$$\mathcal{A}(Y) \leq \mathbb{E}(YZ) - \mathcal{A}^+(Z).$$

For $W \in \mathcal{W}$, this implies that
$\mathbb{E}(WZ^*) = \mathcal{A}(W) \leq \mathbb{E}(WZ) - \mathcal{A}^+(Z)$. This inequality can hold for all $W \in \mathcal{W}$ only if $\mathbb{E}(WZ) = \mathbb{E}(WZ^*)$ for all $Z \in dom(\mathcal{A}^+)$, i.e. $dom(\mathcal{A}^+) \subseteq \mathcal{W}^\perp + Z^*$. Then

$$\begin{aligned}
\mathcal{A}(Y + W) &= \inf\{\mathbb{E}[(Y + W)Z] - \mathcal{A}^+(Z) : Z \in L_q\} \\
&= \inf\{\mathbb{E}[YZ] + \mathbb{E}[WZ^*] - \mathcal{A}^+(Z) : Z \in L_q\} \\
&= \mathcal{A}(Y) + \mathbb{E}(WZ^*).
\end{aligned}$$

Examples for supergradients

- $$\partial[-\|Y\|_p] = -\text{sgn }(Y)|Y|^{p-1}\|Y\|_p^{p-1}.$$
- $$\partial[-\|Y\|_p^p] = -\text{sgn }(Y)p|Y|^{p-1}.$$
- $$\partial[-\|[Y]_-\|_p] = \mathbb{1}_{\{Y<0\}}|Y|^{p-1}\|Y\|_p^{p-1}.$$
- $$\partial[-\|[Y]_-\|_p^p] = \mathbb{1}_{\{Y<0\}}p|Y|^{p-1}.$$

Here $[Y]_- = -\min(Y,0)$.

Order relations

Definition: Orderings (Fishburn (1980).
Let $Y^{(1)}, Y^{(2)}$ be profit&loss variables, not necessarily defined on the same probability space.

(i) $Y^{(2)}$ dominates $Y^{(1)}$ in the first order sense (in symbol $Y^{(1)} \prec_{FSD} Y^{(2)}$, if

$$\mathbb{E}[U(Y^{(1)})] \leq \mathbb{E}[U(Y^{(2)})]$$

for all nondecreasing utility functions U, for which both integrals exist.

(ii) $Y^{(2)}$ dominates $Y^{(1)}$ in the second order sense (in symbol $Y^{(1)} \prec_{SSD} Y^{(2)}$, if

$$\mathbb{E}[U(Y^{(1)})] \leq \mathbb{E}[U(Y^{(2)})]$$

for all nondecreasing concave U, for which both integrals exist.

Trivially

$$Y^{(1)} \prec_{FSD} Y^{(2)} \quad \text{implies that} \quad Y^{(1)} \prec_{SSD} Y^{(2)}.$$

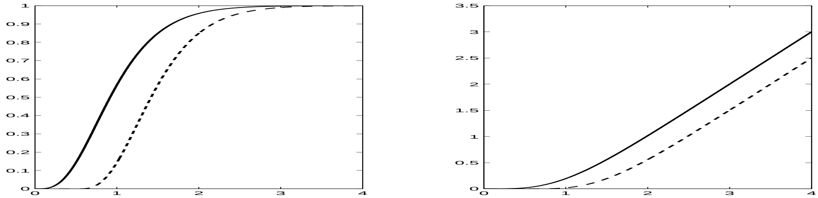


Figure: Left: the distribution functions G_1 of $Y^{(1)}$ (solid) and G_2 of $Y^{(2)}$ (dashed) Right:the integrated distribution functions $\mathcal{G}_1(x) = \int_{\infty}^{x} G_1(t)\,dt$ of $Y^{(1)}$ (solid) and $\mathcal{G}_2$ of $Y^{(2)}$ (dashed); The relation $Y^{(1)} \prec_{FSD} Y^{(2)}$ holds.

