Asymptotic analysis of hedging errors in models with jumps

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Abstract

Most authors who studied the problem of option hedging in incomplete markets, and, in particular, in models with jumps, focused on finding the strategies that minimize the residual hedging error. However, the resulting strategies are usually unrealistic because they require a continuously rebalanced portfolio, which is impossible to achieve in practice due to transaction costs. In reality, the portfolios are rebalanced discretely, which leads to a 'hedging error of the second type', due to the difference between the optimal portfolio and its discretely rebalanced version. In this paper, we analyze this second hedging error and establish a limit theorem for the renormalized error, when the discretization step tends to zero, in the framework of general Itô processes with jumps. The results are applied to the problem of hedging an option with a discontinuous payoff in a jump-diffusion model.

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1 Introduction

The problem of hedging an option in an incomplete market, and in particular, in a market where stock prices may jump, has been studied by many authors starting with Föllmer and Schweizer [6] up to more recent works [4, 14, 17, 20]. All these authors made the assumption that the hedging portfolio is rebalanced continuously, which may be a good approximation in very liquid markets but cannot be satisfied completely due to the presence of transaction costs. Taking into account the discrete nature of hedging is particularly important in illiquid markets where transaction costs are high and it is not always possible to find a counterparty instantaneously.

The observation that discrete hedging leads to an additional source of error is not new (this risk is sometimes referred to as *gamma risk* by market practitioners) but this error is not easy to quantify because the tools of stochastic calculus are not available in discrete time. In [3], Bertsimas, Kogan and Lo introduced an asymptotic approach allowing to tackle the error due to discrete hedging in a continuous-time framework. Their result can be briefly summarized as follows. Suppose that the stock price is a Markovian diffusion

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and we want to hedge a European option with payoff $h(S_T)$. Then it is well known that the market is complete and the optimal strategy is the delta hedging given by $\phi_t = \frac{\partial C}{\partial S}$, where ϕ_t is the number of stocks to hold at time t and C is the option price as a function of time and spot price. If the portfolio were rebalanced continuously, this strategy would yield perfect hedging, however, in practice, the strategy ϕ_t is replaced with a discrete strategy $\phi^n_t := \phi_{h[t/h]}$, $h = T/n$, resulting in a residual hedging error (the only error in this simple setting).

This discretization error is given by

$$\varepsilon^n_T = h(S_T) - \int_0^T \phi^n_t dS_t$$

Then clearly $\varepsilon^n_T \to 0$ as $n \to \infty$ but the interesting question is at what rate this convergence takes place. Bertsimas, Kogan and Lo have shown that the renormalized error $\sqrt{n}\varepsilon^n_T$ converges in law to a non-zero limit given by

$$\sqrt{\frac{T}{2}} \int_0^T \frac{\partial^2 C}{\partial S^2} S_t^2 \sigma(t, S_t)^2 d\overline{W}_t,$$

where $\overline{W}$ is a Brownian motion independent of W.

Apart from its mathematical beauty, this result is very important for practical purposes: it provides a complete first-order characterization of the hedging error and leads to a number of important insights such as

- The hedging error is proportional to the square root of the rebalancing interval: to decrease the error by a factor of 2, one must rebalance 4 times as often.
- Since $\overline{W}$ is independent from W, the hedging error is orthogonal to the stock price process.
- The amplitude of the error is determined by the gamma of the option: $\frac{\partial^2 C}{\partial S^2}$.

Hayashi and Mykland [11] extended the work of Bertsimas, Kogan and Lo in many directions. Among other things, they reinterpreted the discrete hedging error as the error of approximating the "ideal" hedging portfolio $\int_0^T \phi_t dS_t$ with a feasible hedging portfolio $\int_0^T \phi_t^n dS_t$. This formulation makes sense in incomplete markets, even if the price of the continuously rebalanced hedging portfolio does not coincide with the payoff of the option. They considered both the stock price process and the hedging strategy as general continuous Itô processes of the form

$$
\begin{aligned}
d\phi_t &= \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t, \\
dS_t &= \mu_t dt + \sigma_t dW_t,
\end{aligned}
$$

and proved the weak convergence in law in the Skorohod topology of the hedging error *process*

$$
\begin{aligned}
\sqrt{n}\varepsilon^n &\Rightarrow \sqrt{\frac{T}{2}} \int_0^\cdot \tilde{\sigma}_s \sigma_s d\overline{W}_s, \\
\text{where} \quad \varepsilon_t^n &:= \int_0^t (\phi_t - \phi_t^n) dS_t.
\end{aligned}
$$

The study of error from discrete hedging then reduces to that of the error of approximating a stochastic integral with an appropriate Riemann sum. The general problem of approximating a stochastic integral is not new and goes back at least to [19]. More recently, Geiss [7, 8, 9] studied weak and L^2 approximations with non-uniform time steps, making the connection with discrete time hedging. Another paper worth citing in this respect is [10] where the authors study the L^2 discrete hedging errors for options with irregular payoffs (such as binary). A related problem arises in the context of asymptotic behavior of realized volatility and power variation [1, 2, 13].

In parallel with the development of the theory of discrete-time hedging (and sometimes well ahead of it), similar asymptotic results have been obtained for the approximation error of Euler schemes for stochastic differential equations of the form

$$X_t = x_0 + \int_0^t f(X_{s-}) dY_s,$$

where f is a matrix of functions and Y a vector of semimartingales. The Euler scheme is defined by

$$\bar{X}_{t_{i+1}} = \bar{X}_{t_i} + f(\bar{X}_{t_i})(Y_{t_{i+1}} - Y_{t_i})$$

with respect to some partition $0 = t_0 < t_1 < \cdots < t_n = T$. The problem of analyzing the error of this scheme is on one hand more difficult than that of the discrete hedging error, because the integrand also involves an approximation, but on the other hand it is simpler, since the integrand is an explicit function, whereas in the hedging problem the integrand is a hedging strategy resulting from some optimization procedure. The rate of convergence for the Euler scheme for a continuous diffusion was established in [18]. These results were later extended to Lévy-driven SDE's in [12, 15].

In this paper, we generalize the weak convergence results of Hayashi and Mykland [11] to semimartingales with jumps using the methods developed by Jacod and Protter [15] in the context of Euler schemes for Lévy-driven SDE's. We prove a limit theorem for the error arising from discrete hedging and characterize the limiting law. This result takes a particular importance, because price jumps are often a sign of low market liquidity, meaning that transaction costs will be high and the discretization error will be important.

In presence of price jumps, the market is typically incomplete and there are two types of hedging errors: the first one is due to market incompleteness and the second one is due to the discrete nature of the hedging portfolio. One of the main insights of our paper is that these two errors have a very different behavior: while the first one is due to jumps, the second one is dominated by the diffusion component of the price process.

In incomplete markets, the fundamental problem, of course, is how to choose the hedging strategy. One approach is to pick a martingale measure using an ad hoc criterion, or by calibrating it to option prices, and then use, say, delta hedging (as we do in the example at the end of section 3) or the optimal quadratic hedging [4], under this martingale measure. Another approach is to compute the variance-optimal quadratic hedge ratio directly under the historical measure [17]. In illiquid markets, the hedge ratio may be chosen based on risk preferences of the investor, expressed via an utility function [5]. Different choices of the strategy yield different hedging errors, but our treatment of the limiting behavior of the discretization error is not linked to a particular strategy. In Theorem 1 we show that the rate of convergence of the discretization error to zero is invariant with respect to the choice of the hedging strategy. Moreover, Proposition 1 suggests that it is also invariant with respect to the choice of the option which is being hedged. This is specific to weak convergence: for the L^2 error, for instance, different options and different hedging strategies may lead to different convergence rates [10]. The convergence rates may also be modified (and improved) by using non uniformly spaced rebalancing dates, and/or by including additional assets into the hedging portfolio. We plan to address these questions in future research.

The rest of our paper is structured as follows. In section 2, we define our model, state the main hypotheses and introduce the relevant notion of convergence. In section 3 we state the main result on the weak convergence of the renormalized hedging error and provide an example which shows that our theory applies to the delta-hedging of a binary option in a jump diffusion model. The proof of the main theorem is postponed to section 4.

2 Preliminaries

First, let us recall the definition of *stable convergence* which is a type of weak convergence particularly adapted for studying the renormalized error processes and used among other authors by Rootzén [19] in the approximation of stochastic integrals and by Jacod and Protter [15] to analyze the discretization error of the Euler scheme.

Let X_n be a sequence of random variables with values in a Polish space E, all defined on the same probability space $(\Omega, \mathcal{F}, P)$. We say that X_n converges stably in law to X, written $X_n \stackrel{\text{stably}}{\Longrightarrow} X$ if X is an E-valued random variable defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of the original space and if

$$\lim_n E[Uf(X_n)] = \tilde{E}[Uf(X)] \tag{1}$$

for every bounded continuous $f : E \to \mathbb{R}$ and all bounded measurable random variables U.

