Proportional Rank Aggregation with Generalized Kemeny Utilities

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Abstract

In rank aggregation, the goal is to combine multiple weighted input rankings into a single output ranking. Recently, the concept of proportional representation has been studied for this setting, which postulates that each input ranking should receive a utility at least proportional to its weight from the output ranking. Existing works formalize this condition via Kemeny utilities, which count the number of pairs of candidates on which the input and output rankings agree. However, Kemeny utilities fails to capture that higher-ranked positions in the output ranking are typically more valuable than lower-ranked ones. In this paper, we hence study proportional rank aggregation under more general utility functions. Specifically, we introduce two broad classes of utility functions, so-called weighted Kemeny utilities and scoring Kemeny utilities, and analyze when proportional rankings exist under these models. For weighted Kemeny utilities, we fully characterize when our proportionality axioms are guaranteed to be satisfiable. To this end, we introduce a new SWF that returns a proportional ranking whenever such a ranking is guaranteed to exist for the given weighted Kemeny utility function. Further, for scoring Kemeny utilities, we generalize the proportional sequential Borda rule of Lederer (2026) to arbitrary score systems and analyze when the resulting rules return proportional rankings.

1 Introduction

A ubiquitous problem in AI is *rank aggregation*: multiple weighted input rankings need to be aggregated into a single output ranking. For example, this task arises when ranking AI models with respect to their performance on multiple tasks (Maura-Rivero et al. 2025; Lanctot et al. 2025), when sorting hotels based on multiple criteria such as price or user rating (Lederer, Peters, and Wąs 2024), or when combining the preferences of a hiring committee into an overall ranking of the applicants (Kuhlman and Rundensteiner 2020). Moreover, rank aggregation also constitutes a fundamental problem of social choice theory since, e.g., Arrow's impossibility theorem (Arrow 1959) has originally been stated for this setting. Following this line of work, we formalize rank aggregation via *social welfare functions (SWFs)*, which map each profile of weighted input rankings to a single output ranking.

Specifically, we are interested in *proportional rank aggregation*, where the goal is to find an output ranking that represents the input rankings proportionally to their weights. Such proportional rankings are desirable for many applications, as they guarantee that each input ranking is taken into account fairly instead of ignoring low-weight rankings. However, as observed by Lederer, Peters, and Wąs (2024) and Lederer (2026), all classical SWFs fail to produce proportional rankings because they are majoritarian, i.e., they primarily cater towards rankings with large weights. In more detail, these authors formalize proportional rankings based on Kemeny utilities: the utility of an input ranking for an output ranking is the number of pairs of candidates on which these rankings agree. Then, proportionality requires that each input ranking receives a utility at least proportional to its weight from the output ranking. Moreover, Lederer (2026) proposes SWFs which always select proportional rankings in this sense.

However, Kemeny utilities do not align well with how rankings are typically used. In particular, these utilities treat the top and the bottom of a ranking symmetrically, although the top positions are usually much more important than the bottom ones. For example, when producing an overall ranking over hotels based on multiple criteria, users may not even consider hotels at the bottom of the ranking, thus making Kemeny utilities unsuitable for this application. This drawback is widely acknowledged in the recommender systems literature (e.g., Järvelin and Kekäläinen 2002; Kumar and Vassilvitskii 2010; Ricci, Rokach, and Shapira 2022), and Lederer (2026, page 18) even poses the question of generalizing his results to more top-focused utility functions.

Contribution. In this paper, we study proportional rank aggregation for utility functions that model that higher-ranked positions of the output ranking are more valuable than lower-ranked ones. To this end, we first generalize the proportionality axioms of Lederer (2026) to novel utility functions, which measure how well an input ranking is represented by an output ranking. Specifically, our main axiom is called *unanimous justified proportional representation (uPJR)* and requires that each input ranking receives a share of its maximal possible utility that is at least proportional to its weight. Further, we also study *strong proportional justified representation (sPJR)*, which extends the reasoning of uPJR to groups of rankings. We then analyze when rankings satisfying these axioms are guaranteed to exist for two classes of utility functions. Table 1 gives a summary of our results.

In the first class, so-called *weighted Kemeny utility functions*, we generalize the Kemeny utility function by introducing weights for the positions of the output ranking. More specifically, each weighted Kemeny utility function is defined by a weight vector $\lambda = (\lambda_1, \dots, \lambda_m)$ and each λ_i specifies the importance of the pairwise comparisons between the i-th candidate of the output ranking and lower-ranked ones. Hence, by choosing large weights for higher-ranked positions, we can model that these positions are more important than lower-ranked ones. For weighted Kemeny utility functions, we fully characterize when rankings satisfying uPJR and sPJR are guaranteed to exist. To this end, we introduce a condition called tail-boundedness, which restricts how much larger a weight λ_i can be compared to the "tail" weights $\lambda_{i+1}, \dots, \lambda_m$. We then show the following results:

- For every tail-bounded weighted Kemeny utility function, both uPJR and sPJR are always satisfiable. To prove this claim, we introduce the *Proportional Sequential Eating (PSE) rule*, a new SWF parameterized by a weight vector λ. Further, we show that the output ranking of this SWF satisfies sPJR for the weighted Kemeny utility function induced by λ if λ is tail-bounded (Theorem 1).
- For every weighted Kemeny utility function that is not tail-bounded, there is a profile where no output ranking satisfies uPJR (and thus sPJR). This means that tail-boundedness is necessary to guarantee that our proportionality notions are satisfiable (Theorem 2).

As the second class, we consider *scoring Kemeny utility functions*, which generalize weighted Kemeny utility functions by replacing the linear weights with arbitrary score functions. In more detail, these utility functions are defined by score systems $s = (s_i)_{i \in \{1,\dots,m\}}$, where $s_i(j)$ specifies the utility an input ranking derives from the i-th candidate of the output ranking when its marginal contribution to the Kemeny utility is j. Scoring Kemeny utilities even allow to express non-linear relations between a candidate's position in an input ranking and its generated utility, thereby offering even more flexibility. To analyze when proportional rankings exist for these utilities, we generalize the proportional sequential Borda rule of Lederer (2026) to arbitrary score systems, resulting in the class of *Proportional Sequential Scoring (PSS) rules*. Then, we show the following results:

- We identify a condition called ratio-boundedness which extends the idea of tail-boundedness to score systems. Further, we show that, if a score system s is ratio-bounded, the ranking returned by the corresponding PSS rule satisfies uPJR for every profile and the scoring Kemeny utility function induced by s (Theorem 3).
- For a subclass of score systems, so-called balanced score systems, we show that PSS rules can return rankings that fails uPJR if the underlying score system fails ratio-boundedness. This means that, for balanced score systems, ratio-boundedness is both sufficient and necessary for PSS rules to return proportional rankings (Theorem 4).