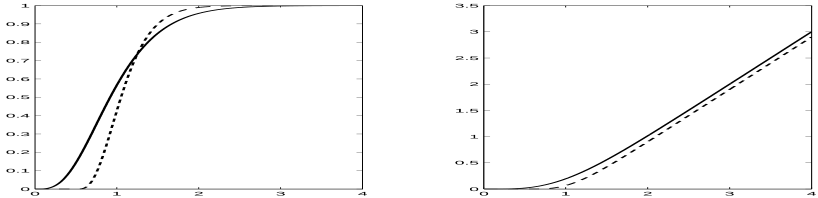


Figure: $Y^{(1)} \prec_{SSD} Y^{(3)}$ holds, but $Y^{(1)} \prec_{FSD} Y^{(3)}$ does not hold.

Coupling

(i) The FSD-coupling: If $Y^{(1)} \prec_{FSD} Y^{(2)}$, then one may construct a pair $\tilde{Y}^{(1)}, \tilde{Y}^{(2)}$ of random variables with the same marginal distributions as $Y^{(1)}, Y^{(2)}$, such that

$$\tilde{Y}^{(1)} \leq \tilde{Y}^{(2)} \qquad a.s.$$

(iii) The SSD-coupling. If $Y^{(1)} \prec_{SSD} Y^{(2)}$, then one may construct a pair $\tilde{Y}^{(1)}, \tilde{Y}^{(2)}$ of random variables with the same marginal distributions as $Y^{(1)}, Y^{(2)}$, such that

$$\tilde{Y}^{(2)} \geq \mathbb{E}(\tilde{Y}^{(1)} | \tilde{Y}^{(2)}) \qquad a.s.$$

(Strassen 1965, Stoyan and Mueller 2002).

Definition. The version-independent functional $\mathcal{A}$ is called *monotonic w.r.t. to FSD*, if

$$Y^{(1)} \prec_{FSD} Y^{(2)} \quad \text{implies that} \quad \mathcal{A}(Y^{(1)}) \le \mathcal{A}(Y^{(2)}).$$

It is called *monotonic w.r.t to SSD*, if

$$Y^{(1)} \prec_{SSD} Y^{(2)} \quad \text{implies that} \quad \mathcal{A}(Y^{(1)}) \le \mathcal{A}(Y^{(2)})$$

Remark. SSD-monotonicity implies FSD-monotonicity.

Comonotone coupling and compounding

$Y^{(1)}$ and $Y^{(2)}$ are *comonotone coupled*, if

$$\mathbb{P}\{Y^{(1)} \le u, Y^{(2)} \le v\} = \min(G_1(u), G_2(v)).$$

(the copula is the upper Fréchet bound).
The *compound* of $Y^{(1)}$ and $Y^{(2)}$ is

$$\mathcal{C}(Y^{(1)}, Y^{(2)}, \lambda) = \begin{cases} Y^{(1)} & \text{w.pr. } \lambda \\ Y^{(2)} & \text{w.pr. } 1-\lambda \end{cases}$$

The distribution function of $\mathcal{C}(Y^{(1)}, Y^{(2)}, \lambda)$ is

$$\lambda G_1(u) + (1-\lambda) G_2(u).$$

More generally, let $K(\cdot|u)$ be a Markov kernel (d.f.) and $G(u)$ be a further distribution function. The *compound distribution* function $K \circ G$ is defined as

$$(K \circ G)(v) = \int K(v|u)\, dG(u).$$

Notice the difference: The compound variable $\mathcal{C}(Y^{(1)}, Y^{(2)}, 1/2)$ has distribution

$$\frac{1}{2}G_1(u) + \frac{1}{2}G_2(u).$$

The comonotone average of $Y^{(1)}$ and $Y^{(2)}$, has quantile function

$$\frac{1}{2}G_1^{-1}(p) + \frac{1}{2}G_1^{-1}(p),$$

Definition. The functional $\mathcal{A}$ is comonotone additive, if

$$\mathcal{A}(Y^{(1)} + Y^{(2)}) = \mathcal{A}(Y^{(1)}) + \mathcal{A}(Y^{(2)})$$

for comonotone $Y^{(1)}, Y^{(2)}$.