Remark 1. As for weak convergence, the bounded continuous functions in the above definition may be replaced with a *convergence determining class*; in particular, we can suppose f to be bounded and uniformly continuous. This implies in particular that if E is endowed with a metric $\rho(\cdot, \cdot)$ and if (Y_n) is another sequence of E-valued random variables defined on $(\Omega, \mathcal{F}, P)$ such that $\rho(X_n, Y_n) \to 0$ in probability, then $Y_n \stackrel{\text{stably}}{\Longrightarrow} X$. In particular, if X_n and Y_n are càdlàg processes viewed as random variables with values in $\mathbb{D}([0, T])$, and $(X_n - Y_n)^* \to 0$ in probability, then $X_n \stackrel{\text{stably}}{\Longrightarrow} X$ implies $Y_n \stackrel{\text{stably}}{\Longrightarrow} X$. Here and in the following, for any process X, we use the notation

$$X^* = \sup_{t \in [0,T]} |X_t|.$$

Remark 2. Suppose that the σ-field $\mathcal{F}$ is generated by a random variable Y. Then (1) is equivalent to

$$\lim_n E[g(Y)f(X_n)] = \tilde{E}[g(Y)f(X)] \tag{2}$$

for every bounded continuous f and every bounded measurable g. However, for a bounded measurable g, one can find a sequence (g_m) of bounded continuous functions such that $g_m(Y) \to g(Y)$ in $L^1(P)$. Hence, it is sufficient to show (2), with g bounded and continuous.

Remark 3. Let $(\Omega_m)_{m \geq 1}$ be a sequence of subsets of Ω with $\lim_m P[\Omega_m] \to 1$. If, for every m,

$$\lim_n E[Uf(X_n)1_{\Omega_m}] = \tilde{E}[Uf(X)1_{\Omega_m}]$$

for every bounded continuous $f : E \to \mathbb{R}$ and all bounded measurable random variables U, then $X_n \stackrel{\text{stably}}{\Longrightarrow} X$.

We fix a time horizon $T < \infty$ (the maturity of the option) and consider all processes up to this horizon.

We start with a one-dimensional standard Brownian motion W and a Poisson random measure J on $[0,T] \times \mathbb{R}$ with intensity measure $dt \times \nu(dx)$ defined on a probability space $(\Omega, \mathcal{F}, P)$, where ν is a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. $\tilde{J}$ denotes the compensated version of J:

$$\tilde{J}(dt \times dz) = J(dt \times dz) - dt \times \nu(dz).$$

Let $(\mathcal{F}_t)_{t\geq 0}$ stand for the natural filtration of W and J completed with null sets.

A Poisson random measure is a sum of a countable number of point masses, and we denote by $(T_i, \Delta J_i)_{i\geq 1}$ the coordinates of these point masses enumerated in any order.

In this paper, we will work with the following class of processes

Definition 1. A Lévy-Itô process is a process X with the representation

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|z|\leq 1} \gamma_s(z)\tilde{J}(ds \times dz) + \int_0^t \int_{|z|>1} \gamma_s(z) J(ds \times dz).$$

where coefficients μ and σ are càdlàg $(\mathcal{F}_t)$-adapted processes and the jump size γ is a random function $\Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ such that the mapping $(\omega, z) \mapsto \gamma_t(z)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable for every t and the mapping $t \to \gamma_t(z)$ is càglàd (left-continuous with right limits) for every ω and z. Furthermore, it satisfies

$$\gamma_t(z)^2 \leq A_t \rho(z)$$

where ρ is a positive deterministic function decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with $\int_{|z|\leq 1} \rho(z)\nu(dz) < \infty$ and A is a càglàd $(\mathcal{F}_t)$-adapted process.

Main assumptions Throughout the paper except for the example at the end of section 3, we suppose that the asset price process S is a Lévy-Itô process satisfying the assumptions of Definition 1, whose coefficients are denoted by μ, σ, γ.

We suppose that there exists a continuous-time trading strategy F which is the strategy that the agent would follow if continuous-time hedging was possible. In incomplete markets, this strategy need not lead to perfect replication, and can be chosen in many different ways; here we do not discuss the relative advantages of different choices of F and suppose simply that it is given by another Lévy-Itô process satisfying the assumptions of Definition 1 whose coefficients are denoted by $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{\gamma}$.

In addition, throughout the paper, we suppose without loss of generality that the interest rates are zero (one can always choose the bank account as numeraire).

Reduction to the case of bounded coefficients Let

$$\mu_t^{(n)} := (-n) \vee \mu_t \wedge n; \qquad \sigma_t^{(n)} := (-n) \vee \sigma_t \wedge n;$$
$$\gamma_t^{(n)}(z) := (-\sqrt{n\rho(n)}) \vee \gamma_t(z) \wedge \sqrt{n\rho(n)};$$

and define

$$S_t^{(n)} = S_0 + \int_0^t \mu_s^{(n)} ds + \int_0^t \sigma_s^{(n)} dW_s + \int_0^t \int_{|z|\leq 1} \gamma_s^{(n)}(z) \tilde{J}(ds \times dz) + \int_0^t \int_{n \geq |z| > 1} \gamma_s^{(n)}(z) J(ds \times dz). \quad (3)$$

$S^{(n)}$ is then a Lévy-Itô process with bounded coefficients and bounded jumps which coincides with S on the set

$$\Omega_n := \{\omega : \sup_{0\leq t\leq T} \max(|\mu_t|, |\sigma_t|, |A_t|) \leq n; J([0,T] \times ((-\infty, -n) \cup (n, \infty)) = 0\}.$$

Since all processes are supposed càdlàg, $P[\Omega_n] \to 1$. Exactly the same logic can be applied to the process F. Given that in this paper we study convergence in law of various processes, by Remark 3 we can and will suppose with no loss of generality that μ, σ, A, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{A}$ are bounded and the processes S and F have bounded jumps. Similarly, we will suppose $\int_{\mathbb{R}} \rho(z)\nu(dz) < \infty$. In this case, the representation (3) can be simplified to

$$S_t = S_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \gamma_s(z) \tilde{J}(ds \times dz); \quad (4)$$

$$F_t = F_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \int_{\mathbb{R}} \tilde{\gamma}_s(z) \tilde{J}(ds \times dz). \quad (5)$$

Discrete hedging and related notation Since continuously rebalancing one's portfolio is unfeasible in practice, we assume that the hedging portfolio is rebalanced at equally spaced dates $t_i = iT/n$, $i = 0, \dots, n$. For $t \in (0, T]$ we denote by $\phi_n(t)$ the rebalancing date immediately before t and by $\psi_n(t)$ the rebalancing date immediately after t:

$$\phi_n(t) = \sup\{t_i, t_i < t\}, \qquad \psi_n(t) = \inf\{t_i, t_i \geq t\}.$$

The trading strategy is therefore piecewise constant and it is assumed to be given by $F_{\phi_n(t)}$.

Unless explicitly defined otherwise, here and in the rest of the paper, adding a superscript n to the process means taking the difference between the process and its discretized version: $X_t^n := X_t - X_{\phi(t)}$. In particular, F_t^n denotes the difference between the continuously rebalanced strategy and the discretely rebalanced one: $F_t^n := F_t - F_{\phi(t)}$.

The value of the hedging portfolio at time t is $V_0 + \int_0^t F_{s-} dS_s$ with continuous hedging and $V_0 + \int_0^t F_{\phi_n(s)} dS_s$ with discrete hedging. In this paper, we study the asymptotic distribution (when $n \to \infty$) of the difference between discrete and continuous hedging

$$U_t^n = \int_0^t (F_{s-} - F_{\phi_n(s)}) dS_s \equiv \int_0^t F_{s-}^n dS_s. \tag{6}$$

The integral $\int_0^t F_{\phi_n(s)} dS_s$ is nothing but a Riemann sum for the stochastic integral $\int_0^t F_{s-} dS_s$, and it is clear that $U_t^n \to 0$ uniformly on compacts in probability. To obtain a nontrivial limiting distribution, this process will therefore be suitably renormalized.

3 Asymptotic distribution of hedging error

Define the renormalized hedging error process by

$$Z_t^n = \sqrt{n} U_t^n = \sqrt{n} \int_0^t F_{s-}^n dS_s. \tag{7}$$

Sometimes we will also need the piecewise constant process

$$\bar{Z}_t^n = Z_{\phi_n(t)}^n.$$

To describe the limiting law of Z^n and $\bar{Z}^n$, let $\overline{W}$ be a standard Brownian motion independent from W and J, and let $(\xi_k)_{k\geq 1}$ and $(\xi'_k)_{k\geq 1}$ be two sequences of independent standard normal random variables and $(\zeta_k)_{k\geq 1}$ a sequence of independent uniform random variables on $[0,1]$, such that the three sequences are independent from each other and all other random elements. Define the process Z by

$$Z_t = \sqrt{\frac{T}{2}} \int_0^t \sigma_s \tilde{\sigma}_s d\overline{W}_s + \sqrt{T} \sum_{i:T_i \leq t} \Delta F_{T_i} \sqrt{\zeta_i} \xi_i \sigma_{T_i} + \sqrt{T} \sum_{i:T_i \leq t} \Delta S_{T_i} \sqrt{1-\zeta_i} \xi'_i \tilde{\sigma}_{T_i -},$$

where $(T_i)_{i\geq 1}$ is an enumeration of the jump times of J.

Theorem 1.

(a) The process $\bar{Z}^n$ converges stably in law to Z on the Skorohod space $\mathbb{D}([0,T])$.

(b) The process Z^n converges stably in finite-dimensional laws to Z.

(c) Suppose that the hedging strategy F and the stock price process S have no diffusion components: $\sigma \equiv \tilde{\sigma} \equiv 0$. Then $(Z^n)^* \to 0$ and $(\bar{Z}^n)^* \to 0$ in probability.

Remark 4. The Skorohod convergence fails for the "interpolated" process Z roughly because we cannot control its behavior between the discretization dates in a uniform fashion. This phenomenon was discovered in [15] in the context of Euler schemes for jump processes, and we refer the reader to this reference for more detailed explanations of its origins.

