Extending Time-consistent Convex Risk Measures from Discrete to Continuous Time: a Convergence Approach

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9th Winter School on Mathematical Finance

January 19th, 2010

Outline

- How to measure risk?
  - Convex risk measures in a one-period setting (static)
  - Dynamic convex risk measures
    - in a multiperiod setting
    - in a continuous time setting
- Convex risk measures in continuous time may be interpreted as limits of certain compound one-period convex risk measures.
- Extending discrete-time risk measures like AV@R, the semi-deviation and the Gini risk measure to continuous time

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What is risk?

Let Ω be a fixed set of scenarios. A financial position is described by a mapping $X : \Omega \to \mathbb{R}$ where $X(\omega)$ is the discounted net worth of the position at the end of the trading period if the scenario $\omega \in \Omega$ is realized. What is the risk of X?

Value at Risk

One notion of risk used by banks and insurances is *Value at Risk* (V@R). For the cdf F_X of the random variable X define

$$q_X^+(s) = \inf\{x \in \mathbb{R} | F_X(x) > s\}.$$

Then for fixed $\alpha \in (0,1]$ the Value at Risk of X to a level α is defined by

$$Risk(X) = V@R_\alpha(X) = -q_X^+(\alpha) = \inf\{m \in \mathbb{R} | P(X+m < 0] \leq \alpha\}.$$

In other words $V@R_\alpha(X)$ is the minimal amount of money I have to add to my position X such that with a probability greater than $1-\alpha$ I will not encounter any losses.

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Axiomatic approach: static (one-period) convex risk measure (Artzner, Delbaen, Heath (1999), Föllmer, Schied/Frittelli, Rosazza Gianin (2002))

Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose that the set of all possible payoffs is given by $L^\infty(\Omega, \mathcal{F}, P)$. A mapping $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$, is a *convex risk measure* if it has the following properties:

- *Normalization*: $\rho(0) = 0$
- *Translation Invariance*: $\rho(X + m) = \rho(X) - m$ for all $m \in \mathbb{R}$
- *Monotonicity*: If $X \leq Y$ a.s., then $\rho(X) \geq \rho(Y)$
- *Convexity*: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $0 \leq \lambda \leq 1$
- *Lower-Semicontinuity:* If (X_n) is a bounded sequence which converges to X a.s. then

$$\rho(X) \leq \liminf_n \rho(X_n).$$

Examples

Typical examples for one-period risk measures are

- Average Value at Risk:

$$AV@R^{\alpha}(X) = \frac{1}{\alpha}\int_0^{\alpha} V@R^{\lambda}(X)d\lambda, \quad \alpha \in (0,1].$$

  If the distribution of X is continuous
  $AV@R^{\alpha}(X) = \mathbb{E}[-X|X \leq q_X^{+}(\alpha)]$.

- *Semi-deviation risk measure:*

$$S_{t_i}^{\lambda,p}(X) = \mathbb{E}[-X]+\lambda||(X-\mathbb{E}[X])_-||_p, \quad \lambda \in [0,1], \quad p \in [1,\infty).$$

- *Gini risk measure:*

$$V^{\theta}(X) = \sup_{Q<<P}\left\{E_Q[-X] - \frac{1}{2\theta}C(Q|P)\right\}, \quad \theta > 0$$

  where

$$C(Q|P) = \mathbb{E}\Big[\Big(\frac{dQ}{dP} - 1\Big)^2\Big].$$

Dynamic risk measures

Given $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Let I be the set of time instances in which the agent is allowed to udate his risk. We call a family of mappings $\rho_{s,t} : L^\infty(\mathcal{F}_t) \to L^\infty(\mathcal{F}_s)$, $s, t \in I$ and $s \leq t$, a *dynamic risk measure* if it has the following properties for $X, Y \in L^\infty(\mathcal{F}_t)$:

- *Normalization*: $\rho_{s,t}(0) = 0$
- *Monotonicity*: If $X \leq Y$, then $\rho_{s,t}(X) \geq \rho_{s,t}(Y)$ a.s.
- *$\mathcal{F}_s$-Translation Invariance*: $\rho_{s,t}(X + m) = \rho_{s,t}(X) - m$ for all $m \in L^\infty(\mathcal{F}_s)$
- *$\mathcal{F}_s$-Convexity*: $\rho_{s,t}(\lambda X + (1-\lambda)Y) \leq \lambda \rho_{s,t}(X) + (1-\lambda)\rho_t(Y)$ for all $\lambda \in L^\infty(\mathcal{F}_s)$ such that $0 \leq \lambda \leq 1$
- *$\mathcal{F}_s$-Lower-Semicontinuity:* For any $\mathcal{F}_t-$adapted bounded sequences X_n converging a.s. to X we have $\rho_{s,t}(X) \leq \liminf_n \rho_{s,t}(X_n)$ a.s.