Related Work. Our paper belongs to a recent line of work that studies proportional representation in rank aggregation (Lederer, Peters, and Wąs 2024; Aziz et al. 2025; Lederer 2026). Specifically, Lederer, Peters, and Wąs (2024) and Lederer (2026) design proportional SWFs for Kemeny utilities and thus are the direct predecessors of our work. Moreover, Aziz et al. (2025) study proportional committee voting rules that satisfy committee monotonicity. However, their rules can be interpreted as SWFs and, in this context, guarantee that every prefix of the output ranking satisfies a proportionality notion called PSC. Similar prefix-based proportionality notions have been analyzed by Skowron et al. (2017) and Brill and Israel (2025) for aggregating approval ballots into rankings. Lastly, proportional rank aggregation is also studied in the recommender systems literature (e.g., Kuhlman and Rundensteiner 2020; Chakraborty et al. 2022; Wei et al. 2022), but with a different meaning. These works assume that candidates have protected attributes and aim to fairly represent these attributes, irrespective of the input profile.

	Positive results	Negative results
WKU	PSE rule satisfies sPJR if the weight vector λ is tail-bounded (Thm. 1)	There are profiles for which uPJR is unsatisfiable if λ is not tail-bounded (Thm. 2)
SKU	PSS rules satisfy uPJR if the score system s is ratio-bounded (Thm. 3)	PSS rules may fail uPJR if s is balanced but not ratio-bounded (Thm. 4)

Table 1: Summary of results. The first row states our results for weighted Kemeny utilities (WKU) and the second one for scoring Kemeny utilities (SKU).

More broadly, our paper is connected to the study of proportional representation in approval-based committee voting and participatory budgeting (Lackner and Skowron 2022; Rey, Schmidt, and Maly 2025). In particular, our proportionality notions and techniques draw on ideas from this literature (e.g., Aziz et al. 2017; Sánchez-Fernández et al. 2017; Peters and Skowron 2020; Brill et al. 2023). Indeed, Lederer (2026) noted that his axioms are logically related to a notion called proportional justified representation in committee voting, and the conditions studied in this paper connect to variants of this notion in participatory budgeting (Peters, Pierczyński, and Skowron 2021; Los, Christoff, and Grossi 2022).

Further, our paper loosely relates to the large literature on axiomatic rank aggregation in social choice theory. Typical works in this area study consistency properties of SWFs, such as population consistency or independence of irrelevant alternatives, and prove characterizations of SWFs (e.g., Arrow 1959; Smith 1973; Young 1974; Young and Levenglick 1978; Young 1988; Lederer 2024; Boehmer, Bredereck, and Peters 2026). For an overview on the social choice literature on SWFs, we refer to the books by Arrow, Sen, and Suzumura (2011) and Brandt et al. (2016).

Lastly, we note that many weighted variants of the Kemeny distance and thus of the Kemeny utility function have been suggested (e.g., Kumar and Vassilvitskii 2010; Farnoud and Milenkovic 2014; Can 2014; Plaia, Buscemi, and Sciandra 2021). All of these works share the idea that higher-ranked positions of the output ranking are more valuable than lower-ranked ones. However, to our knowledge, our generalizations of the Kemeny utility are distinct from prior works.

2 Preliminaries

Let $C = \{x_1, \dots, x_m\}$ denote a set of m candidates. A *ranking* $\succ$ is a strict linear order over C and we write rankings as sequences of candidates. For example, $\succ = x_1 x_2 x_3$ means that x_1 is the best, x_2 the second-best, and x_3 the worst candidate. The set of all rankings over C is denoted by $\mathcal{R}$. A *(ranking) profile* R maps every ranking $\succ \in \mathcal{R}$ to a weight $R(\succ) \in [0, 1]$ such that $\sum_{\succ \in \mathcal{R}} R(\succ) = 1$. These weights could capture the importance of a ranking in a multi-criteria decision-making problem or the fraction of voters that report a given ranking in a voting setting.

Given a ranking profile, our goal is to choose an aggregate ranking. To this end, we will use *social welfare functions (SWFs)*, which map every ranking profile to a single output ranking. To ease readability, we will refer to input rankings by $\succ$ and to output rankings by $\rhd$. Since SWFs always choose a single ranking, tie-breaking will be necessary. We thus assume that ties are broken according to a fixed order $>$ over the candidates.

2.1 Utility Functions over Rankings

To formalize proportional representation for SWFs, we first need to define how well an input ranking is represented by an output ranking. To this end, we will use *utility functions* $u : \mathcal{R} \times \mathcal{R} \to \mathbb{N}_0$, where $u(\succ, \rhd)$ quantifies how well $\succ$ is represented by $\rhd$. Throughout the paper, we will assume that each ranking $\succ$ is best represented by itself (i.e., $u(\succ, \succ) = \max_{\rhd \in \mathcal{R}} u(\succ, \rhd)$ for all $\succ \in \mathcal{R}$) and that all input rankings have the same maximal utility (i.e., $u(\succ, \succ) = u(\succ', \succ')$ for all $\succ, \succ' \in \mathcal{R}$). We denote this common maximal utility of a utility function u by $\max(u)$. Further, we will consider the following three (classes of) utility functions.

Kemeny utility function. The Kemeny utility function has been studied by Lederer (2026) and counts the number of pairs of candidates on which two rankings agree. Thus, this utility function is defined by $u_K(\succ, \rhd) = |\{(x, y) \in C^2 \colon x \succ y \wedge x \rhd y\}|$. Alternatively, the Kemeny utility can also be defined in terms of the *marginal Kemeny utility* $\hat{u}(\succ, x, X) = |\{y \in X \setminus \{x\} \colon x \succ y\}|$, which counts how many candidates in X are lower-ranked than x by $\succ$. Using this expression, the Kemeny utility of an input ranking $\succ$ for an output ranking $\rhd = x_1 \dots x_m$ is given by

$$u_K(\succ, \rhd) = \sum_{i=1}^{m} \hat{u}(\succ, x_i, \{x_i, \dots, x_m\}).$$

Weighted Kemeny utility functions. Weighted Kemeny utility functions generalize the Kemeny utility function by introducing weights for the positions of the output ranking. To this end, we say a vector $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{N}_0^m$ is a *weight vector* if $\lambda_m = 0$, $\lambda_{m-1} = 1$, and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. Intuitively, each λ_i quantifies how important the i-th position of the output ranking is for the input rankings. Thus, given a weight vector λ, the induced *weighted Kemeny utility* of an input ranking $\succ$ for an output ranking $\rhd = x_1 \dots x_m$ is

$$u_K^\lambda(\succ, \rhd) = \sum_{i=1}^{m} \lambda_i \cdot \hat{u}(\succ, x_i, \{x_i, \dots, x_m\}).$$

This model allows us to express that higher-ranked positions of the output ranking are more valuable, thereby addressing the central flaw of the Kemeny utility function. Further, the weighted Kemeny utility function induced by $\lambda = (1, \dots, 1, 0)$ corresponds to the unweighted case.