Remark 5. A similar limiting process appears in the study of the asymptotic behavior of the quadratic variation [13, Theorem 2.11]. This is because the stochastic integral $\int X_{t-}dX_t$ and the quadratic variation $[X,X]$ are closely related via the integration by parts formula.

Example: discrete delta hedging in a Lévy jump-diffusion model In this example we suppose that under the historical probability the stock price follows an exponential Lévy model with nonzero diffusion part and a finite Lévy measure:

$$S_t = S_0 e^{X_t} \quad X_t = bt + \Sigma W_t + \int_{[0,t]\times\mathbb{R}} zJ(ds \times dz)$$
$$S_t = S_0 + \int_0^t (b+\Sigma^2/2)S_s ds + \int_0^t \Sigma S_s dW_s + \int_{[0,t]\times\mathbb{R}} S_{s-}(e^z-1)J(ds\times dz)$$
$$\Rightarrow \quad \mu_t = S_t\left(b+\Sigma^2/2+\int_{|z|\leq 1}(e^z-1)\nu(dz)\right), \quad \sigma_t = \Sigma S_t, \quad \gamma_t = S_{t-}(e^z-1).$$

where $\Sigma > 0$ and J is a Poisson random measure with intensity $ds \times \nu(dz)$ with $\nu(\mathbb{R}) < \infty$. This shows that S is a Lévy-Itô process in the sense of Definition 1 with $A_t = S_{t-}^2$ and $\rho(z) = (e^z-1)^2$.

We assume that the option price may be computed as the expectation of the pay-off under the risk-neutral probability Q, under which X is again a Lévy process. For simplicity, we suppose that X has the same Lévy measure under Q as under P.

$$X_t = \tilde{b}t + \Sigma W_t^Q + \int_{[0,t]\times\mathbb{R}} zJ(ds\times dz),$$

where W^Q is a Q-brownian motion, $\tilde{b}$ is chosen so that e^{X_t} is a martingale and J is a Poisson random measure under Q with compensator $ds \times \nu(dz)$.

We study the hedging of a European option with pay-off function H using the popular delta hedging strategy. The option price is given by

$$C(t,S) = E^Q[H(S_T)|S_t = S] = E^Q[H(Se^{X_{T-t}})]$$

and the strategy is $F_t = \frac{\partial C(t,S_t)}{\partial S}$. This is by far the most widely used hedging strategy and it has an additional merit of being mathematically tractable which makes it a convenient choice for our example. It is not optimal in presence of jumps, but if the jumps are not very violent, it is reasonably close to being optimal.

We impose strong conditions on the Lévy measure, because we want to illustrate the power of our main result based on weak convergence for irregular option pay-offs. In particular, our hypotheses cover the delta hedging of a binary option in the Merton jump-diffusion model. The notation is the same as in Theorem 1.

Proposition 1. *Suppose*

- *The option pay-off function H is bounded and piecewise C^∞ with at most a finite number of discontinuities;*
- *The diffusion coefficient is positive: $\Sigma > 0$;*
- *The Lévy measure ν is finite, has a C^∞ bounded density (also denoted by ν) with $\int_{\mathbb{R}} |x|\nu(x)dx < \infty$ and such that for every $k \geq 1$, $|\nu^{(k)}|$ is integrable.*

Then the renormalized discrete delta-hedging error as defined by (7) *converges stably in finite-dimensional laws to the process*

$$Z_t = \sqrt{\frac{T}{2}} \int_0^t \Sigma^2 S_s^2 \frac{\partial^2 C}{\partial S^2} d\overline{W}_s + \sqrt{T} \sum_{s \leq t: \Delta S_t \neq 0} \left(\frac{\partial C}{\partial S}(s, S_s) - \frac{\partial C}{\partial S}(s, S_{s-}) \right) \sqrt{\zeta_i}\xi_i \Sigma S_s$$
$$+ \sqrt{T} \sum_{s \leq t: \Delta S_t \neq 0} \Delta S_s \sqrt{1-\zeta_i}\xi_i' \Sigma S_{s-} \frac{\partial^2 C}{\partial S^2}(s, S_{s-}) \quad (8)$$

Proof. By Proposition 2 in [4], the option price $C \in C^\infty([0,T) \times \mathbb{R})$ and one can apply the Itô formula to show that F_t has the decomposition

$$dF_t = d\frac{\partial C(t, S_t)}{\partial S} = \left\{ \frac{\partial^2 C}{\partial t \partial S} + (b + \Sigma^2/2)\frac{\partial^2 C}{\partial S^2} S_t + \frac{\Sigma^2}{2}\frac{\partial^3 C}{\partial S^3} S_t^2 \right\} dt$$
$$+ \Sigma \frac{\partial^2 C}{\partial S^2} S_t dW_t + \int_{\mathbb{R}} \left(\frac{\partial C}{\partial S}(t, S_{t-}e^z) - \frac{\partial C}{\partial S} C(t, S_{t-}) \right) J(dt \times dz)$$

We now need to check that the coefficients of this decomposition satisfy the hypotheses of Theorem 1, and the essential point is to show that they do not explode as $t \to T$. We will use the following lemma.

Lemma 1. *Under the assumptions of Proposition 1, for all $k \geq 0$, almost surely, the processes*

$$\frac{\partial^k C(t, S_t)}{\partial S^k} \quad \text{and} \quad \frac{\partial^{k+1} C(t, S_t)}{\partial t \partial S^k}$$

have left limits at T.

Proof of lemma. Let $h(x) := H(e^x)$ and $c(t,x) := C(t, e^x)$, and denote by

$$p_t(x) = \frac{1}{\Sigma\sqrt{2\pi t}} e^{-\frac{(x-\tilde{b}t)^2}{2\Sigma^2 t}} \quad (9)$$

the (risk-neutral) density of $X_t^c := \tilde{b}t + \Sigma W_t$, by $\lambda = \nu(\mathbb{R})$ the jump intensity of X and by $\mu := \nu/\lambda$ the density of its jump size distribution. Then, the following representation for $c(t,x)$ holds true:

$$c(t,x) = e^{-\lambda(T-t)} h * p_{T-t}(x) + \lambda \int_t^T ds\, e^{-\lambda(s-t)} \int_{\mathbb{R}} dz\, \mu(z-x) p_{s-t} * c(s, \cdot)(z)$$

For every $t < T$, since the corresponding derivatives are bounded and integrable, we have for every k,

$$\frac{\partial^k c(t,x)}{\partial x^k} = e^{-\lambda(T-t)} h * \frac{\partial^k p_{T-t}}{\partial x^k}(x) + \lambda \int_t^T ds\, e^{-\lambda(s-t)} \int_{\mathbb{R}} dz (-1)^k \mu^{(k)}(z-x) p_{s-t} * c(s,\cdot)(z). \quad (10)$$

The second term above satisfies

$$\lambda \int_t^T ds e^{-\lambda(s-t)} \int_{\mathbb{R}} dz (-1)^k \mu^{(k)}(z-x) p_{s-t} * c(s,\cdot)(z) = O(T-t),$$

therefore we only need to study the convergence of the first term as $t \to T$. This is done in several small steps:

- Since X_T has absolutely continuous density and h has at most a finite number of discontinuities, for almost all trajectories of X, X_T is not a discontinuity point of h. Fix any such trajectory. Then there exists $\delta > 0$ such that h is smooth on the set $\{x : |X_T - x| < \delta\}$.
- Since $X_t \to X_T$ a.s. as $t \to T$, there exists $t_0 < T$ such that $|X_t - X_T| < \frac{\delta}{2}$ for all $t > t_0$.
- Fix $\varepsilon > 0$. The explicit form of the gaussian density p_t enables us to find t_1 with $T > t_1 \geq t_0$ such that for all $t > t_1$,

$$\int_{|x|>\frac{\delta}{2}} \left| \frac{\partial^k p_{T-t}(x)}{\partial x^k} \right| \sup_z h(z) dx < \frac{\varepsilon}{2}$$

- Therefore, we can find $\tilde{h}$ which is smooth, bounded with bounded derivatives, coincides with h on the set $\{x : |x - X_T| < \frac{\delta}{2}\}$ (which does not contain discontinuities of h) and satisfies

$$\left| \int \frac{\partial^k p_{T-t}(x)}{\partial x^k} h(X_t + x) dx - \int \frac{\partial^k p_{T-t}(x)}{\partial x^k} \tilde{h}(X_t + x) dx \right| < \frac{\varepsilon}{2},$$

for all $t > t_1$. Note that the function $\tilde{h}$ and the size of the set δ depend on the trajectory of X which was fixed above.