Definition. The version-independent functional $\mathcal{A}$ is compound convex, if

$$\mathcal{A}(K \circ G) \leq \int \mathcal{A}(K(v|u))\, dG(u).$$

Conditional risk functionals

Let $\mathcal{F}_1$ be a σ-field contained in $\mathcal{F}$. A mapping $\mathcal{A}_{\mathcal{F}_1} : L_p(\mathcal{F}) \to L_{p'}(\mathcal{F}_1)$ is called *conditional acceptability mapping* (with observable information $\mathcal{F}_1$) if the following conditions are satisfied for all Y, $\lambda \in [0,1]$:

(CA1) $\mathcal{A}_{\mathcal{F}_1}(Y + Y^{(1)}) = \mathcal{A}_{\mathcal{F}_1}(Y) + Y^{(1)}$ (*predictable translation-equivariance*),

(CA2) $\mathcal{A}_{\mathcal{F}_1}(\lambda Y + (1-\lambda)\tilde{Y}) \geq \lambda \mathcal{A}_{\mathcal{F}_1}(Y) + (1-\lambda)\mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$ *(*concavity*),*

(CA3) $Y \leq \tilde{Y}$ *implies* $\mathcal{A}_{\mathcal{F}_1}(Y) \leq \mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$ *(*monotonicity*).*

We write $\mathcal{A}_{\mathcal{F}_1}(\cdot)$ or $\mathcal{A}(\cdot|\mathcal{F}_1)$, whatever is more convenient.

Theorem. A mapping $\mathcal{A}_{\mathcal{F}_1}$ is a conditional acceptability mapping if and only if for all $B \in \mathcal{F}_1$ the functional $Y \mapsto \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B)$ is an acceptability functional, which has the $(L_p(\mathcal{F}_1), \mathbb{1}_B)$ translation property, that is

$$\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y + Y^{(1)})\mathbb{1}_B) = \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) + \mathbb{E}(Y^{(1)}\mathbb{1}_B)$$

for all $Y^{(1)} \in L_p(\mathcal{F}_1)$.
One may apply the Theory of concave mappings with values in Banach lattices. The extension of the Fenchel Moreau Theorem to $L_{p'}$-valued functionals leads to a representation of the form

$$\mathcal{A}_{\mathcal{F}_1}(Y) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) - \mathcal{A}^+_{\mathcal{F}_1}(Z) : Z \in \mathcal{Z}\},$$

(PhD Thesis of R. Kovacevic).

Example: The entropic functional

- *Primal form*

$$\mathcal{A}(Y) = -\frac{1}{\gamma} \log \mathbb{E}[\exp(-\gamma Y)].$$

- *Dual form*

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + \frac{1}{\gamma}\mathbb{E}(Z \log Z) : \mathbb{E}(Z) = 1, Z \geq 0\}.$$

- *Conditional form*

$$\mathcal{A}(Y|\mathcal{F}_1) = -\frac{1}{\gamma} \log \mathbb{E}[\exp(-\gamma Y)|\mathcal{F}_1].$$

- *Dual conditional form*

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \frac{1}{\gamma}\mathbb{E}(Z \log Z|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1, Z \geq 0\}$$

Example: The average value-at-risk

- *Primal form.* $\mathbb{AV@R}_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha G_Y^{-1}(p)\,dp$

$$\mathbb{AV@R}_0(Y) = \mathrm{ess-}\inf(Y).$$

- *Dual form*

$$\mathbb{AV@R}_\alpha(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(Z) = 1, 0 \le Z \le 1/\alpha\}.$$

- *Conditional form*

$$\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1) = \mathbf{sup}\{X - \frac{1}{\alpha}\mathbb{E}([Y-X]_-) : X \lhd \mathcal{F}_1\}.$$

- *Dual conditional form*

$$\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1, 0 \le Z \le 1/\alpha\}.$$

Other names for this functional: *conditional value-at-risk* (Rockefellar and Uryasev (2002)), *expected shortfall* (Acerbi and Tasche (2002)) and *tail value-at-risk* (Artzner et al. (1999)). The name average value-at-risk is due to Föllmer and Schied (2004).

More about the conditional average value-at-risk

The conditional $\mathbb{A}\mathrm{V@R}_\alpha(Y|\mathcal{F}_1)$ has the following properties:

(i) It is version-independent and maps $L_1(\mathcal{F})$ to $L_1(\mathcal{F}_1)$.