Scoring Kemeny utility functions. As our last class of utility functions, we will consider scoring Kemeny utilities, which apply score functions to $\hat{u}(\succ, x_i, \{x_i, \dots, x_m\})$ to derive the final utility. To formalize this, we define a *score system* $s = (s_i)_{i \in \{1, \dots, m\}}$ as a family of functions $s_i : \{0, \dots, m - j\} \to \mathbb{N}_0$ such that $0 = s_i(0) \leq s_i(1) \leq \dots \leq s_i(m-i)$ for all $i \in \{1, \dots, m\}$ and $s_i(0) < s_i(m-i)$ if $i < m$. Intuitively, $s_i(j)$ specifies the utility a ranking obtains from the i-th candidate of the output ranking if it creates a marginal Kemeny utility of j. Given a score system $s = (s_i)_{i \in \{1, \dots, m\}}$, the *scoring Kemeny utility* of an input ranking $\succ$ for an output ranking $\rhd = x_1 \dots x_m$ is

$$u_K^s(\succ, \rhd) = \sum_{i=1}^{m} s_i(\hat{u}(\succ, x_i, \{x_i, \dots, x_m\})).$$

Scoring Kemeny utilities generalize weighted Kemeny utilities since every weighted Kemeny utility function u_K^λ is equal to the scoring Kemeny utility function u_K^s induced by $s_i(j) = \lambda_i \cdot j$ for all $i \in \{1, \dots, m\}$, $j \in \{0, \dots, m - i\}$. Moreover, scoring Kemeny utility functions allow for a non-linear dependency on $\hat{u}(\succ, x_i, \{x_i, \dots, x_m\})$. For example, we can define a scoring Kemeny utility function based on the Plurality score system given by $s_i^{PL}(j) = 1$ if $j = m - i > 0$ and $s_i^{PL}(j) = 0$ otherwise.

2.2 Axioms for Proportional Representations

We next generalize the two proportionality axioms of Lederer (2026) to arbitrary utility functions. The first one, unanimous justified proportional representation (uPJR), requires that each input ranking obtains a share of the maximal utility $\max(u)$ that is at least proportional to its weight. Formalizing this idea results in the following definition.

Definition 1 (uPJR). A ranking $\rhd$ satisfies *unanimous proportional justified representation (uPJR)* for a utility function u and profile R if $u(\succ, \rhd) \geq \lfloor R(\succ) \cdot \max(u) \rfloor$ for all rankings $\succ \in \mathcal{R}$.

We note, however, that uPJR only gives guarantees to individual rankings instead of groups of rankings. To address this issue, Lederer introduced another axiom called strong proportional justified representation (sPJR). For the Kemeny utility function, this condition requires that, for every group of ranking $S \subseteq \mathcal{R}$, the number of pairs on which the output ranking agrees with some input ranking in S should be at least proportional to the total weight of S. To make this more formal, we define by $K(\succ, \rhd) = \{(x, y) \in C^2 \colon x \succ y \wedge x \rhd y\}$ the set of pairs of candidates on which two rankings $\succ$ and $\rhd$ agree. Then, sPJR for u_K requires of a ranking $\rhd$ that $|\bigcup_{\succ \in S} K(\succ, \rhd)| \geq \lfloor \sum_{\succ \in S} R(\succ) \cdot \max(u_K) \rfloor$ for all sets of rankings $S \subseteq \mathcal{R}$. However, this definition uses that the Kemeny utility function is additively separable over pairs of candidates, which is, e.g., not true for all scoring Kemeny utilities. We will hence introduce sPJR only for weighted Kemeny utility functions, which can be equivalently defined by $u_K^\lambda(\succ, \rhd) = \sum_{(x_i,x_j)\in K(\succ,\rhd)} \lambda_i$ for all rankings $\succ$ and $\rhd = x_1 \dots x_m$. Using this formulation, we derive the following definition of sPJR.

Definition 2 (sPJR)**.** A ranking $\rhd = x_1 \dots x_m$ satisfies *strong proportional justified representation (sPJR)* for a weighted Kemeny utility function u_K^λ and profile R if it holds for all sets of rankings $S \subseteq \mathcal{R}$ that

$$\sum_{(x_i,x_j)\in\bigcup_{\succ\in S} K(\succ,\rhd)} \lambda_i \geq \lfloor \sum_{\succ\in S} R(\succ) \cdot \max(u_K^\lambda) \rfloor.$$

We note that sPJR implies uPJR for every weighted Kemeny utility function u_K^λ and profile R. Finally, we will discuss an example to illustrate the interplay of our utility functions and proportionality axioms. For ease of exposition, we only consider uPJR in our example. The logic of sPJR is analogous except that it also considers groups of rankings.

Example 1 (uPJR)**.** We consider the profile R shown below, where the rankings $\succ_1 = x_1x_2x_3x_4x_5x_6$ and $\succ_2 = x_6x_5x_4x_3x_2x_1$ have weight 0.6 and 0.4, respectively.

$$0.6\colon \succ_1 = x_1x_2x_3x_4x_5x_6$$
$$0.4\colon \succ_2 = x_6x_5x_4x_3x_2x_1$$

We first discuss the Kemeny utility function u_K. In our example, it holds that $\max(u_K) = \binom{m}{2} = 15$ since each ranking induces $\binom{m}{2}$ pairwise comparisons. Thus, uPJR guarantees utilities of $\lfloor 0.6 \cdot 15 \rfloor = 9$ and $\lfloor 0.4 \cdot 15 \rfloor = 6$ to $\succ_1$ and $\succ_2$, respectively. The ranking $\rhd_1 = x_1x_2x_6x_5x_4x_3$ satisfies uPJR for R and u_K since $u_K(\succ_1, \rhd_1) = 9$ and $u_K(\succ_2, \rhd_1) = 6$. However, $\rhd_1$ suffers from the problem described in the introduction: $\succ_2$ only obtains utility from the bottom-ranked candidates in the output ranking, which questions whether $\rhd_1$ can truly be considered proportional.

Next, we analyze the weighted Kemeny utility function u_K^λ induced by $\lambda = (16, 8, 4, 2, 1, 0)$. For this utility function, it holds that $\max(u_K^\lambda) = \sum_{i=1}^m \lambda_i \cdot (m-i) = 129$, so uPJR guarantees utilities of $\lfloor 0.6 \cdot 129 \rfloor = 77$ and $\lfloor 0.4 \cdot 129 \rfloor = 51$ to $\succ_1$ and $\succ_2$. This means that $\rhd_1 = x_1x_2x_6x_5x_4x_3$ fails uPJR for u_K^λ because $u_K^\lambda(\succ_2, \rhd_1) = 4\cdot 3 + 2\cdot 2 + 1 \cdot 1 = 17$. However, it is still possible to satisfy uPJR for R and u_K^λ: for example, the ranking $\rhd_2 = x_3x_4x_2x_1x_5x_6$ satisfies our axiom as $u_K^\lambda(\succ_1, \rhd_2) = 77$ and $u_K^\lambda(\succ_2, \rhd_2) = 52$.