- By integration by parts we conclude

$$\int \frac{\partial^k p_{T-t}(x)}{\partial x^k} \tilde{h}(X_t + x) dx = (-1)^k \int p_{T-t}(x) \frac{\partial^k \tilde{h}(X_t + x)}{\partial x^k} dx \xrightarrow{t \to T} (-1)^k \frac{\partial^k \tilde{h}(X_T)}{\partial x^k} = (-1)^k \frac{\partial^k h(X_T)}{\partial x^k}$$

Therefore, for $t > t_1$,

$$\left| \int \frac{\partial^k p_{T-t}(x)}{\partial x^k} h(X_t + x) dx - (-1)^k \frac{\partial^k h(X_T)}{\partial x^k} \right| < \varepsilon.$$

To handle the convergence of the time derivative, one can use the same method but the notation is a little heavier. Differentiating the right-hand side of (10) term by term, we obtain

$$\begin{aligned}\frac{\partial^{k+1}c(t,x)}{\partial x^k\partial t} &= \lambda e^{-\lambda(T-t)}h*\frac{\partial^k p_{T-t}}{\partial x^k}(x)+e^{-\lambda(T-t)}h*\frac{\partial^{k+1}p_{T-t}}{\partial x^k\partial t}(x)\\ &+\lambda^2\int_t^T dse^{-\lambda(s-t)}\int_{\mathbb{R}}dz(-1)^k\mu^{(k)}(z-x)p_{s-t}*c(s,\cdot)(z)\\ &-\int_{\mathbb{R}}(-1)^k\mu^{(k)}(z-x)c(t,z)dz\\ &+\lambda\int_t^T dse^{-\lambda(s-t)}\int_{\mathbb{R}}dz(-1)^k\mu^{(k)}(z-x)\frac{\partial}{\partial t}p_{s-t}*c(s,\cdot)(z). \quad (11)\end{aligned}$$

The convergence of the terms which do not contain the derivative of p_{T-t} with respect to t is easily shown either in the same way as above or using dominated convergence. It remains then to prove the convergence of the second term and of the last term. To get rid of the derivatives with respect to t, we use the Fokker-Planck equation for the Gaussian density:

$$\frac{\partial p_t(x)}{\partial t}=\frac{\Sigma^2}{2}\frac{\partial^2 p_t(x)}{\partial x^2}-\tilde{b}\frac{\partial p_t(x)}{\partial x}.$$

The second term in the RHS of (11) then becomes

$$e^{-\lambda(T-t)}h*\frac{\partial^{k+1}p_{T-t}}{\partial x^k\partial t}(x)=-\frac{\Sigma^2}{2}e^{-\lambda(T-t)}h*\frac{\partial^{k+2}p_{T-t}}{\partial x^{k+2}}(x)+\tilde{b}e^{-\lambda(T-t)}h*\frac{\partial^{k+1}p_{T-t}}{\partial x^{k+1}}(x)$$

and its convergence follows from the argument given in the first part of the proof of this lemma. After integration by parts, the last term in the RHS of (11) becomes

$$\begin{aligned}\lambda\int_t^T dse^{-\lambda(s-t)}\int_{\mathbb{R}}dz(-1)^k\left\{-\frac{\sigma^2}{2}\mu^{(k+2)}(z-x)+\tilde{b}\mu^{(k+1)}(z-x)\right\}p_{s-t}*c(s,\cdot)(z)\\ =O(|T-t|),\end{aligned}$$

hence it converges to zero and the proof of the lemma is completed. □

Proof of Proposition 1 continued. Since $\frac{\partial C}{\partial S}(t,S_t)$ is a.s. càdlàg, the same argument that was used in section 2 to reduce to the case of bounded coefficients, can be used here to replace the strategy F with the strategy $F^{(n)}$ defined by

$$\begin{aligned}dF_t^{(n)} &= \left\{\frac{\partial^2 C}{\partial t\partial S}+(b+\Sigma^2/2)\frac{\partial^2 C}{\partial S^2}S_t+\frac{\Sigma^2}{2}\frac{\partial^3 C}{\partial S^3}S_t^2\right\}dt\\ &+\Sigma\frac{\partial^2 C}{\partial S^2}S_t dW_t+\int_{\mathbb{R}}\left(\zeta_n\left(\frac{\partial C}{\partial S}(t,S_{t-}e^z)\right)-\zeta_n\left(\frac{\partial C}{\partial S}C(t,S_{t-})\right)\right)J(dt\times dz)\end{aligned}$$

with $\zeta_n(x) := (-n) \vee x \wedge n$. Then $F^{(n)}$ is a Lévy-Itô process with coefficients

$$\begin{aligned}
\tilde{\mu}_t^n &= \frac{\partial^2 C}{\partial t \partial S} + (b + \Sigma^2/2)\frac{\partial^2 C}{\partial S^2} S_t + \frac{\Sigma^2}{2}\frac{\partial^3 C}{\partial S^3} S_t^2 \\
&\quad + \int_{|z|\leq 1} \left(\zeta_n \left(\frac{\partial C}{\partial S}(t, S_t e^z) \right) - \zeta_n \left(\frac{\partial C}{\partial S} C(t, S_t) \right) \right) \nu(dz), \\
\tilde{\sigma}_t^n &= \Sigma \frac{\partial^2 C}{\partial S^2} S_t, \\
\tilde{\gamma}_t^n(z) &= \zeta_n \left(\frac{\partial C}{\partial S}(t, S_{t-} e^z) \right) - \zeta_n \left(\frac{\partial C}{\partial S} C(t, S_{t-}) \right),
\end{aligned}$$

which satisfy the hypothesis of Theorem 1 with $\rho(z) \equiv 4n^2$ and $\tilde{A}_t \equiv 1$. Therefore, the desired convergence holds for the strategy $F^{(n)}$ for every n and hence, for the strategy F. □

Estimating Value at Risk of a hedged option position Finally, we would like to illustrate, by a simple heuristic computation, how our result may be useful for managing the risks of a hedged option position. Since we have characterized the *weak* limit of the renormalized hedging error, our method allows to approximate various bounded continuous functionals of the hedging error for a finite time step. As an example, we compute an estimate for the probability that a hedged option position exceeds a given value.

We place ourselves under the hypotheses of Proposition 1 and suppose, in addition, that either the option pay-off is sufficiently regular, or the option is sold before maturity, so that the limiting process Z_t is square integrable (this, of course, does not imply the convergence of $E[(Z_t^n)^2]$). Then, for any $\varepsilon > 0$ the non-renormalized hedging error U_t^n satisfies

$$P[|U_t^n| \geq \varepsilon] \leq E\left[\frac{1}{\varepsilon}|U_t^n| \wedge 1\right] \approx E\left[\frac{1}{\varepsilon}\left|\frac{Z_t}{\sqrt{n}}\right| \wedge 1\right] \leq \frac{1}{\varepsilon\sqrt{n}} E[Z_t^2]^{1/2}, \tag{12}$$

which leads to an upper bound for the Value at Risk at the level δ:

$$\mathrm{VaR}_\delta(U_t^n) \lesssim \frac{1}{\delta\sqrt{n}} E[Z_t^2]^{1/2}. \tag{13}$$

Using the explicit form of the limiting process, $E[Z_t^2]$ can be computed as follows:

$$\begin{aligned}
E[Z_t^2] &= \frac{T}{2}\int_0^t E\left[\Sigma^4 S_s^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right] ds \\
&\quad + \frac{T}{2}\int_0^t \int E\left[\Sigma^2 S_s^2 e^{2z} \left(\frac{\partial C}{\partial S}(s, S_s e^z) - \frac{\partial C}{\partial S}(s, S_s)\right)^2\right] \nu(dz) ds \\
&\quad + \frac{T}{2}\int_0^t \int E\left[\Sigma^2 S_s^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2 (e^z - 1)^2\right] \nu(dz) ds.
\end{aligned}$$

If the jumps are small, then, performing a Taylor development of C, we obtain the following compact formula

$$E[Z_t^2] \approx \frac{T}{2}\int_0^t E\left[S_s^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\right](\Sigma^4 + \Sigma^2 \int (e^z-1)^2(e^{2z}+1)\nu(dz))ds. \quad (14)$$

The estimate (12)–(14) may be compared with the mean square error due to market incompleteness given by [4]:

$$\begin{aligned} E[\epsilon_t^2] &= \int_0^t ds \int \nu(dz) E\left[(C(s,S_s e^z) - C(s,S_s) - S_s(e^z-1)\frac{\partial C}{\partial S})^2\right] \\ &\approx \frac{1}{4}\int_0^t E\left[S_s^4\left(\frac{\partial^2 C}{\partial S^2}\right)^2\right]\int (e^z-1)^4\nu(dz)ds. \quad (15)\end{aligned}$$

We see that the two types of error have very different behavior: while the market incompleteness error (15) is due to jumps and disappears in a model with continuous paths, the discretization error (12)–(14) is due mainly to the presence of a diffusion component; if a diffusion component is absent, this error does not disappear but converges to zero at a faster rate.

4 Proof of the main result

Throughout the proof, C denotes a generic constant which may change from line to line.

