Lecture III
Derivatives pricing in energy markets – an infinite dimensional approach

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Overview

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1. Representing energy forwards in Hilbert space

2. Analysis of options on energy forwards
Overview

1. Representing energy forwards in Hilbert space

2. Analysis of options on energy forwards
Representing energy forwards in Hilbert space

- So far dealt with forward contracts delivering at a fixed time
  - Forward price $t \mapsto f(t, x)$, $x$ time to delivery
- Energy markets: forwards deliver over a period
  - Power, gas, temperature
  - Delivery of gas and power over an agreed period, a month say
  - Measurement of temperature index over an agreed period (CDD, HDD, CAT)
- Interpreted $t \mapsto f(t)$ as Hilbert-valued stochastic process
- **Question**: can energy forward prices be viewed as Hilbert-valued stochastic processes?
  - ...or rather HOW?
■ Power forwards/futures: delivery over period \([T_1, T_2]\)
■ Assume constant risk-free interest rate \(r > 0\)
■ Forward-style: settlement at \(T_2\)

\[
\mathcal{F}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, T) dT , t \leq T_1
\]

■ Futures-style: balancing (margin) account during settlement

\[
\mathcal{F}(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{e^{-rT}}{\int_{T_1}^{T_2} e^{-rs} ds} F(t, T) dT , t \leq T_1
\]

■ NordPool: both forward- and futures-style contracts traded
  ■ Forwards when long delivery period, futures when short
Temperature futures on CDD, HDD and CAT indices

\[ \mathcal{F}(t, T_1, T_2) = \int_{T_1}^{T_2} F(t, T) \, dT, \quad t \leq T_1 \]

- CAT = cumulative average temperature
  - Daily average: average of minimum and maximum
- CDD = cooling degree day
  \[ \text{CDD}(t) = \max(T(t) - 18^\circ, 0) \]

- HDD = heating degree day
  - HDD "call option" on temperature with strike 18°
  - CDD "put option"
General expression for energy forward/futures prices

$$\mathcal{F}\tilde{\omega}(t, T_1, T_2) = \int_{T_1}^{T_2} \tilde{\omega}(T, T_1, T_2) F(t, T) \, dT, \quad t \leq T_1$$

$T \mapsto \tilde{\omega}(T, T_1, T_2)$ weight function

$$\tilde{\omega}(T, T_1, T_2) = 1, \text{CAT, CCC, HDD, gas}$$

$$\tilde{\omega}(T, T_1, T_2) = \frac{1}{T_2 - T_1}, \text{power forward}$$

$$\tilde{\omega}(T, T_1, T_2) = \frac{e^{-rT}}{\int_{T_1}^{T_2} e^{-rs} \, ds}, \text{power futures}$$
Let $\ell = T_2 - T_1$: length of delivery, and $x = T_1 - t \geq 0$, time to start of delivery

With $f(t, y) := F(t, t + y)$, $y \geq 0$

$$F_{\ell}^{\omega}(t, x) := \mathcal{F}\tilde{\omega}(t, t + x, t + x + \ell) = \int_{x}^{x+\ell} \omega_{\ell}(t, x, y)f(t, y) \, dy$$

Weight function

$$\omega_{\ell}(t, x, y) = \tilde{\omega}_{\ell}(t + y, t + x, t + x + \ell)$$

Example: power futures

$$\omega_{\ell}(t, x, y) = \frac{1}{1 - e^{-r\ell}} e^{-r(y-x)}$$
Suppose $\omega_\ell(x, y) := \omega_\ell(y - x)$, and assume $z \mapsto \omega_\ell(z)$ is positive, bounded and measurable.

Musiela representation of energy forward

\[
F_\ell^\omega(t, x) = \int_x^{x+\ell} \omega_\ell(y - x)f(t, y) \, dy
\]

$F_\ell^\omega$ representable as a linear operator on $H_w$, which is want we analyse next:

Simple integration-by-parts

\[
F_\ell^\omega(t, x) = \mathcal{W}_\ell(\ell)f(t, x) + \int_0^\infty q_\ell^\omega(x, y)\partial_y f(t, y) \, dy
\]
Define

\[ \mathcal{W}_\ell(u) = \int_0^u \omega_\ell(v) \, dv, \quad u \geq 0 \]

\[ q^{\omega}_\ell(x, y) = (\mathcal{W}_\ell(\ell) - \mathcal{W}_\ell(y - x)) \, 1_{[0,\ell]}(y - x) \]

Consider the integral operator \( I^{\omega}_\ell \)

\[ I^{\omega}_\ell(g) = \int_0^\infty q^{\omega}_\ell(\cdot, y)g'(y) \, dy \]
Proposition

$I^\omega_\ell$ is a bounded linear operator on $H_w$

Proof.

