Valuation of Guaranteed Annuity Options using a Stochastic Volatility Model for Equity Prices

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Motivation

- A Guaranteed Annuity Option (GAO) gives the holder the right to receive at the retirement data T either a cash payment equal to the investment in the equity fund $S(T)$ or a life annuity of this investment against the guaranteed rate g.
- Terminal payoff

$$\begin{aligned} H(T) &= \left(gS(T) \sum_{i=0}^{n} c_i P(T, t_i) - S(T) \right)^+ \\ &= gS(T) \left(\sum_{i=0}^{n} c_i P(T, t_i) - K \right)^+ \end{aligned}$$

$P(T, t_i)$: discount factor,
c_i: probability of survival till time t_i, independent of $S(T)$.

- GAOs were a common feature in retirement savings contracts in the UK.
- Currently, similar options are frequently sold as Guaranteed Minimum Income Benefit (GMIB) in the U.S. and Japan as part of variable annuity offerings. These markets have witnessed an explosively over expansion the last past years, and a growth in Europe is also expected, e.g. see Wyman (2007).
- A vast literature on the pricing and risk management of deferred annuity products has emerged.
  - The risk management and hedging of GAOs and GMIBS by Dunbar (1999), Yang (2001), Wilkie et al. (2003) and Pelsser (2003).
  - Approaches for the pricing of GAOs are in van Bezooyen et al. (1998), Milevsky and Promislow (2001), Ballotta and Haberman (2003), Boyle and Hardy (2003), Biffis and Millossovich (2006), Chu and Kwok (2007), Bauer et al. (2008) and Marshall et al. (2009).

Stochastic Volatility

- Generally a geometric Brownian motion is assumed for equity prices, e.g. the Black-Scholes-Hull-White (BSHW) model

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_S dW_S^{\mathbb{Q}}(t),$$

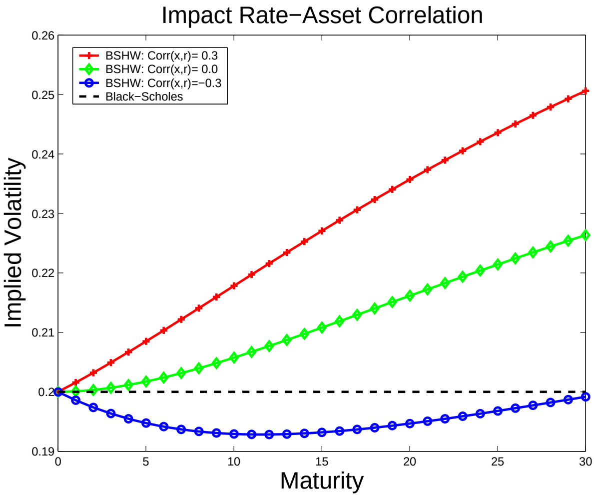
with the short interest rate $r(t)$ according Hull and White (1993).

- To grasp the impact of stochastic volatility, we consider the Schöbel-Zhu-Hull-White (SZHW) model:

$$\frac{dS(t)}{S(t)} = r(t)dt + \nu(t)dW_S^{\mathbb{Q}}(t)$$
$$\nu(t) = \kappa\Big(\psi - \nu(t)\Big)dt + \tau dW_\nu^{\mathbb{Q}}(t)$$

- Full correlation structure between all underlying processes and with closed-form pricing formulas for vanilla options using Fourier inversion techniques, see van Haastrecht et al. (2008).
- Closed-form prices formulas are a big advantage for the calibration of the model.

- Having a realistic correlation structure is of practical importance for the pricing and hedging of long-term exotic options, such as GAOs.
- Correlation between the equity index and the interest rates, for instance, gives additional flexibility for the at-the-money implied volatility structure:



Calibration

- By calibrating the BSHW and SZHW model to 10-year European call options, end of July 2007, we obtain the following implied volatility fits:

strike	Market	SZHW	BSHW
80	27.8%	27.9%	26.4%
90	27.1%	27.1%	26.4%
95	26.7%	26.7%	26.4%
100	26.4%	26.4%	26.4%
105	26.0%	26.0%	26.4%
110	25.7%	25.7%	26.4%
120	25.1%	25.1%	26.4%

- As expected, a stochastic volatility model, does a better job fitting the market prices.
- For calculating the replication/hedging costs, this is extremely important.