Part (c) We can represent S_t as $S_t = S_0 + B_t + M_t^\varepsilon + P_t^\varepsilon$, where

$$B_t = \int_0^t \mu_s ds, \qquad M_t^\varepsilon = \int_0^t \int_{|z|\le\varepsilon} \gamma_{s-}(z)\tilde{J}(ds\times dz)$$
$$\text{and} \quad P_t^\varepsilon = \int_0^t \sigma_s dW_s + \int_0^t\int_{|z|>\varepsilon}\gamma_{s-}(z)\tilde{J}(ds\times dz).$$

The hedging strategy F is represented in a similar way:

$$F_t = F_0 + \tilde{B}_t + \tilde{M}_t^\varepsilon + \tilde{P}_t^\varepsilon.$$

We can now write

$$Z_t^n = \sqrt{n}\int_0^t (\tilde{B}_s^n + \tilde{M}_{s-}^{n,\varepsilon} + \tilde{P}_{s-}^{n,\varepsilon})d(B_s + M_s^\varepsilon + P_s^\varepsilon).$$

This integral may be decomposed in the following way:

$$Z_t^n = J_{1,t}^{n,\varepsilon} + J_{2,t}^{n,\varepsilon} + K_t^{n,\varepsilon} + L_t^n,$$

where

$$J_{1,t}^{n,\varepsilon} = \sqrt{n}\int_0^t \tilde{M}_{s-}^{n,\varepsilon} d(M_s^\varepsilon + P_s^\varepsilon), \tag{16}$$

$$J_{2,t}^{n,\varepsilon} = \sqrt{n}\int_0^t (\tilde{B}_s^n + \tilde{P}_{s-}^{n,\varepsilon}) dM_s^\varepsilon, \tag{17}$$

$$K_t^{n,\varepsilon} = \sqrt{n}\int_0^t (\tilde{B}_s^n + \tilde{P}_{s-}^{n,\varepsilon}) dP_s^\varepsilon, \tag{18}$$

$$L_t^n = \sqrt{n}\int_0^t F_s^n dB_s. \tag{19}$$

We first want to show that

$$\lim_{\varepsilon\downarrow 0}\limsup_n E[(J_{1,t}^{n,\varepsilon})^*] = 0, \tag{20}$$

$$\lim_{\varepsilon\downarrow 0}\limsup_n E[(J_{2,t}^{n,\varepsilon})^*] = 0, \tag{21}$$

$$\lim_n E[(L_t^n)^*] = 0. \tag{22}$$

This will be used in the proof of part (a). Then, we will make the hypothesis $\sigma_s \equiv 0$ and $\tilde{\sigma}_s \equiv 0$ and prove that in this case

$$\lim_{\varepsilon\downarrow 0}\limsup_n E[(K_t^{n,\varepsilon})^*] = 0. \tag{23}$$

as well. By Chebyshev's inequality, this will imply $\lim_n \mathbb{P}((Z^n)^* > \eta) = 0$ $\forall \eta > 0$ and the proof of part (c) will be completed.

Let us consider the first expectation. Applying Jensen's and Doob's inequalities, we get

$$\mathbb{E}[(J_1^{n,\varepsilon})^*] \le \sqrt{\mathbb{E}[(J_1^{n,\varepsilon})^{*2}]} \le 2\sqrt{\mathbb{E}(J_{1,T}^{n,\varepsilon})^2} = 2\sqrt{\mathbb{E}\langle J_1^{n,\varepsilon}, J_1^{n,\varepsilon}\rangle_T}. \tag{24}$$

By definition of $J_1^{n,\varepsilon}$,

$$\langle J_1^{n,\varepsilon}, J_1^{n,\varepsilon}\rangle_T = n\int_0^T (\tilde{M}_t^{n,\varepsilon})^2 \left(\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)\right) dt.$$

By our assumptions,

$$\int_{\mathbb{R}} \gamma_t^2(z)\nu(dz) \le A_t \int \rho(z)\nu(dz) \le C, \qquad \forall t \in [0,T]$$

and σ_t is also bounded. Therefore,

$$\mathbb{E}\langle J_1^{n,\varepsilon}, J_1^{n,\varepsilon}\rangle_T \le Cn\int_0^T \mathbb{E}(\tilde{M}_t^{n,\varepsilon})^2 dt. \tag{25}$$

By the same reasoning, we obtain

$$\mathbb{E}(\tilde{M}_t^{n,\varepsilon})^2 = \mathbb{E}\int_{\phi_n(t)}^t \int_{|z|\le\varepsilon} \tilde{\gamma}_u^2(z)\nu(dz)du$$
$$\le \int_{\phi_n(t)}^t \mathbb{E}[A_t]du \int_{|z|\le\varepsilon} \rho(z)\nu(dz) \le \frac{C}{n}\int_{|z|\le\varepsilon} \rho(z)\nu(dz). \quad (26)$$

Combining (24), (25), and (26) yields

$$\lim_{\varepsilon\to 0}\limsup_n \mathbb{E}[(J_1^{n,\varepsilon})^*] \le \lim_{\varepsilon\to 0} C\left(\int_{|z|\le\varepsilon}\rho(z)\nu(dz)\right)^{1/2} = 0. \quad (27)$$

The term with $J_2^{n,\varepsilon}$ is treated in a similar way:

$$\mathbb{E}[(J_2^{n,\varepsilon})^*] \le 2\sqrt{\mathbb{E}\langle J_2^{n,\varepsilon}, J_2^{n,\varepsilon}\rangle_T}$$
$$= 2\left(n\mathbb{E}\int_0^T (\tilde{B}_t^n + \tilde{P}_t^{n,\varepsilon})^2 \left(\int_{|z|\le\varepsilon}\gamma_t^2(z)\nu(dz)\right)dt\right)^{1/2}$$
$$\le C\left(\int_{|z|\le\varepsilon}\rho(z)\nu(dz)\right)^{1/2}\left(n\int_0^T \mathbb{E}(\tilde{B}_t^n + \tilde{P}_t^{n,\varepsilon})^2 dt\right)^{1/2}. \quad (28)$$

We have

$$\mathbb{E}(\tilde{B}_t^n + \tilde{P}_t^{n,\varepsilon})^2 \le 2(\mathbb{E}(\tilde{B}_t^n)^2 + \mathbb{E}(\tilde{P}_t^{n,\varepsilon})^2). \quad (29)$$

The drift term satisfies:

$$|\tilde{B}_t^n| = \int_{\phi_n(t)}^t |\tilde{\mu}_s|ds \le \frac{C}{n} \quad \text{and therefore} \quad \mathbb{E}(\tilde{B}_t^n)^2 \le \frac{C}{n^2} \quad (30)$$

due to the assumption that $\tilde{\mu}$ is bounded. For the second expectation in (29), we obtain

$$\mathbb{E}(\tilde{P}_t^{n,\varepsilon})^2 = \mathbb{E}\int_{\phi_n(t)}^t \left(\tilde{\sigma}_s^2 + \int_{|z|>\varepsilon}\tilde{\gamma}_s^2(z)\nu(dz)\right)ds \le \frac{C}{n}. \quad (31)$$

The estimates (28), (29), and (31) imply

$$\lim_{\varepsilon\to 0}\limsup_n \mathbb{E}[(J_2^{n,\varepsilon})^*] \le \lim_{\varepsilon\to 0} C\left(\int_{|z|\le\varepsilon}\rho(z)\nu(dz)\right)^{1/2} = 0. \quad (32)$$

Let us now consider the finite variation process L_t^n. We have

$$\mathbb{E}[(L^n)^*] \le \mathbb{E}[\int_0^T |dL_t^n|] = \sqrt{n}\mathbb{E}[\int_0^T |F_t^n||\mu_t|dt] \le C\sqrt{n}\int_0^T \mathbb{E}|F_t^n|dt. \quad (33)$$

We can write

$$\mathbb{E}|F_t^n| \le \mathbb{E}|\tilde{B}_t^n| + \mathbb{E}|\tilde{M}_t^{n,\varepsilon}| + \mathbb{E}|\tilde{P}_t^{n,\varepsilon}|. \tag{34}$$

Putting together (26), (30), (31), we obtain

$$\limsup_n \mathbb{E}[(L^n)^*] \le C\left(\int_{|z|\le\varepsilon} \rho(z)\nu(dz)\right)^{1/2} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore,

$$\lim_n \mathbb{E}[(L^n)^*] = 0. \tag{35}$$

This finishes the proof of (20)–(22) in the general case.

From now and until the end of proof of part (c), we suppose that the processes S and F have no diffusion component: $\sigma_s \equiv \tilde{\sigma}_s \equiv 0$. Consider now the term $K^{n,\varepsilon}$ which is a finite variation process:

$$\begin{aligned}\mathbb{E}[(K^{n,\varepsilon})^*] \le \mathbb{E}[\int_0^T |dK_t^{n,\varepsilon}|] &\le \sqrt{n}\mathbb{E}[\int_0^T |\tilde{B}_t^n + \tilde{P}_{t-}^{n,\varepsilon}||dP_t^\varepsilon|] \\ &\le \sqrt{n}\left(\frac{C}{n}\mathbb{E}\int_0^T |dP_t^\varepsilon| + \mathbb{E}\int_0^T |\tilde{P}_{t-}^{n,\varepsilon}||dP_t^\varepsilon|\right). \end{aligned} \tag{36}$$

To bound the terms involving $\tilde{P}_t^{n,\varepsilon}$ and P_t^ε, let us introduce the following Poisson process:

$$N_t^\varepsilon = \int_0^t \int_{|z|>\varepsilon} J(dt\times dz) \qquad \text{with} \qquad \lambda(\varepsilon) = \frac{1}{t}\mathbb{E}N_t^\varepsilon = \int_{|z|>\varepsilon} \nu(dz). \tag{37}$$

We will use the following type of estimate:

$$\begin{aligned}|dP_t^\varepsilon| &\le \int_{|z|>\varepsilon} |\gamma_{t-}(z)||\tilde{J}(dt\times dz)| \\ &\le C\left(\int_{|z|>\varepsilon} J(dt\times dz) + \int_{|z|>\varepsilon} \nu(dz)dt\right) = C(dN_t^\varepsilon + \lambda(\varepsilon)dt). \end{aligned} \tag{38}$$

This implies, in particular,

$$\mathbb{E}\int_0^T |dP_t^\varepsilon| \le 2C\lambda(\varepsilon)T \le C(\varepsilon). \tag{39}$$

We also have

$$|\tilde{P}_t^{n,\varepsilon}| \le C\left[N_t^{n,\varepsilon} + \lambda(\varepsilon)(t-\phi_n(t))\right] \tag{40}$$

and

$$\mathbb{E}|\tilde{P}_t^{n,\varepsilon}| \le 2C\lambda(\varepsilon)(t-\phi_n(t)) \le \frac{C(\varepsilon)}{n}. \tag{41}$$