Finally, we discuss the scoring Kemeny utility function u_K^s induced by the Plurality score system $s = (s_i)_{i\in\{1,\dots,m\}}$ given by $s_i(j) = 1$ if $j = m - i > 0$ and $s_i(j) = 0$ otherwise. Less formally, under s, a ranking $\succ$ obtains a point from a candidate x_i if $\succ$ ranks x_i first among all lower-ranked candidates in the output ranking. This implies that $\max(u_K^s) = \sum_{i=1}^m s_i(m-i) = 5$, so uPJR guarantees utilities of $\lfloor 0.6 \cdot 5 \rfloor = 3$ and $\lfloor 0.4 \cdot 5 \rfloor = 2$ to $\succ_1$ and $\succ_2$. While both $\rhd_1$ and $\rhd_2$ fail uPJR for u_K^s, the ranking $\rhd_3 = x_1x_6x_2x_5x_3x_4$ now satisfies our fairness axiom.

3 Results for Weighted Kemeny Utilities

In this section, we will analyze for which weighted Kemeny utility functions uPJR and sPJR are guaranteed to be satisfiable. As we will show, a condition called tail-boundedness will be crucial for this. Specifically, we say a weight vector $\lambda \in \mathbb{N}_0^m$ is *tail-bounded* if, for all $i \in \{1, \dots, m-1\}$, it holds that $\lambda_i \leq 1 + \sum_{j=i+1}^m \lambda_j \cdot (m-j)$. Further, a weighted Kemeny utility function u_K^λ is tail-bounded if its weight vector λ satisfies this condition.

We will show that tail-boundedness characterizes the weighted Kemeny utility functions for which uPJR and sPJR are always satisfiable. In more detail, in Section 3.1, we present a new SWF called the Proportional Sequential Eating rule. This rule is parameterized by a weight vector λ and, if λ is tail-bounded, it always returns an output ranking that satisfies sPJR for the weighted Kemeny utility function u_K^λ. Further, in Section 3.2, we prove that, for every weighted Kemeny utility function that is not tail-bounded, there is a profile where no output ranking satisfies uPJR.

Remark 1. To better understand tail-boundedness, it is useful to consider the maximal tail-bounded weight vector λ^*, which is derived by setting $\lambda_i = 1 + \sum_{j=i+1}^m \lambda_j \cdot (m-j)$ for all $i \in \{1, \dots, m-1\}$. This vector is given by $\lambda^* = ((m-1)!, (m-2)!, \dots, 1!, 0!)$, so tail-boundedness permits even factorial growth of the weight vector. This shows that tail-bounded weighted Kemeny utilities are very flexible.

3.1 The Proportional Sequential Eating Rule

We will now show that sPJR can be satisfied for every profile and tail-bounded weighted Kemeny utility function. To this end, we will introduce a new SWF, the so-called Proportional Sequential Eating Rule ($\texttt{PSE}$), which is parameterized by a weight vector λ and returns for every profile an output ranking that satisfies sPJR for u_K^λ if λ is tail-bounded.

As a subroutine, $\texttt{PSE}$ uses an eating process similar to those of the Simultaneous Veto rule by Kizilkaya and Kempe (2023) or the Probabilistic Serial rule by Bogomolnaia and Moulin (2001). In this process, rankings continuously "eat" their bottom-ranked candidates and the last fully eaten candidate will be selected. Formally, we assume that each ranking $\succ \in \mathcal{R}$ has a weight $w(\succ)$ and there is a set of active candidates Z. Further, let $Y(t) \subseteq Z$ be the set of candidates that are not fully eaten at time t, let $\mu(\succ, x, t)$ be the share of x eaten by $\succ$ by time t, and let $\texttt{B}(\succ, Y(t))$ be the bottom-ranked candidate of $\succ$ in $Y(t)$. At time $t = 0$, we have $Y(0) = Z$ and $\mu(\succ, x, 0) = 0$ for all $\succ \in \mathcal{R}$ and $x \in Z$. Now, each ranking $\succ$ continuously eats its current bottom-ranked candidate $\texttt{B}(\succ, Y(t))$ at rate $w(\succ)$, so $\mu(\succ, \texttt{B}(\succ, Y(t)), t)$ increases at rate $w(\succ)$. Whenever a candidate x is fully eaten, meaning that $\sum_{\succ\in\mathcal{R}} \mu(\succ, x, t) = 1$, we delete this candidate from $Y(t)$ and the rankings move on to their new bottom-ranked candidates. This process continues until all candidates are fully eaten, and we return the last such candidate (with ties broken by our tie-breaking order $>$).

Now, the $\texttt{PSE}$ rule repeatedly uses this eating process and combines it with budgets to guarantee proportionality. Specifically, we assume that, in each round $i \in \{1, \dots, m\}$, each input ranking $\succ$ has a budget $b_i(\succ)$, the total budget is $B_i = \sum_{\succ\in\mathcal{R}} b_i(\succ)$, and X_i denotes the set of remaining candidates. In the first round, it holds that $b_1(\succ) = R(\succ) \cdot \max(u_K^\lambda)$ for all $\succ \in \mathcal{R}$, $B_1 = \max(u_K^\lambda)$, and $X_1 = C$, where R is the input profile and λ the considered weight vector. In each round i, we execute the eating process with $Z = X_i$ and $w(\succ) = b_i(\succ)$ for all $\succ \in \mathcal{R}$ (i.e., the weights of the rankings are their current budgets) to compute the last fully eaten candidate x^* and the shares $\mu_i(\succ, x)$ specifying how much $\succ$ has eaten of x. Then, we put x^* at the i-th position of the output ranking and remove it from the set of remaining candidates (i.e., $X_{i+1} = X_i \setminus \{x^*\}$). Next, to ensure proportionality, we reduce the budget of each ranking $\succ$ to $b_{i+1}(\succ) = b_i(\succ) - \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \sum_{x \in X_i \setminus \{x^*\}} \mu_i(\succ, x)$. After this, we continue with the next round until all candidates are placed in the output ranking.

We next present an example to illustrate the PSE rule. Also, pseudocode for this SWF can be found in Appendix A.1.

Example 2 (PSE rule)**.** We compute PSE for the following profile R and the weight vector $\lambda = (5, 5, 2, 1, 0)$.

$$0.5\colon \succ_1 = x_1x_2x_3x_4x_5$$
$$0.25\colon \succ_2 = x_5x_1x_2x_3x_4$$
$$0.25\colon \succ_3 = x_5x_4x_3x_2x_1$$

We first observe that the maximal utility under the weighted Kemeny utility function induced by λ is $\max(u_K^\lambda) = 40$. Hence, the initial budgets of the input rankings are $b_1(\succ_1) = 20$ and $b_1(\succ_2) = b_1(\succ_3) = 10$. Next, the eating process of PSE starts with $Z = C$. First, at time $t = \frac{1}{20}$, $\succ_1$ has fully eaten x_5. From this point on, both $\succ_1$ and $\succ_2$ consume x_4, which will be fully eaten at $t = \frac{4}{60}$ (with $\mu(\succ_1, x_4, t) = \frac{1}{3}$ and $\mu(\succ_2, x_4, t) = \frac{2}{3}$). After this, both $\succ_1$ and $\succ_2$ eat x_3, while $\succ_3$ still eats x_1. At $t = \frac{1}{10}$, both x_3 and x_1 will be fully consumed with $\mu(\succ_1, x_3, t) = \frac{2}{3}$, $\mu(\succ_2, x_3, t) = \frac{1}{3}$, and $\mu(\succ_3, x_1, t) = 1$. Hence, x_2 will be the last eaten candidate and top-ranked by PSE. Lastly, to update the budgets of our rankings we note that $\min(\frac{B_1}{|X_1|}, \lambda_1) = \lambda_1 = 5$ and that $\succ_1$ consumed a total of 2 from candidates in $C \setminus \{x_2\}$, whereas both $\succ_2$ and $\succ_3$ consumed a total of 1. Thus, $b_2(\succ_1) = 20 - 5 \cdot 2 = 10$ and $b_2(\succ_2) = b_2(\succ_3) = 10 - 5 \cdot 1 = 5$.