(ii) It is concave in the following sense: For any $\mathcal{F}_1$ measurable Λ, $0 \le \Lambda \le 1$,

$$\begin{aligned} & \mathbb{A}\mathrm{V@R}_\alpha(\Lambda\, Y^{(1)} + (1-\Lambda)Y^{(2)}|\mathcal{F}_1) \\ \geq\ & \Lambda\, \mathbb{A}\mathrm{V@R}_\alpha(Y^{(1)}|\mathcal{F}_1) + (1-\Lambda)\mathbb{A}\mathrm{V@R}_\alpha(Y^{(2)}|\mathcal{F}_1). \end{aligned}$$

(iii) For any non-negative, bounded $\mathcal{F}_1$-measurable Λ

$$\mathbb{A}\mathrm{V@R}_\alpha(\Lambda\, Y|\mathcal{F}_1) = \Lambda\, \mathbb{A}\mathrm{V@R}_\alpha(Y|\mathcal{F}_1).$$

(iv) If $Y^{(1)} \le Y^{(2)}$, then $\mathbb{A}\mathrm{V@R}_\alpha(Y^{(1)}|\mathcal{F}_1) \le \mathbb{A}\mathrm{V@R}_\alpha(Y^{(2)}|\mathcal{F}_1)$.

(v) If $Y \lhd \mathcal{F}_1$, then $\mathbb{A}\mathsf{V@R}_\alpha(Y|\mathcal{F}_1) = Y$.

(vi) If $\mathcal{F}_0 = (\Omega, \emptyset)$, then $\mathbb{A}\mathsf{V@R}_\alpha(Y|\mathcal{F}_0) = \mathbb{A}\mathsf{V@R}_\alpha(Y)$.

(vii) If $\alpha_1 \leq \alpha_2$, then

$$\mathbb{A}\mathsf{V@R}_{\alpha_1}(Y|\mathcal{F}_1) \leq \mathbb{A}\mathsf{V@R}_{\alpha_2}(Y|\mathcal{F}_1).$$

(viii)

$$\mathbb{A}\mathsf{V@R}_1(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1).$$

(ix) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then

$$\mathbb{A}\mathsf{V@R}_\alpha(Y|\mathcal{F}_1) \leq \mathbb{E}[\mathbb{A}\mathsf{V@R}_\alpha(Y|\mathcal{F}_2)|\mathcal{F}_1] \leq \mathbb{A}\mathsf{V@R}_\alpha(\mathbb{E}(Y|\mathcal{F}_2)|\mathcal{F}_1).$$

Risk corrected expectation I

$\mathcal{A}(Y) = \mathbb{E}(Y) - \mathcal{D}(Y)$, where $\mathcal{D}$ is a convex, translation-invariant functional.

Let h be a nonnegative convex function on $\mathbb{R}$ with $h(0) = 0$ and $h^*(u) = \sup\{uv - h(v) : v \in \mathbb{R}\}$ be its Fenchel conjugate.

- *Primal form.* $\mathcal{A}(Y) = \mathbb{E}Y - \mathbb{E}[h(Y - \mathbb{E}Y)]$.
- *Dual form.* $\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + D_{h^*}(Z) : \mathbb{E}Z = 1\}$, where $D_{h^*}(Z) = \inf\{\mathbb{E}[h^*(Z - a)] : a \in \mathbb{R}\}$.
- *Conditional form*

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1) - \mathbb{E}[h(Y - \mathbb{E}(Y|\mathcal{F}_1))|\mathcal{F}_1].$$

- *Dual conditional form* $\mathcal{A}(Y|\mathcal{F}_1) =$

$$\mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \mathbf{inf}\{\mathbb{E}[h^*(Z - A)|\mathcal{F}_1] : A \lhd \mathcal{F}_1\} : \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

For every $\delta > 0$, there are random variables $Y^{(1)}$ and $Y^{(2)}$ such that $Y^{(1)} \prec_{FSD} Y^{(2)}$, but $\mathbb{E}Y^{(1)} - \delta \mathbb{V}\mathrm{ar}Y^{(1)} > \mathbb{E}Y^{(2)} - \delta \mathbb{V}\mathrm{ar}Y^{(2)}$.