• $I^\omega_\ell$ well-defined on $H_w$: By Cauchy-Schwartz,

$$\left| \int_0^\infty q^\omega_\ell(x, y) g'(y) \, dy \right|^2 \leq \int_0^\infty w^{-1}(y)(q^\omega_\ell(x, y))^2 \, dy \int_0^\infty w(y)(g'(y))^2 \, dy$$

First term finite since $\omega_\ell$ is bounded. Second term finite since $g \in H_w$.

• $I^\omega_\ell \in H_w$ for $g \in H_w$: Let $\xi(x) := I^\omega_\ell(g)(x)$,

$$\xi(x) = \int_x^{x+\ell} (W_\ell(\ell) - W_\ell(y - x)) g'(y) \, dy$$
Proof.

Proof cont’d....

Direct calculation shows that $\xi$ has weak derivative

$$
\xi'(x) = \int_x^{x+\ell} \omega(y-x)g'(y) \, dy - \mathcal{W}_\ell(x)g'(x)
$$

By boundedness of $\omega$, it follows from Cauchy-Schwartz,

$$
|\mathcal{I}_\ell^\omega(g)|_w \leq C|g|_w < \infty
$$

for some constant $C > 0$. 

■
Wrapping up: energy forwards

- Given a model for $t \mapsto f(t) \in H_W$
  - Fixed-delivery forward price curve
  - Recall models in Lecture II
- Realize dynamics for energy forwards in $H_W$

$$F^\omega_\ell(t) = \mathcal{W}_\ell(\ell)f(t) + \mathcal{I}^\omega_\ell(f(t))$$

- More compact notation

$$F^\omega_\ell(t) = D^\omega_\ell(f(t)), \quad D^\omega_\ell = \mathcal{W}_\ell(\ell)\text{Id} + \mathcal{I}^\omega_\ell \in L(H_W)$$
Overview

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Analysis of options on energy forwards

- European options on energy forwards:
  - Energy forward price $t \mapsto \mathcal{F}^\omega(t, T_1, T_2), t \leq T_1$
  - Exercise time $0 < \tau \leq T_1$
  - Payoff at exercise: $p : \mathbb{R} \to \mathbb{R}$ measurable function of at most linear growth
    \[ p(\mathcal{F}^\omega(\tau, T_1, T_2)) \]

- Recall representation of $\mathcal{F}^\omega(t, T_1, T_2)$, in compact form
  \[ \mathcal{F}^\omega(t, T_1, T_2) := F_{T_2-T_1}^\omega(t, T_1 - t) \]
  where, for $f(t) \in H_w$,
  \[ F_{\ell}^\omega(t) = D_{\ell}^\omega(f(t)) \]
Lemma

Define $\mathcal{P}_t^\omega : \mathbb{R}_+ \times H_w \rightarrow \mathbb{R}$ as

$$\mathcal{P}_t^\omega (x, g) = p \circ \delta_x \circ D_t^\omega (g)$$

Then

$$\sup_{x \geq 0} |\mathcal{P}_t^\omega (x, g)| \leq c (1 + |g|_w)$$

for a constant $c > 0$ Moreover,

$$p(\mathcal{F}_t^\omega (\tau, T_1, T_2)) = \mathcal{P}_{T_2 - T_1}^\omega (T_1 - \tau, f(\tau))$$

Note: $\mathcal{P}_t^\omega (x, \cdot)$ is a nonlinear functional on $H_w$. 

Proof.

By linear growth of $p$:

$$|\mathcal{P}_\ell^\omega(x, g)| \leq c(1 + |\mathcal{D}_\ell^\omega(g)(x)|)$$

Recall from Lecture II, proof of $H_w$ being Banach algebra, the sup norm is bounded by $H_w$-norm. Since $\mathcal{D}_\ell^\omega \in L(H_w)$, the result follows. 

■
Assume $\mathbb{E}[|f(t)|_w] < \infty$ for all $t \geq 0$

Arbitrage-free option price dynamics for $t \leq \tau$

$$V(t) = e^{-r(\tau-t)}\mathbb{E}[\rho(\mathcal{F}_{\tilde{\omega}}(\tau, T_1, T_2)) | \mathcal{F}_t]$$

$$= e^{-r(\tau-t)}\mathbb{E}[\mathcal{P}_{T_2-T_1}^{\omega}(T_1 - \tau, f(\tau)) | \mathcal{F}_t]$$