In the second round, we have $X_2 = X_1 \setminus \{x_2\} = \{x_1, x_3, x_4, x_5\}$. Hence, we execute our eating process on the following reduced and budget-weighted profile R':

$$10\colon x_1x_3x_4x_5$$
$$5\colon x_5x_1x_3x_4$$
$$5\colon x_5x_4x_3x_1$$

For this profile, it can be checked that x_5 will be eaten first, x_4 second, and x_1 and x_3 at the same time. Assuming that our tie-breaking prefers x_1, we will place this candidate second in the output ranking. Further, it can be checked that $\succ_1$ has eaten a total of 2 among candidates in $X_2 \setminus \{x_1\}$, $\succ_2$ a total of 1, and $\succ_3$ a total of 0. Since $\min(\frac{B_2}{X_2}, \lambda_2) = 5$, the new budgets are $b_3(\succ_1) = 10 - 5 \cdot 2 = 0$, $b_3(\succ_2) = 5 - 5 \cdot 1 = 0$, and $b_3(\succ_3) = 5 - 5 \cdot 0 = 5$. Finally, since $\succ_3$ is now the only ranking with non-zero budget, PSE will order the remaining candidates according to $\succ_3$, so it returns $x_2x_1x_5x_4x_3$.

We will next show, that for every profile R and tail-bounded weight vector λ, the output ranking returned by PSE satisfies sPJR for R and the corresponding weighted Kemeny utility function u_K^λ. We defer the full proof of this result to Appendix A.1 and discuss a proof sketch instead.

Theorem 1. *For every profile R and tail-bounded weight vector λ, the ranking returned by the Proportional Sequential Eating rule satisfies sPJR for R and the weighted Kemeny utility function u_K^λ.*

Proof Sketch. To prove this result, we follow the strategy of Lederer (2026) and separate our proof into two steps. First, we generalize Lederer's notion of pair-priceability to weighted Kemeny utility functions and show that this condition implies sPJR under our more general utility functions. Roughly, a ranking $\rhd = x_1 \dots x_m$ is pair-priceable for a profile R and weighted Kemeny utility function u_K^λ if there is a payment scheme π from the input rankings $\succ \in \mathcal{R}$ to the pairs of candidates (x_i, x_j) with $x_i \rhd x_j$ such that

(1) no input ranking pays more than $R(\succ) \cdot \max(u_K^\lambda)$,
(2) each input ranking pays only for pairs of candidates (x_i, x_j) with $x_i \succ x_j$,
(3) for each pair of candidates (x_i, x_j) with $x_i \rhd x_j$, the input rankings pay at most λ_i in total, and
(4) the total payment of the input rankings to the pairs of candidates is more than $\max(u_K^\lambda) - 1$.

Secondly, we prove that the ranking $\rhd = x_1 \dots x_m$ returned by PSE for a given profile R is pair-priceable if λ is tail-bounded. To show this, we denote by $X_i = \{x_i, \dots, x_m\}$ the remaining candidates in the i-th round of PSE and by $\mu_i(\succ, x)$ the share that $\succ$ has eaten of x in round i. Then, we prove that the payment scheme π given by $\pi(\succ, (x_i, x_j)) = \min(\lambda_i, \frac{B_i}{|X_i|}) \cdot \mu_i(\succ, x_j)$ satisfies all conditions of pair-priceability. This means that the ranking chosen by PSE satisfies pair-priceability and thus also sPJR. $\square$

Remark 2. We require in the definition of weight vectors λ that λ is non-increasing, $\lambda_{m-1} = 1$, and $\lambda_m = 0$. This captures that higher-ranked positions are more valuable and that each position of the output ranking but the last one has some value to the input rankings. However, with one small twist, the output rankings of PSE satisfy sPJR even for tail-bounded weighted Kemeny utilities that fail these assumptions. Specifically, the only new issue is that all rankings may run out of budget before all candidates are ranked, which results in an ill-defined eating process. However, in this case, we may order the remaining candidates arbitrarily since the output ranking of PSE satisfies sPJR if the budget is fully spent.

3.2 Optimality of PSE

Next, we prove that uPJR and thus also sPJR can be unsatisfiable for weighted Kemeny utility functions that are not tail-bounded. Hence, the PSE rule satisfies sPJR for all weighted Kemeny utility functions for which this condition is guaranteed to be satisfiable. The full proof of the following theorem can be found in Appendix A.2.

Theorem 2. *For every weighted Kemeny utility function that is not tail-bounded, there is a profile for which no ranking satisfies uPJR.*

Proof Sketch. To prove this theorem, we fix a weighted Kemeny utility function u_K^λ whose weight vector λ fails tail-boundedness. This means that there is an index $i \in$

$\{1, \ldots, m-1\}$ such that $\lambda_i > 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m-j)$. Further, we fix two reverse rankings $\succ_1 = x_1 \ldots x_m$ and $\succ_2 = x_m \ldots x_1$ and consider the profile R with $R(\succ_1) = \frac{\lambda_i - 1}{\max(u_K^\lambda)}$ and $R(\succ_2) = 1 - R(\succ_1)$. For this profile, no ranking satisfies uPJR. The reason for this is that uPJR guarantees to $\succ_2$ a utility of at least $\max(u_K^\lambda) - \lambda_i + 1$, which implies that the first i candidates of the output ranking must agree with $\succ_2$. However, this means that $\succ_1$ only obtains a utility from the candidates at positions $i+1, \ldots, m$, whose value is at most $\sum_{j=i+1}^{m} \lambda_j \cdot (m-j) < \lambda_i - 1$. Since uPJR entitles $\succ_1$ to a utility of $\lambda_i - 1$, this axiom is violated by $\succ_1$. $\square$

Remark 3. There is another condition under which uPJR can be unsatisfiable: for each utility function u such that $\max(u) \geq m!$ and $u(\succ, \succ^{-1}) = 0$ for all rankings $\succ = x_1 \ldots x_m$ and their reverse rankings $\succ^{-1} = x_m \ldots x_1$, there is a profile for which no ranking satisfies uPJR. Specifically, this holds for the profile R where each input ranking $\succ$ has a weight of $R(\succ) = \frac{1}{m!}$. Indeed, if $\max(u) \geq m!$, uPJR guarantees each ranking a utility of at least 1 in R. However, the output ranking must be the reverse of some input ranking, which then has a utility of 0. By contrast, for the "maximal" tail-bounded weighted Kemeny utility function $u_K^{\lambda^*}$ given by $\lambda^* = ((m-1)!, \ldots, 1!, 0!)$, it holds that $\max(u_K^{\lambda^*}) = m! - 1$ and uPJR is always satisfiable by Theorem 1.