Risk corrected expectation II

- *Primal form*

$$\mathcal{A}(Y) = \mathbb{E}Y - \inf\{\mathbb{E}[h(Y-a)] : a \in \mathbb{R}\}.$$

- *Dual form*

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + \mathbb{E}[h^*(1-Z)] : \mathbb{E}(Z) = 1\}$$

where $D_{h^*}(Z) = \inf\{\mathbb{E}[h^*(Z-a)] : a \in \mathbb{R}\}$.

- *Conditional form*

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbb{E}(Y|\mathcal{F}_1) - \mathbf{inf}\{\mathbb{E}[h(Y-A)|\mathcal{F}_1] : A \lhd \mathcal{F}_1\}.$$

- *Dual conditional form*

$$\mathcal{A}(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \mathbb{E}[h^*(1-Z)|\mathcal{F}_1] : \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

Orlicz-type functionals

The Minkowski gauge is defined as
$M_h(Y) = \inf\{a \geq 0 : \mathbb{E}[h(\frac{Y}{a})] \leq h(1)\}$.

- *Primal form*

$$\mathcal{A}(Y) = \mathbb{E}(Y) - M_h(Y - \mathbb{E}Y).$$

- *Dual form*

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(Z) = 1, \inf_a\{D^*_{h^*}(Z-a)\} \leq 1\}$$

where $D^*_{h^*}(Z) = \sup\{\mathbb{E}(Z\,V) : \mathbb{E}[h^*(V)] \leq h^*(1)\}$.

- *Dual conditional form* $\mathcal{A}(Y) =$

$$\mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1,$$
$$\inf_a\{\sup\{\mathbb{E}[(Z-a)\,V|\mathcal{F}_1] : \mathbb{E}[h(V)|\mathcal{F}_1] \leq h(1)\} \leq 1\}.$$

Distortion functionals

Distortion functionals were introduced independently as insurance pricing principles (Deneberg (1989), Wang (2000)) and by Yaari (1987) (Yaari's dual functionals).

- *Primal form*

$$\mathcal{A}(Y) = \int_0^1 G_Y^{-1}(p)\, k(p)\, dp$$

where G_Y is the distribution function of Y.

- *Dual form*

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(\phi(Z)) \leq \int \phi(k(u))\, du, \phi \text{ convex }, \phi(0) = 0$$

- *Dual conditional form* $\mathcal{A}(Y|\mathcal{F}_1) =$

$$\mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : \mathbb{E}(\phi(Z)|\mathcal{F}_1) \leq \int \phi(k(u))\, du, \phi \text{ convex }, \phi(0) = 0\}.$$

Making a functional translation-equivariant

The *sup-convolution* of two functions f and g is defined as

$$f \bar{*} g(y) = \sup\{f(x) + g(y-x) : x \in \mathbb{R}\}.$$

If f and g are concave, then so is $f \bar{*} g$.
Let $\pi(Y)$ a convex functional (an insurance premium principle), which is not necessarily translation-equivariant. The translation-equivariant extension (Filipovic) of $\pi([Y]_-)$ is

$$\mathcal{A}_\pi(Y) = \sup\{x - \pi([Y-x]_-) : x \in \mathbb{R}\}.$$

If π has the dual representation $\pi([Y]_-) = \sup\{\mathbb{E}(Y\,Z) - p(Z) : Z \in L_q\}$, then $\mathcal{A}_\pi$ has the representation

$$\mathcal{A}_\pi = \inf\{\mathbb{E}(Y\,Z) + p(-Z) : \mathbb{E}(Z) = 1, -Z \in L_q\}.$$

If π is monotonic, then $\mathcal{A}_\pi$ is isotonic w.r.t. $\prec_{FSD}$. If π is positively homogeneous, then $\mathcal{A}_\pi$ is also positively homogeneous. If π is compound concave, then $\mathcal{A}_\pi$ is compound convex.