Consider the second term in (36):

$$\mathbb{E}\int_0^T |\tilde{P}^{n,\varepsilon}_{t-}||dP^\varepsilon_t| \le C\mathbb{E}\int_0^T [N^{n,\varepsilon}_{t-} + \lambda(\varepsilon)(t-\phi_n(t))][dN^\varepsilon_t + \lambda(\varepsilon)dt]$$
$$\le C\left[\mathbb{E}\int_0^T N^{n,\varepsilon}_{t-}dN^\varepsilon_t + \frac{C(\varepsilon)}{n}\right]. \quad (42)$$

The expectation in (42) may be computed explicitly. Indeed,

$$\mathbb{E}\int_0^T N^{n,\varepsilon}_{t-}dN^\varepsilon_t = \sum_{i=1}^n \mathbb{E}\int_{t_{i-1}}^{t_i}(N^\varepsilon_{t-} - N^\varepsilon_{t_{i-1}})dN^\varepsilon_t = n\mathbb{E}\int_0^{T/n} N^\varepsilon_{t-}dN^\varepsilon_t. \quad (43)$$

If M denotes the (random) number of jumps in the interval $[0,T/n]$, then

$$\mathbb{E}\int_0^T N^{n,\varepsilon}_{t-}dN^\varepsilon_t = n\mathbb{E}\sum_{j=1}^M (j-1) = n\mathbb{E}\left[\frac{M(M-1)}{2}\right] = \frac{(\lambda(\varepsilon)T)^2}{2n}. \quad (44)$$

We conclude that

$$\mathbb{E}\int_0^T |\tilde{P}^{n,\varepsilon}_{t-}||dP^\varepsilon_t| \le \frac{C(\varepsilon)}{n} \quad (45)$$

and

$$\mathbb{E}[(K^{n,\varepsilon})^*] \le \frac{C(\varepsilon)}{\sqrt{n}}, \quad (46)$$

which implies

$$\lim_{\varepsilon\to 0}\limsup_n \mathbb{E}[(K^{n,\varepsilon})^*] = 0. \quad (47)$$

Part (a) **Step 1**: Removing small jumps. Fix $\varepsilon > 0$ and define

$$S^\varepsilon_t = S_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t\int_{|z|>\varepsilon} \gamma_{s-}(z)\tilde{J}(ds\times dz);$$
$$F^\varepsilon_t = F_0 + \int_0^t \tilde{\mu}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t\int_{|z|>\varepsilon} \tilde{\gamma}_{s-}(z)\tilde{J}(ds\times dz);$$
$$\bar{Z}^{\varepsilon,n}_t = \sqrt{n}\int_0^{\phi_n(t)} F^{\varepsilon,n}_{s-}dS^\varepsilon_s; \qquad Z^{\varepsilon,n}_t = \sqrt{n}\int_0^t F^{\varepsilon,n}_{s-}dS^\varepsilon_s;$$
$$Z^\varepsilon_t = \sqrt{\frac{T}{2}}\int_0^t \sigma_s\tilde{\sigma}_s d\overline{W}_s + \sqrt{T}\sum_{i:T_i\le t} 1_{|\Delta J_i|>\varepsilon}\Delta F^\varepsilon_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i}$$
$$+ \sqrt{T}\sum_{i:T_i\le t} 1_{|\Delta J_i|>\varepsilon}\Delta S^\varepsilon_{T_i}\sqrt{1-\zeta_i}\xi'_i\tilde{\sigma}_{T_i-}.$$

From equations (20)–(22), it follows that

$$\lim_{\varepsilon\to 0}\limsup_n P[(Z^n - Z^{\varepsilon,n})^* > \eta] = 0, \quad \forall \eta > 0. \tag{48}$$

Then, we clearly have

$$\lim_{\varepsilon\to 0}\limsup_n P[(\bar Z^n - \bar Z^{\varepsilon,n})^* > \eta] = 0, \quad \forall \eta > 0.$$

If we are now able to prove that

$$\lim_{\varepsilon\to 0} P[(Z - Z^{\varepsilon})^* > \eta] = 0, \quad \forall \eta > 0; \tag{49}$$

$$\bar Z^{\varepsilon,n} \stackrel{\text{stably}}{\Longrightarrow} Z^{\varepsilon}, \tag{50}$$

then, for any bounded random variable U and any bounded uniformly continuous function $f : \mathbb{D}([0,T]) \to \mathbb{R}$, we can write

$$\begin{aligned}\lim_n E[U(f(\bar Z^n) - f(Z))] &= \lim_{\varepsilon\to 0}\lim_n E[U(f(\bar Z^n) - f(\bar Z^{\varepsilon,n}))] \\ &+ \lim_{\varepsilon\to 0}\lim_n E[U(f(\bar Z^{\varepsilon,n}) - f(Z^{\varepsilon}))] + \lim_{\varepsilon\to 0} E[U(f(Z^{\varepsilon}) - f(Z))] = 0.\end{aligned}$$

and the proof of part (a) will be completed.

Proof of (49) By construction of Z and Z^{ε},

$$Z_t - Z^{\varepsilon}_t = \sqrt{T}\sum_{i:T_i\le t} 1_{|\Delta J_i|\le\varepsilon}\Delta F^{\varepsilon}_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i} + \sqrt{T}\sum_{i:T_i\le t} 1_{|\Delta J_i|\le\varepsilon}\Delta S^{\varepsilon}_{T_i}\sqrt{1-\zeta_i}\xi'_i\tilde\sigma_{T_i-}.$$

We will prove the convergence for the first term in the right-hand side, denoted by $Z^{1,\varepsilon}$, the second term can be treated in the same fashion. Conditionnally on the sigma-field $\mathcal{G}$, generated by J and W, $Z^{1,\varepsilon}$ is a martingale, therefore, by Doob's inequality,

$$E[(Z^{1,\varepsilon})^{*2}|\mathcal{G}] \le 4E[(Z^{1,\varepsilon}_T)^2|\mathcal{G}] = 2T\sum_i 1_{|\Delta J_i|\le\varepsilon}(\Delta F^{\varepsilon}_{T_i}\sigma_{T_i})^2.$$

Further, from the boundedness of σ and A,

$$\begin{aligned}E[(Z^{1,\varepsilon})^{*2}] &\le 2TCE\left[\sum_i 1_{|\Delta J_i|\le\varepsilon}(\Delta F^{\varepsilon}_{T_i})^2\right] \\ &= 2TCE\left[\int_0^T\int_{|z|\le\varepsilon}\tilde\gamma^2_s(z)\nu(dz)\right] \le 2C'T\int_{|z|\le\varepsilon}\rho(z)\nu(dz) \to 0,\end{aligned}$$

where C and C' are constants.

This finishes step 1 and it remains to prove (50).

Step 2. We fix $\varepsilon > 0$ and write

$$\begin{aligned}
S_t^\varepsilon &= S_0 + S_t^d + S_t^c + S_t^j,\\
S_t^d &= \int_0^t (\mu_s - \int_{|z|>\varepsilon} \gamma_s(z)\nu(dz))ds := \int_0^t b_s ds,\\
S_t^c &= \int_0^t \sigma_s dW_s,\\
S_t^j &= \int_0^t \int_{|z|>\varepsilon} \gamma_{s-}(z)J(ds \times dz),
\end{aligned}$$

and similarly $F_t^\varepsilon = F_0 + F_t^d + F_t^c + F_t^j$. We would like to show that

$$\begin{aligned}
P[(\bar{Z}^{\varepsilon,n} - \sqrt{n}\int_0^{\phi_n(\cdot)} F^{c,n} dS^c - \sqrt{n}\int_0^{\phi_n(\cdot)} F^{j,n} dS^c \\
- \sqrt{n}\int_0^{\phi_n(\cdot)} F^{c,n} dS^j)^* > \eta] \to 0, \quad \forall \eta > 0. \quad (51)
\end{aligned}$$

Suppose this is proven. Then instead of $\bar{Z}^{\varepsilon,n}$ it is sufficient to study the convergence of the process

$$\tilde{Z}_t^{\varepsilon,n} = \sqrt{n}\int_0^{\phi_n(t)} F^{c,n} dS^c + \sqrt{n}\int_0^{\phi_n(t)} F^{j,n} dS^c + \sqrt{n}\int_0^{\phi_n(t)} F^{c,n} dS^j$$

(see Remark 1). These three terms correspond to the three terms in the definition of the limiting process Z^ε.