4 Results for Scoring Kemeny Utilities

We now turn to the analysis of scoring Kemeny utility functions, for which we generalize the proportional sequential Borda rule of Lederer (2026) to arbitrary score systems. This results in the class of Proportional Sequential Scoring (PSS) rules and we analyze when these SWFs return proportional rankings. Similar to our analysis of weighted Kemeny utilities, the following regularity condition on the score systems turns out to be crucial: a score system $s = (s_i)_{i \in \{1,\ldots,m\}}$ is *ratio-bounded* if it holds for all $i \in \{1, \ldots, m-1\}$ that

$$\frac{s_i(m-i)}{1 + \sum_{j=i}^{m} s_j(m-j)} \leq \frac{\frac{1}{m-i+1} \sum_{j=0}^{m-i} s_i(j)}{s_i(m-i)}.$$

Intuitively, this condition bounds the ratio between the maximal utility in round i and the total utility in all remaining rounds (left hand side) by the ratio of the average and the maximum utility in round i (right hand side).

In Section 4.1, we show that, for every ratio-bounded score system s, the corresponding PSS rule always returns rankings satisfying uPJR for the scoring Kemeny utility u_K^s. Furthermore, while scoring Kemeny utilities are too flexible to fully characterize when uPJR is satisfiable, we present in Section 4.2 a subclass of these utility functions for which tail-boundedness is necessary for PSS rules to satisfy uPJR.

Remark 4. To illustrate ratio-boundedness, we discuss two examples. First, consider the Plurality score system $s = (s_i)_{i \in \{1,\ldots,m\}}$ given by $s_i(m-i) = 1$ for all $i < m$ and $s_i(j) = 0$ for all $i \in \{1, \ldots, m\}$ and $j < m-i$. This score system gives 1 point to a ranking if it receives the maximal possible Kemeny utility in a step and 0 otherwise. The Plurality score system is ratio-bounded because $\frac{s_i(m-i)}{1+\sum_{j=i}^{m} s_j(m-j)} = \frac{1}{m-i+1}$ and $\frac{\sum_{j=0}^{m-i} s_i(j)}{(m-i+1) \cdot s_i(m-i)} = \frac{1}{m-i+1}$ for all $i \in \{1, \ldots, m-1\}$. Second, it can be checked that the score system $\hat{s} = (\hat{s}_i)_{i \in \{1,\ldots,m\}}$ given by $\hat{s}_i(j) = (m-i+1)! - (m-i)!$ for all $j > 0$ and $i < m$, and $\hat{s}_i(0) = 0$ for all $i \in \{1, \ldots, m\}$, is ratio-bounded as well. Intuitively, in round i, this score system gives utility $(m-i+1)! - (m-i)!$ to each ranking that obtains some Kemeny utility.

4.1 Proportional Sequential Scoring Rules

In this section, we introduce Proportional Sequential Scoring (PSS) rules and show that, if their score system s is ratio-bounded, their output ranking is guaranteed to satisfy uPJR for u_K^s. Intuitively, the PSS rule induced by a score system s sequentially computes the winning ranking from top to bottom and the rankings repeatedly use budgets to buy the next candidate. In more detail, in each step, we will choose the candidate maximizing the score with respect to the current budgets, put it at the next-best position of the output ranking, and reduce the budget of each input ranking proportional to its contribution to the total score of the selected candidate.

Formally, in each round i of the PSS rule induced by a score system $s = (s_j)_{j \in \{1,\ldots,m\}}$, each ranking $\succ$ has a budget $b_i(\succ)$ and there is a set of remaining candidates X_i. In the first round, it holds that $b_1(\succ) = R(\succ) \cdot \max(u_K^s)$ for all $\succ \in \mathcal{R}$ and $X_1 = C$, where R is the input profile. Now, in each round $i \in \{1, \ldots, m\}$, we select the candidate x^* maximizing the total score $\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ, x, X_i))$ among all candidates in X_i (with ties broken by $>$), put x^* at the i-th position of the output ranking, and remove it from the set of available candidates (i.e., $X_{i+1} = X_i \setminus \{x^*\}$). Finally, we assume that the price of x^* is $s_i(m-i)$, i.e., the maximal utility a ranking could receive from x^*. Each ranking pays a share of this cost that is either proportional to its contribution to the total score of x^* or its entire remaining budget. Formally, we set $b_{i+1}(\succ) = b_i(\succ) - \min(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x^*, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x^*, X_i))})$ for each $\succ \in \mathcal{R}$. After updating the budgets, we proceed with the next round until all candidates are placed in the output ranking.

We observe that the PSS rule corresponding to the Borda score system (given by $s_i(j) = j$ for all i and j) is precisely the proportional sequential Borda rule of Lederer (2026). We next consider an example illustrating PSS rules and note that pseudocode for these SWFs is given in Appendix A.4.

Example 3 (Plurality PSS rule)**.** We compute the PSS rule induced by the Plurality score system s (see Remark 4) on the following profile.

$$\begin{aligned} 0.5\colon\ & \succ_1 = x_1 x_2 x_3 x_4 x_5 \\ 0.25\colon\ & \succ_2 = x_2 x_4 x_3 x_5 x_1 \\ 0.25\colon\ & \succ_3 = x_4 x_5 x_2 x_3 x_1 \end{aligned}$$

First, we observe that $\max(u_K^s) = 4$ for the Plurality score system s, so the initial budgets of our rankings are $b_1(\succ_1) = 2$ and $b_1(\succ_2) = b_1(\succ_3) = 1$. Further, we set $X_1 = C$. In the first round, x_1 maximizes the total score as $\sum_{\succ \in \mathcal{R}} b_1(\succ) \cdot s_1(\hat{u}(\succ, x_1, X_1)) = 2$. Since only $\succ_1$ obtains utility from x_1, it pays its entire cost of $s_1(m-1) = 1$.

Hence, the new budgets are $b_2(\succ_1) = b_2(\succ_2) = b_2(\succ_3) = 1$ and we set $X_2 = X_1 \setminus \{x_1\} = \{x_2, \dots, x_5\}$. In the second round, x_2 maximizes the total score with $\sum_{\succ \in \mathcal{R}} b_2(\succ) \cdot s_2(\hat{u}(\succ, x_2, X_2)) = 2$. Since both $\succ_1$ and $\succ_2$ top-rank this candidate and have the same budget, each of them pays a share of 0.5. Thus, the new budgets are $b_1(\succ_1) = b_2(\succ_2) = 0.5$ and $b_2(\succ_3) = 1$, and we set $X_3 = X_2 \setminus \{x_2\} = \{x_3, x_4, x_5\}$. In the third round, x_4 maximizes the total score with $\sum_{\succ \in \mathcal{R}} b_3(\succ) \cdot s_3(\hat{u}(\succ, x_3, X_3)) = 1.5$. Both $\succ_2$ and $\succ_3$ will contribute to the cost of x_4, so $\succ_2$ pays $\frac{0.5}{1.5} = \frac{1}{3}$ and $\succ_3$ pays $\frac{1}{1.5} = \frac{2}{3}$. Hence, $b_4(\succ_1) = 0.5$, $b_4(\succ_2) = \frac{1}{6}$, and $b_4(\succ_3) = \frac{1}{3}$ and $X_4 = X_3 \setminus \{x_4\} = \{x_3, x_5\}$. By continuing in this manner, it can be checked that the PSS rule induced by $s = (s_i)_{i \in \{1,\dots,5\}}$ selects $x_1 x_2 x_4 x_5 x_3$ for R.