The linear growth of the payoff $\rho$ ensures that $V$ is finite

Assume Markovian HJMM dynamics with Lipschitz parameters

$$df(t) = \partial_x f(t) \, dt + \psi(t, f(t-)) \, dL(t)$$

Recall Lecture II for all assumptions...!
Mild solution for $t \leq s$

$$f(s) = S(s - t)f(t) + \int_t^S S(s - u)\psi(u, f(u-))\,dL(u)$$

Option price $V(t) := V(t, f(t))$, with

$$V(t, g) = e^{-r(\tau-t)}E[\mathcal{P}^\omega_{T_2-T_1}(T_1 - \tau, f(\tau)) \mid f(t) = g]$$
Stability of option prices wrt current forward curve

Proposition

Suppose that the payoff function \( p \) is Lipschitz continuous. Then, for any \( g, \tilde{g} \in H_w \),

\[
\sup_{0 \leq t \leq \tau} |V(t, g) - V(t, \tilde{g})| \leq C|g - \tilde{g}|_w
\]

for a positive constant \( C \) depending on \( \tau \).

- Option price is not sensitive to small errors in the current forward curve
- Note: we have only discrete forward price observations available, and must construct/recover the curve from these
Proof.

By Lipschitz continuity of $p$ and linearity of $\delta_x, D^\omega_l$,

$$|\mathcal{P}^\omega_l(x, g) - \mathcal{P}^\omega_l(x, \tilde{g})| \leq c \|\delta_x\|_{op} |g - \tilde{g}|_w$$

From lecture II, $\|\delta_x\|_{op}^2 = h_x(x) \leq c$,

$$|\mathcal{P}^\omega_l(x, g) - \mathcal{P}^\omega_l(x, \tilde{g})| \leq c |g - \tilde{g}|_w$$

for some positive (generic) constant $c > 0$ independent of $x$. Thus,

$$|V(t, g) - V(t, \tilde{g})| \leq c \mathbb{E}[|f^{t,g}(\tau) - f^{t,\tilde{g}}(\tau)|_w]$$

where $f^{t,g}(t) = g$. ■
Proof.

On $H_w$, the operator norm of $S(t)$ is uniformly bounded in $t$:

$$|f^t,g(\tau) - f^t,\tilde{g}(\tau)|^2_w \leq c|g - \tilde{g}|^2_w$$

$$+ 2\left| \int_t^T S(\tau - s)(\psi(s, f^t,g(s-)) - \psi(s, f^t,\tilde{g}(s-))) \, dL(s) \right|^2_w$$

Using Itô's isometry and Lipschitz of $\psi$

$$\mathbb{E} \left[ \left| \int_t^T S(\tau - s)(\psi(s, f^t,g(s-)) - \psi(s, f^t,\tilde{g}(s-))) \, dL(s) \right|^2_w \right]$$

$$\leq \int_t^T \mathbb{E} \left[ \|S(\tau - s)(\psi(s, f^t,g(s-)) - \psi(s, f^t,\tilde{g}(s-)))\|_{LHS(H_w)}^{1/2} \right]^2 \, ds$$

$$\leq c \int_t^T \mathbb{E} \left[ |f^t,g(s) - f^t,\tilde{g}(s)|^2_w \right] \, ds$$
Proof.

Hence,

\[
\mathbb{E}[|f^{t,g}(\tau) - f^{t,\tilde{g}}(\tau)|_W^2] \leq c|g - \tilde{g}|_W^2 + c \int_t^T \mathbb{E}[|f^{t,g}(s) - f^{t,\tilde{g}}(s)|_W^2] \, ds
\]

We conclude by Gronwall’s inequality,

\[
\mathbb{E}[|f^{t,g}(\tau) - f^{t,\tilde{g}}(\tau)|_W^2] \leq ce^{c(\tau-t)}|g - \tilde{g}|_W^2
\]
Pricing of options in Gaussian case

- Focus on simple Gaussian dynamics; \( L = W \)

\[
f(\tau) = S(\tau - t)f(t) + \int_t^\tau S(\tau - s)\psi(s) \, dW(s)
\]

- Recalling representation analysis in Lecture II

\[
\mathcal{F}^{\tilde{\omega}}(\tau, T_1, T_2) = \delta_{T_1-t} \mathcal{D}_{T_2-T_1}^{\omega} f(t) + \int_t^\tau \sigma_{T_1,T_2}(s) \, dB(s), \ t \leq \tau \leq T_1
\]

with \( B \) being a real-valued Brownian motion and

\[
\sigma_{T_1,T_2}^2(s) = (\delta_{T_1-s} \mathcal{D}_{T_2-T_1}^{\omega} \Psi(s) \mathcal{Q} \Psi^*(s) \mathcal{D}_{T_2-T_1}^{\omega,*} \delta^*_{T_1-s})(1)
\]
Proposition

Suppose $\psi : \mathbb{R}_+ \to L(H_w)$ is deterministic. Then

$$V(t, g) = e^{-r(\tau-t)} \mathbb{E}[p(m(g) + \xi X)]$$

where $X$ is a standard normal distributed random variable,

$$\xi^2 := \int_t^\tau \sigma_{T_1, T_2}^2(s) \, ds, \quad m(g) = \delta_{T_1-t} D_{T_2-T_1}^\omega (g)$$

Proof.