Examples of sup-convoluted acceptability functionals

- $\pi(Y) = c\,\mathbb{E}(Y)$ for some $c > 1$. Then $\mathcal{A}_\pi(Y) = \mathbb{A}\mathsf{V@R}_{1/c}$.
- Let h be a convex, strictly increasing non-negative function on $\mathbb{R}$ with $h(0) = 0$, $h(1) = 1$ and $0 < h(u) < \infty$ for $u \neq 0$. The Orlicz premium $\pi_\alpha(L)$ for $L \neq 0$ is given as the unique solution of
$$\mathbb{E}\Big[h\Big(\frac{L}{\pi_\alpha(L)}\Big)\Big] = 1 - \alpha.$$
The pertaining sup-convoluted acceptability functional is
$$\mathcal{A}_{\pi_\alpha}(Y) = \sup\{x - \pi_\alpha([Y - x]^-) : x \in \mathbb{R}\}.$$
The negative value
$$\rho_\alpha(Y) = -\mathcal{A}_{\pi_\alpha}(Y) = \inf\{u + \pi_\alpha([-Y - u]^+) : u \in \mathbb{R}\}$$
is called *Haezendonck-Goovaerts risk functional*.

Utility type functional

Let U be a concave, strictly monotonic utility function.

$$\mathcal{A}(Y) = U^{-1}(\mathbb{E}[U(Y)]).$$

Then $\mathcal{A}$ is the *certainty equivalent* and

$$\mathcal{D}(Y) = \mathbb{E}Y - \mathcal{A}(Y),$$

is the *risk premium*, i.e. decision maker with utility function U is indifferent between Y and the deterministic value $\mathcal{A}$ in the sense that

$$\mathbb{E}[U(Y)] = U[\mathcal{A}(Y)].$$

This type of functionals is translation-equivariant iff $U(x) = -k\exp(-\gamma x) + d$; $k \geq 0$ or $U(x) = kx + d$. Up to affine transformations, these are the entropic functionals and the expectation itself.

Version-independent conditional functionals

Recall that we consider $(\Omega, \mathcal{F}, P)$ and a sub-sigma algebra $\mathcal{F}_1$ of $\mathcal{F}$.

Definition. The *nested distribution* of a random variable $Y \in L_p(\mathcal{F})$ given $\mathcal{F}_1$ is the distribution of the conditional distributions of Y given $\mathcal{F}_1$. A conditional acceptability (or risk) functional $\mathcal{A}(\cdot|\mathcal{F}_1)$ is version independent, if its distribution depends only on the nested distribution of Y given $\mathcal{F}_1$.

Examples. All presented conditional functionals are version independent.

Choquet representation

Definition. The version-independent functional $\mathcal{A}$ has a *Choquet representation*, if it is representable in the form

$$\mathcal{A}(Y) = \int_0^1 \mathbb{A}\text{V@R}_\alpha(Y)\, dm(\alpha),$$

for some probability measure m on $[0,1]$.

Proposition. The concave, version-independent positively homogeneous acceptability functional $\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : Z \in \mathcal{Z}\}$ has a Choquet representation if and only if it is comonotone additive: The $\mathbb{A}$V@R's are the extremal elements of the (convex) family of all comonotone additive acceptability functionals.

Kusuoka representation

Theorem.(Kusuoka 2001, Fritelli and Rosazza Gianin, 2005). Let $\mathcal{A}$ be concave version-independent acceptability-type functional on $L_p(\Omega, \mathcal{F}, P)$, $1 \leq p < \infty$, where $(\Omega, \mathcal{F}, P)$ is non-atomic.

- Condition C1: $\mathcal{A}$ is positively homogeneous;
- Condition C2: $\mathcal{A}$ is FSD monotonic;
- Condition C3: $\mathcal{A}$ is translation-equivariant.

(i) If (C1), (C2) and (C3) are fulfilled, then $\mathcal{A}$ has the *Kusuoka representation*

$$\mathcal{A}(Y) = \inf\{ \int_0^1 \mathbb{A}\mathrm{V@R}_\alpha(Y) \, dm_G(\alpha) : G \in \mathcal{G}\}$$

where $\{m_G : G \in \mathcal{G}\}$ is a family of prob. measures on (0,1].

(ii) If (C1) and (C2) are fulfilled, then $\mathcal{A}$ has the representation

$$\mathcal{A}(Y) = \inf\{ \int_0^1 \mathbb{A}\mathrm{V@R}_\alpha(Y) \, dm_G(\alpha) : G \in \mathcal{G}\}$$

where $\{m_G : G \in \mathcal{G}\}$ are non-negative measures on (0,1].