We next show that, for ratio-bounded score systems s, the induced PSS rules always return rankings that satisfy uPJR for the corresponding scoring Kemeny utility function u_K^s. The full proof of this claim can be found in Appendix A.4.

Theorem 3. *For every ratio-bounded score system s, the ranking returned by the Proportional Sequential Scoring rule induced by s satisfies uPJR for every profile and the scoring Kemeny utility function u_K^s.*

Proof Sketch. The proof of this theorem roughly follows the same strategy as that of Theorem 1. First, we introduce another notion called rank-priceability and show that this axiom implies uPJR for every profile R and scoring Kemeny utility u_K^s. This condition is a weakening of the pair-priceability condition used in the proof of Theorem 1: a ranking $\rhd = x_1 \dots x_m$ is rank-priceable for a profile R and a scoring Kemeny utility function u_K^s if there is a payment scheme π from the rankings to the candidates such that

(1) each ranking pays in total at most $R(\succ) \cdot \max(u_K^s)$,
(2) each ranking $\succ$ pays at most $s_i(\hat{u}(\succ, x_i, \{x_i, \dots, x_m\}))$ for each candidate x_i,
(3) for each candidate x_i, the rankings pay in total at most $s_i(m-i)$, and
(4) the rankings pay in total strictly more than $\max(u_K^s) - 1$.

As the second step, we show that, if s is ratio-bounded, the output ranking $\rhd = x_1 \dots x_m$ of the PSS rule induced by s is rank-priceable for u_K^s and the input profile R. To prove this claim, we let $b_i(\succ)$ denote the budgets of the input rankings during the execution of our rule. Then, we show that the payment scheme π given by $\pi(\succ, x_i) = b_i(\succ) - b_{i+1}(\succ)$ satisfies all conditions of rank-priceability for R and u_K^s. Hence, $\rhd$ is rank-priceable and thus also satisfies uPJR. $\square$

4.2 Necessity of Ratio-boundedness

Finally, we will show that there is a family of score systems for which ratio-boundedness is necessary to guarantee that the ranking chosen by the corresponding PSS rule satisfies uPJR. Specifically, we subsequently focus on *balanced* score systems $s = (s_i)_{i \in \{1,\dots,m\}}$, which additionally satisfy that, for all $i \in \{1, \dots, m\}$, there is a constant c_i such that $s_i(j) + s_i(m-i-j) = c_i$ for all $j \in \{0, \dots, m-i\}$. For example, the Borda score system is balanced. We next show that, for every balanced score system s that is not ratio-bounded, there is a profile where the output of the corresponding PSS rule fails uPJR for u_K^s. Thus, for balanced score systems, ratio-boundedness is both necessary and sufficient to ensure that the output ranking of PSS rules satisfy uPJR.

Theorem 4. *For every balanced score system s that is not ratio-bounded, there is a profile R such that the ranking chosen by the corresponding Proportional Sequential Scoring rule fails uPJR for R and u_K^s.*

Proof Sketch. To prove this theorem, consider a balanced score system s that is not ratio-bounded. First, we show that our assumptions on s imply that there is an index $j^* \in \{1, \dots, m-1\}$ such that $s_{j^*}(m-j^*) > 1 + \sum_{j=j^*+1}^{m} s_j(m-j)$. Next, we fix two reverse rankings $\succ_1 = x_1 \dots x_m$ and $\succ_2 = x_m \dots x_1$ and the profile R given by $R(\succ_1) = \frac{-1+\sum_{j=1}^{j^*} s_j(m-j)}{\max(u_K^s)}$ and $R(\succ_2) = 1 - R(\succ_1)$. Based on our assumptions on s, we show that the PSS rule induced by s chooses the ranking $\rhd = x_1 \dots x_{j^*} x_m \dots x_{j^*+1}$ for R. However, this ranking fails uPJR because $u_K^s(\succ_2, \rhd) = \sum_{j=j^*+1}^{m} s_j(m-j)$ but uPJR guarantees a utility of $1 + \sum_{j=j^*+1}^{m} s_j(m-j)$ to $\succ_2$. $\square$

Remark 5. Theorem 4 only proves that PSS rules may return rankings that fail uPJR if ratio-boundedness is violated. However, there can be other rankings that satisfy uPJR. For example, the score systems corresponding to weighted Kemeny utility functions are balanced. Moreover, for weighted Kemeny utility functions u_K^λ, ratio-boundedness requires that $\lambda_i \cdot (m-i) \leq 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m-j)$ for all $i \in \{1, \dots, m-1\}$. However, PSE returns rankings satisfying uPJR if u_K^λ is tail-bounded, which only requires that $\lambda_i \leq 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m-j)$ for all $i \in \{1, \dots, m-1\}$.

5 Conclusion

In this paper, we study proportional rank aggregation for two novel classes of utility functions. Specifically, our main axiom is called unanimous justified proportional representation (uPJR) and requires that each input ranking receives a utility at least proportional to its weight from the output ranking. For our first class of utility functions, so-called weighted Kemeny utilities, we characterize when uPJR and a strengthening thereof are guaranteed to be satisfiable based on a condition called tail-boundedness. Moreover, we introduce a new SWF that returns proportional rankings for all tail-bounded weighted Kemeny utilities. For our second class of utility functions, so-called scoring Kemeny utilities, we generalize the proportional sequential Borda rule of Lederer (2026) to arbitrary score systems and give a sufficient condition when the resulting rules satisfy uPJR.

Our paper points to numerous follow-up questions. Firstly, an immediate open problem is which utility functions one should use in practice. Secondly, we believe it would be interesting to study proportional rank aggregation also for positional utility functions (which specify for each pair (i, j) how much utility a ranking gets if its i-th best candidate is at the j-th position of the output ranking) and cardinal models (where we are given numerical values instead of a ranking).

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A Omitted Proofs

A.1 Proof of Theorem 1

In this appendix, we will prove that the ranking returned by the Proportional Sequential Eating rule satisfies sPJR for every profile R and tail-bounded weighted Kemeny utility function u_K^λ. Further, we note that a pseudocode description of this SWF is given in Algorithm 1.