Dynamic risk measures

If T is the time horizon of our model, $\rho_{s,T}$ is often denoted by ρ_s.

- *Time-consistency:* for $X, Y \in L^\infty(\mathcal{F}_t)$ $\rho_{s',t}(X) \leq \rho_{s',t}(Y)$ a.s. for some s' with $t \geq s' \geq s$, implies $\rho_{s,t}(X) \leq \rho_{s,t}(Y)$ a.s.

see for instance Delbaen (2003) or Barrieu and El Karoui (2005). Using time-consistency you can show that for every bounded $\mathcal{F}_t-$measurable X

$$\rho_{s,t}(X) = \rho_{s,T}(X) = \rho_s(X),$$

i.e., $\rho_{s,t} = \rho_s|L^\infty(\mathcal{F}_t)$ and thus the whole family $(\rho_{s,t})_{s,t\in I}$ is uniquely determined by $(\rho_s)_{s\in I}$.

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Duality in a discrete setting

Suppose that we are in a multiperiod discrete setting, i.e., $I = \{t_0, t_1, \ldots, t_k\}$ where $0 = t_0 < t_1 < \ldots < t_k = T$.
For $i = 0, \ldots, k-1$ define the set of one-step transition densities

$$D_{t_i} = \{\xi_{t_{i+1}} \in L^1_+(\mathcal{F}_{t_{i+1}}) \mid \mathbb{E}[\xi_{t_{i+1}}|\mathcal{F}_{t_i}] = 1 \text{ a.s.}\}.$$

We identify a probability measure Q with its density $\xi_{t_{i+1}} \in D_{t_i}$.
Suppose that $\rho_{t_i,t_{i+1}} : L^\infty(\mathcal{F}_{t_{i+1}}) \to L^\infty(\mathcal{F}_{t_i})$ is a one-period risk measure.
Define the *penalty function* on D_{t_i} of a one-period risk measure $\rho_{t_i,t_{i+1}}$ as

$$\phi_{t_i}^{\rho_{t_i,t_{i+1}}}(Q) = \operatorname{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t_{i+1}})}\{\mathbb{E}_Q[-X|\mathcal{F}_{t_i}] - \rho_{t_i,t_{i+1}}(X)\}.$$

Every sequence $\xi \in D_{t_i} \times D_{t_{i+1}} \times \ldots \times D_{t_{k-1}}$ induces a P-martingale

$$M^{\xi}_{t_r} = \begin{cases} \prod_{j=i+1}^{r} \xi_{t_j} \;\; if \;\; r \geq i+1 \\ 1 \;\; if \;\; r \leq i \end{cases}$$

and a probability measure Q^{ξ} by $\frac{dQ^{\xi}}{dP} = M^{\xi}_T$. Set $D = D_{t_0} \times D_{t_1} \times \ldots \times D_{t_{k-1}}$. For $\xi \in D$ define $\phi^{\rho_{t_i,t_{i+1}}}_{t_i}(Q^{\xi}) = \phi^{\rho_{t_i,t_{i+1}}}_{t_i}(\xi_{t_{i+1}})$. Then from Cheridito and Kupper (2006), we obtain the following representation.