- The calibrations imply the following risk-neutral densities:

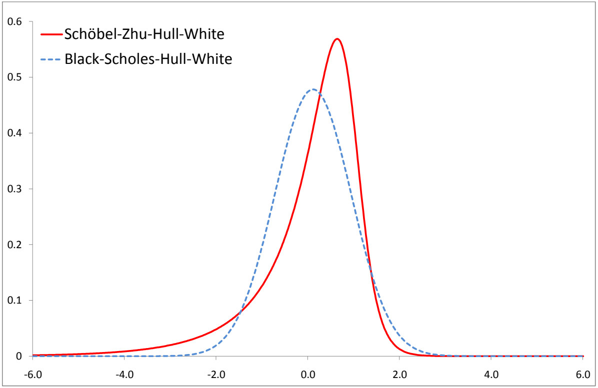


Figure: Risk-neutral density of the log-asset price for the SZHW and BSHW model, calibrated to European Option data (Eurostoxx50).

- Clearly, the SZHW model incorporates the skewness and heavy-tails of the option markets (e.g. see Bakshi et al. (1997)) a lot more realistically than the BSHW model.

Pricing

- The GAO price can be expressed under the risk-neutral measure $\mathcal{Q}$, but also under the equity price measure $\mathcal{Q}^S$, which uses the stock price as numeraire

$$
\begin{aligned}
& {}_x p_r \, \mathbb{E}^{\mathcal{Q}}\left[\exp\left(-\int_0^T r(u)du\right) gS(T)\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+\right] \\
= \quad & {}_x p_r \; gS(0)\mathbb{E}^{\mathcal{Q}^S}\left[\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+\right]
\end{aligned}
$$

- By changing to the equity price measure, the GAO can be viewed as an option on a portfolio of zero-coupon bonds.

- The zero-coupon bond price is a monotone function of its state variable $x(T)$ and there exists an x^* such that the payoff is exactly at the money.
- Following Jamshidian (1989), the option on the portfolio of bonds can hence be written as a portfolio of bond options:

$$\mathbb{E}^{\mathcal{Q}^S}\left[\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+\right] \stackrel{!}{=} \mathbb{E}^{\mathcal{Q}^S}\left[\sum_{i=0}^{n} c_i \left(P(T,t_i) - K_i\right)^+\right]$$

- Under the equity price measure the distribution (log-normal) and first two moments of $P(T,t_i)$ can be derived for the SZHW model using Girsanov and Fubini.

Closed-form Pricing Formulas:

- For **1-factor interest rates**, the GAO price is given by a sum of Black and Scholes (1973) formulas:

$$ {}_x p_r \, g S(0) \sum_{i=0}^{n} c_i \left[F_i N\left(d_1^i\right) - K_i N\left(d_2^i\right) \right] $$

- For **2-factor interest rates**, the GAO price is given a one dimensional integral over a sum of Black and Scholes (1973) formulas multiplied by a Gaussian distribution:

$$ {}_x p_r \, g S(0) \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \left[F_i(x) N\left(h_2(x)\right) - K N\left(h_1(x)\right) \right] dx $$

Impact of stochastic volatility

To investigate the impact of stochastic volatility we consider the following example policy:

- 55 year old male with retirement age 65,
- Survival rates based on the PNMA00 table for male pensioners of the CMI,
- Market Data (swap-rates and EuroStoxx50) per end of July 2007,
- Positive Correlation of 0.347 between stock returns and long-term interest rates.

- The SZHW and BSHW model, calibrated using the same EU option data and terminal correlation coefficient 0.347, give the following GAO prices:

strike g	SZHW	BSHW	Rel. Diff
8.23%	3.82	3.07	+ 24.5%
7%	0.59	0.39	+ 50.7%
8%	2.89	2.26	+28.0%
9%	8.40	7.25	+15.8%
10%	17.02	15.53	+9.6%
11%	27.37	25.69	+6.5%
12%	38.30	36.47	+5.0%
13%	49.35	47.37	+4.2%

- For a positive correlation, the prices for GAOs, using a stochastic volatility model for equity prices are considerably higher in comparison to the constant volatility model, especially for those with out of the money strikes.

- For a positive correlation the GAO prices are higher using a stochastic volatility model (and vice versa for a negative correlation).
- Mathematically, this is induced by a stochastic quanto correction for the process driving the interest rates:

$$dx(t) = -ax(t)dt + \rho_{xS}\sigma\nu(t)dt + \sigma dW_x^{\mathcal{Q}^S}(t)$$

- Looking at the payoff profiles, the stochastic quanto correction produces relatively more higher payoffs, i.e. low interest rates in combination with high equity prices, despite the positive correlation.
- Compared to the linear dependency structure induced by the BSHW model, the stochastic nature of the volatility in combination with a positive correlation, creates a more extreme and skewed dependency structure.