(iii) If (C1) holds, then $\mathcal{A}$ has the representation

$$\begin{aligned}\mathcal{A}(Y) &= \inf\{\int_0^1 \mathbb{A}\mathsf{V@R}_\alpha(Y)\, dm_G^{(1)}(\alpha) \\ &\quad - \int_0^1 \mathbb{A}\mathsf{V@R}_{1-\alpha}(-Y)\, dm_G^{(2)}(\alpha) : G \in \mathcal{G}\},\end{aligned}$$

where $m_G^{(1)}, m_G^{(2)}, G \in \mathcal{G}$ are families of non-negative measures on (0,1].

(iv) In general, $\mathcal{A}$ has the representation

$$\begin{aligned}\mathcal{A}(Y) &= \inf\{\int_0^1 \mathbb{A}\mathsf{V@R}_\alpha(Y)\, dm_G^{(1)}(\alpha) \\ &\quad - \int_0^1 \mathbb{A}\mathsf{V@R}_\alpha(-Y)\, dm_G^{(2)}(\alpha) - \tilde{A}\{G\} : G \in \mathcal{G}\}\end{aligned}$$

where $m_G^{(1)}, m_G^{(2)}$ are as before and $\tilde{A}\{G\}$ is some functional defined on $\mathcal{G}$.

Remark. The Kusuoka representation is not unique. For instance,

$$\mathbb{A}\mathsf{V@R}_\beta = \int \mathbb{A}\mathsf{V@R}_\alpha \, d\delta_\beta(\alpha) = \inf\{ \int \mathbb{A}\mathsf{V@R}_\alpha \, d\delta_{\beta+1/n}(\alpha) : \quad n \geq 0\}.$$

↯ If the underlying space has atoms, a Kusuoka representation does not hold in general. Let $\Omega = \{\omega_1, \omega_2\}$ with $P\{\omega_1) = q$, $P\{\omega_2\} = 1 - q$. The functional $\mathcal{A}(Y) = Y(\omega_1)$ is version-independent, but has no Kusuoka representation.

Examples for Kusuoka representations

Example. The mean absolute deviation corrected mean.

$$\begin{aligned} &\mathbb{E}(Y) - \frac{1}{2}\mathbb{E}|Y - \mathbb{E}Y| \\ = \ &\inf\{\int_{(0,1]} \mathbb{A}\mathsf{V@R}_\alpha(Y)\,dm(\alpha) : m \in \mathcal{P}(0,1]; \int_{(0,1)} \frac{1}{v} dm(v) \le 1\}, \end{aligned}$$

where $\mathcal{P}(0,1]$ is the family of all probability measures on $(0,1]$.

Example. The lower-standard deviation corrected mean.

$$\mathbb{E}(Y) - \mathbb{S}\mathrm{td}^-(Y) = \inf\{\int_{(0,1]} \mathbb{A}\mathsf{V@R}_\alpha(Y) dm(\alpha) : m \in \mathcal{M}\},$$

where

$$\mathcal{M} = \left\{ m \in \mathcal{P}(0,1] : \int_{(0,1)} \int_{(0,1)} \frac{\min(v,w)}{vw}\, dm(v)\, dm(w) = 1 \right\}.$$

Example. The standard deviation corrected mean.

$$\mathbb{E}(Y) - \mathbb{S}\mathrm{td}(Y) = \inf\{\int_{(0,u_0]} \mathbb{A}\mathrm{V@R}_\alpha(Y)\, dm^{(1)}(\alpha) - \int_{(u_0,1]} \mathbb{A}\mathrm{V@R}_{1-\alpha}(Y)\, dm^{(2)}(\alpha) : m^{(1)}, m^{(2)} \in \mathcal{M}\},$$

where $\mathcal{M}$ is the family of pairs of non-negative measures satisfying

$$\int_{(u,u_0]} dm^{(1)} + \int_{(u_0,1]} dm^{(2)} = 1$$

and

$$\int_{(0,u_0]} \int (0, u_0] \frac{\min(v,w)}{v \cdot w}\, dm^{(1)}(v)\, dm^{(2)}(w) + \int_{(u_0,1]} \int_{(u_0,1]} \frac{1 - \max(v,w)}{(1-v)(1-w)}\, dm^{(2)}(v)\, dm^{(2)}(w) \leq 2.$$

for some u_0.