Proposition

Suppose that $(\rho_s)_{s \in I}$ is a discrete-time risk measure. Then

$$\rho_{t_i}(X) = \operatorname{ess\,sup}_{Q \in D} \mathbb{E}_Q\Big[-X - \sum_{j=i}^{k-1} \phi^{\rho_{t_j,t_{j+1}}}_{t_j}(Q) \Big| \mathcal{F}_{t_i} \Big].$$

Dynamic risk measures in continuous time

Now suppose that $I = [0, T]$, i.e., the risk manager is allowed to update his information at *any* time. Risk Modelling can then be done using Backward Stochastic Differential Equations (Barrieu and El Karoui (2005)).
BSDE = backward stochastic differential equation

Definition of a BSDE

Assume that we have a d-dimensional Brownian Motion $(W_t^1, \ldots, W_t^d)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ where $(\mathcal{F}_t)$ is the standard filtration. Let $g : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a function such that

- $z \mapsto g(t, \omega, z)$ is convex for every fixed $(t, \omega) \in [0, T] \times \Omega$
- for every fixed $z \in \mathbb{R}^d$, $(t, \omega) \to g(t, \omega, z)$ is progressively measurable
- there exists a $K > 0$ with $|g(t, \omega, z)| \leq K(1 + |z|^2)$ *a.s.*
- ...

The solution of a BSDE with driver $g(t, \omega, z)$ and terminal condition $X \in L^\infty(\mathcal{F}_T)$ is a pair of (suitably integrable) progressively measurable processes (Y_t, Z_t) with values in $\mathbb{R} \times \mathbb{R}^d$, which satisfy

$$Y_t = X + \int_t^T g(s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

Let $Y^g(-X)$ be the solution (Y_t) of the BSDE with driver g and terminal condition $-X$. Define

$$\rho_t^g(X) = Y_t^g(-X).$$

Then ρ^g is a dynamic risk measure!

Framework: Risk measures in discrete time

Setting: For fixed N let $B^j_{N,l}$ be independent Bernoulli random variable with
$P[B^j_{N,l} = 1] = P[B^j_{N,l} = -1] = \frac{1}{2}$; $j = 1, \ldots, d$, $N \in \mathbb{N}$, $l = 1, \ldots, N$. Let

$$R^{N,j}(t_i) = \sqrt{\frac{T}{N}} \sum_{l=1}^{i} B^j_{N,l}, \quad t_i = iT/N, \;\; i = 1, .., N, \;\; j = 1, .., d$$

and constant on the intervals $[t_i, t_{i+1})$. Let

$$R^N(t_i) = (R^{N,1}(t_i), \ldots, R^{N,d}(t_i)).$$

Denote by $\mathcal{F}^N = (\mathcal{F}^N_t)_{0 \le t \le T}$ the filtration generated by the random walk.
Assume that there exists a standard Brownian motion W_t such that

$$\sup_{0 \le s \le T} |R^N(s) - W_s| \to 0 \text{ in } L^2.$$

A robust way of measuring risk in discrete time

Definition

For a collection of one-period risk measures $(F^N_{t_i})_{i=0,\dots,k-1}$ with penalty functions $(\phi^{F^N_{t_i}}_{t_i})_{i=0,\dots,k-1}$ we define its (tilted) robust extension as

$$\rho^N_{t_i}(X) = \sup_{\mu^N} \hat{\mathbb{E}}^{\mu^N}[-X - \sum_{j=i}^{N-1} \phi^{F^N_{t_j}}_{t_j}\left(1 + \mu^N_{t_i} B^N_{i+1}\right)\Delta t_{j+1} | \mathcal{F}_{t_i}]$$

where for every bounded, $\mathcal{F}^N$-adapted process μ^N, $\hat{P}^{\mu^N}$ is the measure under which $R^N_t - \sum_{t_j \le t} \mu^N_{t_j} \Delta t_{j+1}$ is a martingale.

Goal: Start with one-period risk measures $F^N_{t_i}$ like AV@R, semi-deviation etc.

$\longrightarrow$ Define the robust extension in discrete time.

$\longrightarrow$ Extend it to continuous time by convergence.

Proposition

Suppose we are given a collection of one-period risk measures $(F^N_{t_i})_{i=0,\dots,k-1}$. For $z \in \mathbb{R}^d$, $M \in \mathbb{R}$ let

$$g^N(t_i, z, M) = F^N_{t_i}(-zB^N_{t_{i+1}} - M).$$

Then for every $X^N \in L^\infty(\mathcal{F}^N_T)$ there exists a process Z^N and a martingale M^N orthogonal to R^N such that

$$\rho^N_{t_i}(X^N) = -X^N + \sum_{t_i \le t_j < T} g^N(t_j, Z^N_{t_j}, M^N_{t_{j+1}})(t_{j+1} - t_j) - \sum_{t_i \le t_j < T} Z^N_{t_j}(R^N_{t_{j+1}} - R^N_{t_j}) - (M^N_T - M^N_{t_i}).$$