Proof of (51) Write $\bar{Z}^{\varepsilon,n}$ as

$$\bar{Z}_t^{\varepsilon,n} = \sqrt{n}\int_0^{\phi_n(t)} (F^d + F^c + F^j)^n d(S^d + S^c + S^j).$$

The different terms satisfy:

$$\left|\int_0^{\phi_n(\cdot)} F^{d,n} dS^d\right|^* \le \frac{T^2}{n}\sup|\tilde{b}|\sup|b|; \quad \left|\int_0^{\phi_n(\cdot)} F^{d,n} dS^j\right|^* \le \frac{T}{n}\sup|\tilde{b}|\sup|\Delta S|N,$$

where N is the number of jumps of S^ε. Using Doob's maximal inequality,

$$\begin{aligned}
E\left[\left(\int_0^{\phi_n(\cdot)} F^{d,n} dS^c\right)^*\right] &\le 2E\left[\left(\int_0^T F^{d,n} dS^c\right)^2\right]^{\frac{1}{2}}\\
&= 2E\left[\int_0^T (F_s^{d,n})^2 \sigma_s^2 ds\right]^{\frac{1}{2}} \le \frac{2T^{3/2}}{n}\sup|\tilde{b}|\sup\sigma.
\end{aligned}$$

Further, the expression
$$\int_0^{\phi_n(t)} F_s^{j,n} dS_s^j$$
is different from zero only if there exists a discretization interval containing at least two jumps of J, which is an event with probability of order $\frac{1}{n}$, and
$$|\int_0^{\phi_n(\cdot)} F^{j,n} dS^d|^* \leq \frac{T}{n} N \sup |\Delta F| \sup |b|,$$
because the integral is only nonzero on the intervals on which there is at least one jump of J. The last vanishing term,
$$\sqrt{n}\int_0^{\phi_n(t)} F_s^{c,n} dS_s^d = \sqrt{n}\int_0^{\phi_n(t)} F_s^{c,n} b_s ds$$
is a little more difficult to analyze. First, let us show that b_s in the above expression can be replaced with $b_{\phi_n(s)}$. Indeed,
$$\begin{aligned} E\left|\sqrt{n}\int_0^{\phi_n(\cdot)} F_s^{c,n}(b_s - b_{\phi_n(s)})ds\right|^* &\leq \sqrt{n}\int_0^T E[(F_s^c - F_{\phi_n(s)}^c)^2]^{1/2} E[(b_s - b_{\phi_n(s)})^2]^{1/2} ds \\ &\leq \sqrt{T} \sup \tilde{\sigma} \int_0^T E[(b_s - b_{\phi_n(s)})^2]^{1/2} ds. \end{aligned}$$
Since b is càdlàg and bounded, $b_{\phi_n(s)} \to b_s$ in L^2 and almost everywhere on $[0, T]$, so that the above expression converges to zero. Now, changing the order of integration,
$$\int_0^{\phi_n(t)} F_s^{c,n} b_{\phi_n(s)} ds = \int_0^{\phi_n(t)} b_{\phi_n(s)}(\psi_n(s) - s)\tilde{\sigma}_s dW_s$$
with $\psi_n(t) = \inf\{t_i : t_i > t\}$. Therefore,
$$E\left|\int_0^{\phi_n(\cdot)} F_s^{c,n} b_{\phi_n(s)} ds\right|^* \leq \frac{2T^{3/2}}{n} \sup |b| \sup \tilde{\sigma}$$
and we have (51).

Step 3. From now on, T_i will denote the moments of jumps of J bigger than ε in absolute value. In this step, our goal is to prove that the process $\tilde{Z}^{\varepsilon,n}$ converges to the same limit as the process
$$\check{Z}_t^{\varepsilon,n} = \int_0^t \sigma_s \tilde{\sigma}_s dN_s^n + \sqrt{n} \sum_{i:T_i<t} \Delta F_{T_i} \sigma_{T_i} \int_{T_i}^{\psi_n(T_i)} dW_s + \sqrt{n} \sum_{i:T_i<t} \Delta S_{T_i} (\tilde{\sigma}_{T_i-}) \int_{\phi_n(T_i)}^{T_i} dW_s$$

where $N^n := \sqrt{n}\int_0^\cdot W_s^n dW_s$. Let Ω^n denote the event "on every discretization interval there is at most one jump of J bigger than ε in absolute value". Then Ω^n increases to Ω and on Ω^n

$$\tilde{Z}_t^{\varepsilon,n} = \sqrt{n}\int_0^{\phi_n(t)} F^{c,n} dS^c + \sqrt{n}\sum_{i:T_i<\phi_n(t)} \Delta F_{T_i}\int_{T_i}^{\psi_n(T_i)} \sigma_s dW_s$$
$$+ \sqrt{n}\sum_{i:T_i<\phi_n(t)} \Delta S_{T_i}\int_{\phi_n(T_i)}^{T_i} \tilde{\sigma}_s dW_s \qquad (52)$$

Further, it follows from Lemma 2.2 in [15] that the process in the right-hand side of (52) converges to the same limit as

$$\hat{Z}_t^{\varepsilon,n} = \sqrt{n}\int_0^t F^{c,n} dS^c + \sqrt{n}\sum_{i:T_i<t} \Delta F_{T_i}\int_{T_i}^{\psi_n(T_i)} \sigma_s dW_s + \sqrt{n}\sum_{i:T_i<t} \Delta S_{T_i}\int_{\phi_n(T_i)}^{T_i} \tilde{\sigma}_s dW_s. \qquad (53)$$

To complete step 3, it remains to prove that

$$\lim_{n\to\infty} P[(\check{Z}^{\varepsilon,n} - \hat{Z}^{\varepsilon,n})^* > \delta] = 0 \quad \forall \delta > 0$$

where

$$\check{Z}_t^{\varepsilon,n} - \hat{Z}_t^{\varepsilon,n} = \sqrt{n}\int_0^t \sigma_s dW_s \int_{\phi_n(s)}^s (\tilde{\sigma}_s - \tilde{\sigma}_r) dW_r$$
$$+ \sqrt{n}\sum_{i:T_i<t} \Delta F_{T_i}\int_{T_i}^{\psi_n(T_i)} (\sigma_{T_i} - \sigma_s) dW_s + \sqrt{n}\sum_{i:T_i<t} \Delta S_{T_i}\int_{\phi_n(T_i)}^{T_i} (\tilde{\sigma}_{T_i-} - \tilde{\sigma}_s) dW_s \qquad (54)$$

Using the boundedness of σ and Burkholder's inequality, we obtain, for the first term above:

$$E\left(\sqrt{n}\int_0^t \sigma_s dW_s \int_{\phi_n(s)}^s (\tilde{\sigma}_s - \tilde{\sigma}_r) dW_r\right)^2 \le Cn\int_0^t E\left(\int_{\phi_n(s)}^s (\tilde{\sigma}_s - \tilde{\sigma}_r) dW_r\right)^2 ds$$
$$\le Cn\int_0^t E\left(\int_{\phi_n(s)}^s (\tilde{\sigma}_r - \tilde{\sigma}_{\phi_n(s)}) dW_r\right)^2 ds + Cn\int_0^t E\left((\tilde{\sigma}_s - \tilde{\sigma}_{\phi_n(s)})\int_{\phi_n(s)}^s dW_r\right)^2 ds$$
$$\le Cn\int_0^t \int_{\phi_n(s)}^s E(\tilde{\sigma}_r - \tilde{\sigma}_{\phi_n(s)})^2 dr ds + C\int_0^t \left[E(\tilde{\sigma}_s - \tilde{\sigma}_{\phi_n(s)})^4\right]^{1/2} ds$$
$$= Cn\int_0^t dr(\psi_n(r) - r)E(\tilde{\sigma}_r - \tilde{\sigma}_{\phi_n(r)})^2 + C\int_0^t \left[E(\tilde{\sigma}_s - \tilde{\sigma}_{\phi_n(s)})^4\right]^{1/2} ds \to 0$$

because $\tilde{\sigma}$ is càdlàg and bounded. Let us now turn to the last two terms of (54). Since we can limit the sums above to a finite number of terms and suppose that

ΔF_{T_i} and ΔS_{T_i} are bounded by a deterministic constant, it would be sufficient to show that for each i, the random variables

$$A_i^n = \sqrt{n}\int_{T_i}^{\psi_n(T_i)} (\sigma_{T_i} - \sigma_s)dW_s \quad \text{and} \quad B_i^n = \sqrt{n}\int_{\phi_n(T_i)}^{T_i} (\tilde{\sigma}_{T_i-} - \tilde{\sigma}_s)dW_s$$

converge to zero in probability as $n \to \infty$. Since $\psi_n(T_i)$ is a stopping time, we easily get

$$E[(A_i^n)^2] = nE\left[\int_{T_i}^{\psi_n(T_i)} (\sigma_{T_i} - \sigma_s)^2 ds\right]$$

and then, since $\lim_{s\downarrow T_i}\sigma_s = \sigma_{T_i}$, it follows, using the dominated convergence theorem that $E[(A_i^n)^2] \to 0$ and A_i^n converges to 0 in probability.

With B_i^n the situation is more complicated because $\phi_n(T_i)$ is not a stopping time. Our argument uses the independence of the jump times from the Brownian motion W. First, decompose B_i^n as

$$\begin{aligned}
B_i^n &= B_i'^n + B_i''^n,\\
B_i'^n &= \sqrt{n}(W_{T_i} - W_{\phi_n(T_i)})(\tilde{\sigma}_{T_i-} - \tilde{\sigma}_{\phi_n(T_i)}),\\
B_i''^n &= \sqrt{n}\int_{\phi_n(T_i)}^{T_i} (\tilde{\sigma}_{\phi_n(T_i)} - \tilde{\sigma}_s)dW_s.
\end{aligned}$$

Using the independence of W and the jump times and the dominated convergence theorem,

$$E[|B_i'^n|] \leq E[n(W_{T_i} - W_{\phi_n(T_i)})^2]^{1/2} E[(\tilde{\sigma}_{T_i-} - \tilde{\sigma}_{\phi_n(T_i)})^2]^{1/2} \leq E[(\tilde{\sigma}_{T_i-} - \tilde{\sigma}_{\phi_n(T_i)})^2]^{1/2} \to 0$$

Next, let $\mathcal{F}_t^J$ be the σ-field generated by the trajectory of W up to time t and by the entire trajectory of J. Then, W is a $\mathcal{F}^J$–Brownian motion, and $(\tilde{\sigma}_{\phi_n(T_i)} - \tilde{\sigma}_s)1_{s\geq\phi_n(T_i)}$ is an $\mathcal{F}^J$-adapted process. Therefore, we can apply to $B_i''^n$ the same argument that we used for A_i^n. This finishes step 3.