Properties of acceptability-type functionals

(TE)	Translation-equivariant
(CV)	Concave
(FSD)	Isotonic w.r.t. first order stochastic dominance/pointwise monotonic
(SSD)	Isotonic w.r.t second order stochastic dominance
(PH)	Positive homogeneous
(CCX)	Compound convex
(CLI)	Compound linear

expectation $\mathbb{E}[Y]$	(TE)	(CV)	(SSD)	(PH)	(CLI)
(concave) utility-type $U^{-1}\mathbb{E}[U(Y)]$	-	-	(SSD)	-	(CCX)
distortion functionals $\int_0^1 G_Y^{-1}(p)\, dH(p)$	(TE)	(CV)	(SSD)	(PH)	(CCX)†
sup-convolutions $\mathcal{A}_\pi(Y)$	(TE)	(CV)	(SSD)‡	(PH)‡	(CCX)‡
average value-at-risk $\mathbb{AV@R}$	(TE)	(CV)	(SSD)	(A4)	(CCX)
value-at-risk $\mathbb{V@R}$	(TE)	-	(FSD)	(PH)	-

†if H is a concave probability distribution function,
‡if π has the corresponding property

Properties of translation-invariant functionals

Translation-invariant functionals $\mathcal{D}$ build risk functionals by $\mathcal{A}(Y) = \mathbb{E}(Y) - \mathcal{D}(Y)$

(TI)	Translation-invariant
(CX)	Convex
(FSD)	$\mathbb{E} - \mathcal{D}$ is isotonic w.r.t. first order stochastic dominance
(SSD)	$\mathbb{E} - \mathcal{D}$ is isotonic w.r.t second order stochastic dominance
(PH)	Positive homogeneous
(CCC)	Compound concave

$\mathbb{E}[h(Y - \mathbb{E}Y)]$	(TI)	(CX)	-	-	-
$\|Y - \mathbb{E}Y\|_h$	(TI)	(CX)	-	(PH)	-
$\|[Y - \mathbb{E}Y]^-\|_h$	(TI)	(CX)	(SSD)	(PH)	-
$\mathbb{E}[h(Y - Y')]$	(TI)	(CX)	-	-	-
$\inf\{\mathbb{E}[h(Y - a)] : a \in \mathbb{R}\}$	(TI)	(CX)	(SSD)	-	(CCC)

†If $h(u) > 0$ for $u \neq 0$.

The value-at-risk

We define the value-at-risk as $\mathbb{V}@\mathrm{R}_\alpha(Y) = G_Y^{-1}(\alpha)$ with $G_Y(u) = P\{Y \leq u\}$ Some people define it as $-G_Y^{-1}(\alpha)$.

⚡ $\mathbb{V}@\mathrm{R}$ is not concave, not compound convex, not isotonic w.r.t. SSD, but translation-equivariant, isotonic w.r.t. FSD, positively homogenous and comonotone additive.

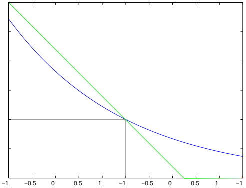
Since the $\mathbb{V}@\mathrm{R}$ is the basic tool in the Basel II formulas, this leads to paradoxic situations.

Convexification of $\mathbb{V}@\mathrm{R}$

The level sets $\{Y : \mathbb{V}@\mathrm{R}_\alpha(Y) \geq q\} = \{Y : \mathbb{E}[\mathbb{1}_{(-\infty,q]}(Y)] \leq \alpha\}$ are not convex. Convex inner approximations are given by

$$\{Y : \mathbb{E}[k(Y)] \leq \alpha\}$$

where k is a convex majorant of $\mathbb{1}_{(-\infty,q]}$.



kinked linear functions $k_a(u) = \frac{1}{a-q}[u-a]^-$: $\mathbb{A}$V@R (Rockafellar and Uryasev)
exponential functions $k_b(u) = e^{b(q-u)}$: Nemirovski and Shapiro