Convergence Theorem for discrete-time risk measures to continuous-time risk measures

Theorem

Let g be a driver function such that for every $z \in \mathbb{R}^d$ *:*

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|g^N(t,z,0)-g(t,z)\big|^2\Big] \overset{N\to\infty}{\to} 0.$$

Then there exists a continuous-time dynamic risk measure $(\rho_s)_{s\in[0,T]}$ *such that for every sequence of discrete payoffs* X^N *which converge in* L^2 *to a continuous-time payoff X we have*

$$\sup_t |\rho_t^N(X^N)-\rho_t(X)| \overset{N\to\infty}{\to} 0 \quad \textit{in } L^2$$

Moreover, ρ *is the solution of a continuous-time BSDE with terminal condition* $-X$ *and driver* g*.*

Examples of discrete-time risk measures extended to continuous time: semi-deviation

Suppose that the one-period risk measures are given by the **semi-deviation:**
The robust extension of semi-deviation converges to ρ_t, where for any terminal condition X, $(\rho_t(X), Z_t)$ is the solution of

$$\rho_t(X) = -X + \int_t^T g(Z_s)ds - \int_t^T Z_s dW_s$$

with $g(z)$ defined by

$$g(z) = \lambda\Bigg(\frac{1}{2^d}\Big(\sum_{l=1,\dots,d,\ k_l\in\{1,2\}} (-1)^{k_l} z^l\Big)_-^p\Bigg)^{1/p},\ \ z = (z^1, \dots, z^d) \in \mathbb{R}^d.$$

Examples of discrete-time risk measures extended to continuous time: Average Value at Risk

Suppose that the one-period risk measures are given by
Average Value at Risk:
Let

$$x_i(z) = i\text{-th largest element of the set } \{(-1)^{k_1}z^1 + \ldots + (-1)^{k_d}z^d | k_l \in \{1,2\}, l = 1,\ldots,d\}.$$

The robust extension of AV@R converges to ρ_t, where $(\rho_t(X), Z_t)$ is the solution of the BSDE with terminal condition $-X$ and driver

$$g(z) = -\frac{1}{\alpha}\left(x_{2^d-\lceil 2^d\alpha\rceil+1}\left(\alpha - \frac{\lceil 2^d\alpha\rceil - 1}{2^d}\right) + \frac{1}{2^d}\sum_{j=1}^{\lceil 2^d\alpha\rceil-1} x_{2^d-j+1}(z)\right).$$

In particular, if $\alpha < 1/2^d$ we have

$$g(z) = |z_1| + |z_2| + \ldots + |z_d|.$$

Examples of discrete risk measures extended to continuous time

Suppose that the one-period risk measures are given by the **Gini risk measure:** Define

$$l(z) = \sup\Big\{ l \in \{2^d, \ldots, 1\} \Big| \text{for all } j \in \{2^d, \ldots, l\} : \quad \frac{1}{\theta(2^d + 1 - j)} > -\frac{\sum_{i=l}^{2^d} x_i(z)}{2^d + 1 - l} + x_j(z) \text{ and } \frac{1}{\theta(2^d + 1 - l)} \leq -\frac{\sum_{i=l}^{2^d} x_i(z)}{2^d + 1 - l} + x_{l-1}(z) \Big\}.$$

The driver for the BSDE of the extension of the Gini risk measure to continuous time is given by

$$g(z) = -\frac{1}{2\theta(2^d+1-l(z))} + \frac{1}{2\theta} - \frac{\sum_{j=l(z)}^{2^d} x_i(z)}{2^d+1-l(z)} - \frac{\theta}{2}\frac{\left(\sum_{j=l(z)}^{2^d} x_i(z)\right)^2}{2^d+1-l(z)} + \frac{\theta}{2}\sum_{j=l(z)}^{2^d} x_j^2(z)$$

for $z = (z^1, \dots, z^d) \in \mathbb{R}^d$. In the special case that $d = 1$ we get

$$g(z) = \begin{cases} |z| - \frac{1}{2\theta}, & \text{if } |z| \geq 1/\theta \\ \frac{\theta}{2} z^2, & \text{if } |z| < 1/\theta. \end{cases}$$