Step 4 Denote

$$\alpha_i^n = \sqrt{n}\int_{T_i}^{\psi_n(T_i)} dW_s, \quad \beta_i^n = \sqrt{n}\int_{\phi_n(T_i)}^{T_i} dW_s, \quad \alpha_i = \sqrt{T\zeta_i}\xi_i, \quad \beta_i = \sqrt{T(1-\zeta_i)}\xi_i'.$$

if $T_i < T$ and $\alpha_i^n = \beta_i^n = \alpha_i = \beta_i = 0$ if $T_i \geq T$.

In this step, we want to show that $(N^n, (\alpha_i^n)_{i\geq 1}, (\beta_i^n)_{i\geq 1})$ converges stably in law to $(\sqrt{\frac{T}{2}}\overline{W}, (\alpha_i)_{i\geq 1}, (\beta_i)_{i\geq 1})$.

First, notice that the Poisson random measure J that we use can be "packed" into a martingale pure jump Lévy process $L_t := \int_0^t x\tilde{J}(dt\times dx)$, that is, a random variable in $\mathcal{F}_T$ can be represented as a measurable function of L and W. The result of this step then follows from Lemma 6.2 in [15] (taking for the Lévy process Y in this lemma the sum of a standard Brownian motion and a Poisson process).

Step 5 In the previous step, we proved the convergence of different quantities which make up $\check{Z}^{\varepsilon,n}$. It remains to assemble them together and prove the convergence of the whole process.

We will use the following obvious properties of the stable convergence.

(a) Let $X_n \stackrel{\text{stably}}{\Longrightarrow} X$, and Y be another random variable with values in a Polish space F. Then $(Y, X_n) \stackrel{\text{stably}}{\Longrightarrow} (Y, X)$ for the product topology on $F \times E$.

(b) If $X_n \stackrel{\text{stably}}{\Longrightarrow} X$, and $f : E \to F$ is a continuous function, then $f(X_n) \stackrel{\text{stably}}{\Longrightarrow} f(X)$ for the topology of F.

First, suppose without loss of generality (Remark 3), that the number of jump times T_i is bounded by a finite number N on the interval $[0, T]$. From stable convergence of $(\alpha_i^n)_{i\geq 1}$ and $(\beta_i^n)_{i\geq 1}$, we can deduce the stable convergence of $(\Delta F_{T_i}\sigma_{T_i}\alpha_i^n)_{i\geq 1}$ and $(\Delta S_{T_i}\tilde{\sigma}_{T_i-}\beta_i^n)_{i\geq 1}$ (using property (a) and property (b) with $f : (x, y) \mapsto xy$ from $\mathbb{R}^2$ to $\mathbb{R}$). Further, the mapping

$$(a_1, \dots, a_N, b_1, \dots, b_N, t_1, \dots t_N) \mapsto \sum_{t_i \leq \cdot} a_i + \sum_{t_i \leq \cdot} b_i$$

is continuous from $\mathcal{B}(\mathbb{R}^{3N})$ to $D([0, T])$ at every point such that $t_1 < t_2 < \dots < t_N$ (see Example VI.1.20 in [16]). Property (b) then implies that

$$\sum_{T_i \leq \cdot} \Delta F_{T_i}\sigma_{T_i}\alpha_i^n + \sum_{T_i \leq \cdot} \Delta S_{T_i}\tilde{\sigma}_{T_i-}\beta_i^n \stackrel{\text{stably}}{\Longrightarrow} \sum_{T_i \leq \cdot} \Delta F_{T_i}\sigma_{T_i}\alpha_i + \sum_{T_i \leq \cdot} \Delta S_{T_i}\tilde{\sigma}_{T_i-}\beta_i \tag{55}$$

Second, fix an integer m and consider the function

$$(x, y) \mapsto \int_0^{\cdot} x_{\phi_m(s)} dy_s \tag{56}$$

from $D([0, T])^2$ to $D([0, T])$. This function can also be written as

$$\int_0^t x_{\phi_m(s)} dy_s = \sum_{i=0}^{m-1} x_{t_i}(y_{t_{i+1}\wedge t} - y_{t_i \wedge t}), \quad t_i = \frac{iT}{m}$$

We recall that a sequence (x_n) of càdlàg functions on $[0, T]$ converges to x in the Skorohod topology if there exists a sequence of time changes λ_n such that $\lambda_n(t) \to t$ and $x_n \circ \lambda_n(t) \to x(t)$ as $n \to \infty$ uniformly on t. Moreover, if x is continuous, convergence to x in the Skorohod topology is equivalent to convergence in the uniform topology.

Function (56) is continuous for the product topology on $D([0, T])^2$ at every point (x, y) such that y is a continuous function and x has no jump times in the set $\{t_i\}_{i=0}^{m-1}$. Indeed, let $(x_n, y_n) \to (x, y)$ in the product topology. Then

$$\int_0^t x^n_{\phi_m(s)} dy^n_s - \int_0^t x_{\phi_m(s)} dy_s$$
$$= \sum_{i=0}^{m-1} x^n_{t_i}(y^n_{t_{i+1}\wedge t} - y^n_{t_i\wedge t} - y_{t_{i+1}\wedge t} + y_{t_i \wedge t}) + \sum_{i=0}^{m-1}(x^n_{t_i} - x_{t_i})(y_{t_{i+1}\wedge t} - y_{t_i\wedge t}).$$

The first term above converges to 0 uniformly in t because y is continuous, and hence y^n converges to y in the uniform topology. The convergence of the second term follows from the convergence of $x^n_{t_i}$ to x_{t_i} for every i (Remark VI.2.3 in [16]). Using again properties (a) and (b), we obtain

$$\int_0^{\cdot} \sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)} dN^n_s \stackrel{\text{stably}}{\Longrightarrow} \sqrt{\frac{T}{2}} \int_0^{\cdot} \sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)} d\overline{W}_s$$

Now, let

$$R^m_t := \int_0^t (\sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)} - \sigma_s\tilde{\sigma}_s) dN^n_s.$$

From Doob's inequality it follows that

$$E[(R^m)^*] \leq 2\sqrt{E[(R^m_T)^2]},$$

and by Burkholder's inequality we have

$$\begin{aligned} E[(R^m_T)^2] &= nE\left[\int_0^T (\sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)} - \sigma_s\tilde{\sigma}_s)^2 (W^n_s)^2 ds\right] \\ &\leq C\int_0^T E\left(\sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)} - \sigma_s\tilde{\sigma}_s)^4\right]^{1/2} ds. \end{aligned}$$

Since $\sigma_s\tilde{\sigma}_s$ is almost everywhere continuous and bounded, we then have

$$\lim_{m\to\infty} \limsup_{n\to\infty} P[(\int_0^{\cdot} (\sigma_s\tilde{\sigma}_s - \sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)}) dN^n_s)^* > \eta] = 0, \quad \forall \eta > 0.$$

Similarly, we obtain

$$\lim_{m\to\infty} P[(\sqrt{\frac{T}{2}} \int_0^{\cdot} (\sigma_s\tilde{\sigma}_s - \sigma_{\phi_m(s)}\tilde{\sigma}_{\phi_m(s)}) d\overline{W}_s)^* > \eta] = 0, \quad \forall \eta > 0.$$

Therefore, by an argument similar to the one used after equations (49)–(50) to remove small jumps,

$$\int_0^{\cdot} \sigma_s\tilde{\sigma}_s dN^n_s \stackrel{\text{stably}}{\Longrightarrow} \sqrt{\frac{T}{2}} \int_0^{\cdot} \sigma_s\tilde{\sigma}_s d\overline{W}_s \tag{57}$$

Finally, the function $(x, y) \mapsto x + y$ from $D([0, T])^2$ to $D([0, T])$ is continuous for the product topology on $D([0, T])^2$ at every point (x, y) such that x is continuous (Proposition VI.1.23 in [16]). Therefore, since the process $\int_0^{\cdot} \sigma_s\tilde{\sigma}_s d\overline{W}_s$ is continuous, combining (55) and (57) we obtain that $\check{Z}^{\varepsilon,n}$ converges stably to Z^{ε}.

Part (b) As in the proof of the part (a), it is sufficient to prove that $Z^{\varepsilon,n} \to Z^{\varepsilon}$ stably in finite-dimensional distributions (see the argument in step 1). We will prove the convergence of the random variable $Z_t^{\varepsilon,n}$ to Z_t^{ε}, the generalization to m-tuples $(Z_{t_1}^{\varepsilon,n}, Z_{t_2}^{\varepsilon,n}, \ldots, Z_{t_m}^{\varepsilon,n})$ being straightforward. By the proof of part (a), step 2, it is sufficient to study the convergence of the random variable

$$\sqrt{n}\int_0^t F^{c,n} dS^c + \sqrt{n}\int_0^t F^{j,n} dS^c + \sqrt{n}\int_0^t F^{c,n} dS^j. \tag{58}$$

Let Ω^n denote the event "on every discretization interval there is at most one jump of J bigger than ε in absolute value and there are no jumps in the interval $[\phi_n(t), t]$". Then, since J is a Poisson random measure and has no fixed jumps, Ω^n increases to Ω. On the other hand, on Ω^n, the random variable defined by expression (58) is equal to $\hat{Z}_t^{\varepsilon,n}$ defined by equation (53), and we have shown that $\hat{Z}_t^{\varepsilon,n} \to Z_t^{\varepsilon}$ stably in law.

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