Efficiency of Pricing Formulas

- A special case of our modeling framework is considered in Chu and Kwok (2007), namely a equity model with constant volatility with two-factor Gaussian interest rates.
- Chu and Kwok (2007) argue that no analytical pricing formula exists and hence propose three approximation methods for the valuation of GAOs:
  - *Method of minimum variance duration*: Approximation of the annuity with a single zero-coupon bond with maturity equal its stochastic duration.
  - *Edgeworth expansion*: Edgeworth approximation of the probability distribution of the value of the annuity.
  - *Affine approximation*: Affine approximation of the exercise region of the underlying annuity.

- The following running times are reported in Chu and Kwok (2007):

method	Monte Carlo	Duration	Edgeworth	Affine
running time	0.4305	0.0016	1.136	0.1812

- Due to long computational times of other methods, the 'minimum variance duration' is favored in Chu and Kwok (2007).
- Our closed-form exact approach relies on the evaluation of a one dimensional integral whose integrand consists of a bounded function against a Gaussian distribution.
- This formula is computational very efficient to compute by using Gauss(-Hermite) quadratures and provides instantaneous, and exact, prices for GAOs.

- Comparison between pricing methods and Monte Carlo prices using 10^6 sample paths:

r0	Strike Level	Closed-form Exact	Min. Var. Duration	Edgeworth Expansion	Affine Approx.	Monte Carlo ($\pm$95% interval)
0.5%	127%	11.8000	11.8100	11.8161	11.7913	11.7921 ($\pm$0.0366)
1.0%	123%	9.7556	9.7714	9.7502	9.7412	9.7487 ($\pm$0.0329)
1.5%	118%	7.8741	7.8958	7.8479	7.8529	7.8678 ($\pm$0.0294)
2.0%	114%	6.1690	6.1946	6.1293	6.1418	6.1633 ($\pm$0.0260)
2.5%	110%	4.6612	4.6860	4.6199	4.6313	4.6555 ($\pm$0.0226)
3.0%	106%	3.3732	3.3911	3.3408	3.3464	3.3678 ($\pm$0.0192)
3.5%	103%	2.3217	2.3273	2.2999	2.3044	2.3174 ($\pm$0.0159)
4.0%	99%	1.5095	1.5008	1.4897	1.5057	1.5065 ($\pm$0.0126)
4.5%	96%	0.9214	0.9008	0.8942	0.9310	0.9198 ($\pm$0.0097)
5.0%	93%	0.5249	0.4984	0.4922	0.5439	0.5244 ($\pm$0.0071)
5.5%	90%	0.2778	0.2517	-	-	0.2775 ($\pm$0.0050)
6.0%	88%	0.1360	0.1150	-	-	0.1354 ($\pm$0.0033)
6.5%	85%	0.0614	0.0471	-	-	0.0609 ($\pm$0.0021)
7.0%	83%	0.0254	0.0171	-	-	0.0251 ($\pm$0.0013)

- Relative differences between exact formula and approximations:

r0	Strike Level	Min. Var. Duration	Edgeworth Expansion	Affine Approx.	Monte Carlo Simulation
0.5%	127%	0.1%	0.1%	-0.1%	-0.1%
1.0%	123%	0.2%	-0.1%	-0.1%	-0.1%
1.5%	118%	0.3%	-0.3%	-0.3%	-0.1%
2.0%	114%	0.4%	-0.6%	-0.4%	-0.1%
2.5%	110%	0.5%	-0.9%	-0.6%	-0.1%
3.0%	106%	0.5%	-1.0%	-0.8%	-0.2%
3.5%	103%	0.2%	-0.9%	-0.7%	-0.2%
4.0%	99%	-0.6%	-1.3%	-0.3%	-0.2%
4.5%	96%	-2.2%	-2.9%	1.0%	-0.2%
5.0%	93%	-5.1%	-6.2%	3.6%	-0.1%
5.5%	90%	-9.4%	-	-	-0.1%
6.0%	88%	-15.4%	-	-	-0.4%
6.5%	85%	-23.3%	-	-	-0.7%
7.0%	83%	-32.8%	-	-	-1.1%

- The approximation methods considered by Chu and Kwok (2007) break down for out-of-the-money GAOs.
- The 'Closed-form Exact' approach is preferable compared to the approaches described in Chu and Kwok (2007), as it gives exact GAO prices over all strike levels whilst being extremely fast.

Conclusion

- The use of a stochastic volatility model, such as the SZHW model, has a significant impact of the valuation and risk management of GAOs.
- Closed form expressions for the price of a GAO can be established under 1- or 2-factor Gaussian interest rates, stochastic volatility and a general correlation structure.
- The numerical results show that our closed-form expression is preferable compared to the approaches described in Chu and Kwok (2007), as it gives exact GAO prices over all strike levels whilst being computational very efficient to compute.

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