Immediate, since Itô integral of the deterministic function $\sigma_{T_1, T_2}(s)$ is centered normally distributed.
Study of the volatility $\sigma_{T_1,T_2}(s)$

- From Lecture II:

$$\delta_{T_1-s}^*(1) = h_{T_1-s}(\cdot) = 1 + \int_0^{(T_1-s)^\wedge} w^{-1}(z) \, dz$$

- Therefore, for $x \geq 0$ and $\ell = T_2 - T_1$,

$$\delta_x D_\ell^{\omega,*} \delta_{T_1-s}^*(1) = D_\ell^{\omega,*}(h_{T_1-s})(x) = \langle D_\ell^{\omega,*}(h_{T_1-s}), h_x \rangle$$

$$= \langle h_{T_1-s}, D_\ell^{\omega}(h_x) \rangle = D_\ell^{\omega}(h_x)(T_1 - s)$$

$$= \mathcal{W}_\ell(\ell) h_{T_1-s}(x) + \int_0^x w^{-1}(z) q_\ell^{\omega}(T_1 - s, z) \, dz$$
Hence,

\[ D_\ell^\omega \delta^*_{T_1-s}(1) = \mathcal{W}_\ell(\ell) h_{T_1-s}(\cdot) + \int_0^\cdot w^{-1}(z) q_\ell^\omega(T_1-s, z) \, dz \in H_w \]

- \( \Sigma(s) := \Psi(s) Q \Psi^*(s) \in L(H_w) \) is the modeller’s choice
  - \( Q \) variance-covariance structure in "spatial" coordinate \( x \)
  - \( \Psi \) space-time volatility scaling

Useful characterization: if \( \mathcal{L} \in L(H_w) \),

\[
\delta_x \mathcal{L}^* g = \langle \mathcal{L}^* g, h_x \rangle = \langle g, \mathcal{L} h_x \rangle \\
= g(0) \mathcal{L}(h_x)(0) + \int_0^\infty (\mathcal{L} h_x)'(y) w(y) g'(y) \, dy
\]

Thus: \( \mathcal{L}^* \) is essentially an integral operator on \( H_w \) ... and the same for \( \mathcal{L} = (\mathcal{L}^*)^* \)
The delta

- ....or, the sensitivity to the current forward curve
- Perturbing the current forward curve in a direction $h \in H_W$.
- Gateaux derivative, $D_h V(t, g)$, $g$ current forward curve

$$
D_h V(t, g) := \frac{d}{d\epsilon} V(t, g + \epsilon h)|_{\epsilon=0}
$$

Proposition

Suppose $\Psi : \mathbb{R}_+ \rightarrow L(H_W)$ is deterministic. For any $h \in H_W$ it holds

$$
D_h V(t, g) = \frac{1}{\xi} m(h) \mathbb{E}[\rho(m(g) + \xi X)X]
$$

with $m$ and $\xi$ as defined earlier.
Proof.

Let \( \phi \) denote the standard normal density function. Change of variables, Fubini and chain rule yield,

\[
D_h V(t, g) = D_h \int_{\mathbb{R}} p(m(g) + \xi x) \phi(x) \, dx
\]

\[
= \frac{1}{\xi} D_h \int_{\mathbb{R}} p(y) \phi((y - m(g))/\xi) \, dy
\]

\[
= \frac{1}{\xi} \int_{\mathbb{R}} p(y) \phi'((y - m(g))/\xi) (-1/\xi) D_h m(g) \, dy
\]

\[
D_h m(g) = \frac{d}{d\epsilon} (m(g) + \epsilon m(h))|_{\epsilon=0} = m(h)
\]
- Extract a smooth curve $g$ from energy forward prices
- Functionals of the smooth curve, over discrete delivery periods
- No unique way to smoothen the forward curve
  - Delta provides a sensitivity measure
At EEX and NordPool: European call and put options on monthly forward contracts

Payoff of a call: \( p(x) = \max(x - K, 0) \)

**Proposition**

The price of a call option with strike \( K \) and exercise time \( \tau \leq T_1 \) is

\[
V(t, g(t)) = \xi \Phi((m(g(t)) - K)/\xi) + (m(g(t)) - K)\Phi((m(g(t)) - K)/\xi)
\]

with \( \Phi \) being the cumulative normal distribution function. Moreover,

\[
D_h V(t, g(t)) = m(h)\Phi((m(g(t)) - K)/\xi)
\]

for any \( h \in H_w \).
References

Fred Espen Benth
Lecture III
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