Theorem 1. *For every profile R and tail-bounded weight vector λ, the ranking returned by the Proportional Sequential Eating rule satisfies sPJR for R and the weighted Kemeny utility function u_K^λ.*

As described in the main body, we will prove this claim in two steps. As the first step, we will introduce an auxiliary property called pair-priceability and show that this condition implies sPJR for every profile R and weighted Kemeny utility function u_K^λ. The idea of pair-priceability is the following: first, we assume that there is a total budget of $\max(u_K^\lambda)$ that is distributed to the rankings proportional to their weights. Then, the rankings use this budget to buy the pairs of candidates (x, y) with $x \rhd y$. Specifically, each ranking $\succ$ is only allow allowed to pay for pairs (x, y) such that $x \succ y$, the rankings can pay at most λ_i for a pair (x, y) (where i is the position of x in the output ranking), each ranking pays at most its share of the budget, and the rankings overall spend almost the entire budget. To formalize this condition, we recall that $K(\succ, \rhd) = \{(x, y) \in C^2 \colon x \rhd y \land x \succ y\}$ denotes the set of pairs on which $\succ$ and $\rhd$ agree on. In particular this means that $K(\rhd, \rhd) = \{(x, y) \in C^2 \colon x \rhd y\}$.

Definition 3 (Pair-priceability for weighted Kemeny utilities)**.** A ranking $\rhd = x_1 \dots x_m$ satisfies pair-priceability for a profile R and a weighted Kemeny utility function u_K^λ if there is a payment scheme $\pi : \mathcal{R} \times K(\rhd, \rhd) \to \mathbb{R}_{\geq 0}$ such that

(1) $\pi(\succ, (x_i, x_j)) = 0$ for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in K(\rhd, \rhd) \setminus K(\succ, \rhd)$,
(2) $\sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) \leq R(\succ) \cdot \max(u_K^\lambda)$ for all $\succ \in \mathcal{R}$,
(3) $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) \leq \lambda_i$ for all $(x_i, x_j) \in K(\rhd, \rhd)$,
(4) $\sum_{\succ \in \mathcal{R}} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) > \max(u_K^\lambda) - 1$.

We note that this definition extends an analogous axiom of Lederer (2026) from Kemeny utilities to weighted Kemeny utilities. Moreover, similar to this author, we will next show that pair-priceability implies sPJR for every profile R and weighted Kemeny utility function u_K^λ.

Lemma 1. *If a ranking $\rhd$ satisfies pair-priceability for a profile R and a weighted Kemeny utility function u_K^λ, then it also satisfies sPJR.*

Proof. Fix a profile R, a weighted Kemeny utility function u_K^λ with weight vector λ, and a ranking $\rhd = x_1 \dots x_m$. Further, we define $X_i = \{x_i, \dots, x_m\}$ for all $i \in \{1, \dots, m\}$. Now, assume for contradiction that $\rhd$ satisfies pair-priceability for R and u_K^λ but not sPJR. Since $\rhd$ fails sPJR, there is a set of rankings $S \subseteq \mathcal{R}$ such that $\sum_{(x_i,x_j) \in \bigcup_{\succ \in S} K(\succ,\rhd)} \lambda_i < \lfloor \sum_{\succ \in S} R(\succ) \cdot \max(u_K^\lambda) \rfloor$. On the other hand, since $\rhd$ satisfies pair-priceability, there is a payment scheme π satisfying all conditions of this axiom for R and u_K^λ. By Condition (2), it holds for the rankings in $\mathcal{R} \setminus S$ that

$$\sum_{\succ \in \mathcal{R} \setminus S} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) \leq (1 - \sum_{\succ \in S} R(\succ)) \cdot \max(u_K^\lambda).$$

Further, by Condition (1) of pair-priceability, the rankings in S are only allowed to pay for pairs $(x_i, x_j) \in \bigcup_{\succ \in S} K(\succ, \rhd)$. Moreover, for each such pair (x_i, x_j), these rankings can pay at most $\sum_{\succ \in S} \pi(\succ, (x_i, x_j)) \leq \lambda_i$ due to Condition (3) of pair-priceability. By summing over all pairs of candidates $(x_i, x_j) \in \bigcup_{\succ \in S} K(\succ, \rhd)$, we derive that

$$\sum_{\succ \in S} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) = \sum_{\succ \in S} \sum_{(x_i,x_j) \in \bigcup_{\succ \in S} K(\succ,\rhd)} \pi(\succ, (x_i, x_j)) \quad \leq \sum_{(x_i,x_j) \in \bigcup_{\succ \in S} K(\succ,\rhd)} \lambda_i.$$

Finally, since the right-hand side is an integer and strictly less than $\lfloor \sum_{\succ \in S} R(\succ) \cdot \max(u_K^\lambda) \rfloor$ by assumption, we conclude that $\sum_{\succ \in S} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) \leq \sum_{\succ \in S} R(\succ) \cdot \max(u_K^\lambda) - 1$. Combining our inequalities now shows that

$$\begin{aligned}\sum_{\succ \in \mathcal{R}} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) &\leq (1 - \sum_{\succ \in \mathcal{R}} R(\succ)) \cdot \max(u_K^\lambda) + \sum_{\succ \in S} R(\succ) \cdot \max(u_K^\lambda) - 1 \\ &= \max(u_K^\lambda) - 1.\end{aligned}$$

This contradicts Condition (4) of pair-priceability, so our initial assumption is wrong and every pair-priceable ranking satisfies sPJR. □

As our second step, we will show that for every tail-bounded weight vector λ and every profile R, the ranking returned by `PSE` is pair-priceable for R and u_K^λ.

Lemma 2. *For each profile R and tail-bounded weight vector λ, the ranking returned by* `PSE` *is pair-priceable for the weighted Kemeny utility function u_K^λ.*

Proof. Consider a profile R, let u_K^λ denote a tail-bounded weighted Kemeny utility function, and let λ be its corresponding weight vector. Moreover, let $\rhd = x_1 \dots x_m$ denote the ranking chosen by the `PSE` rule for R and u_K^λ. For each round i during the execution of `PSE`, we define by $b_i(\succ)$ the remaining budget of each ranking $\succ$, by $B_i = \sum_{\succ \in \mathcal{R}} b_i(\succ)$ the total remaining budget, and by $X_i = \{x_i, \dots, x_m\}$ the set of remaining candidates. Further, we denote by $\mu_i(\succ, x_j)$ how much the ranking $\succ$ consumed of candidates x_j during the eating process of `PSE` in the i-th round. Finally, to show that $\rhd$ is pair-priceable for R and u_K^λ, we consider the payment scheme given by $\pi(\succ, (x_i, x_j)) = \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \mu_i(\succ, x_j)$ for all $\succ \in \mathcal{R}$ and $(x_i, x_j) \in K(\rhd, \rhd)$. We will next show that π satisfies all conditions of pair-priceability.

Condition (1): Fix an input ranking $\succ$ and a pair of candidates $(x_i, x_j) \in K(\rhd, \rhd)$ with $x_j \succ x_i$. We claim that $\mu_i(x_j) = 0$ and thus $\pi(\succ, (x_i, x_j)) = 0$. To see this, we recall that, during the execution of `PSE`, rankings eat always their bottom-ranked candidate and that x_i is the last fully eaten candidate in the i-th round. Hence, if $x_j \succ x_i$, the ranking $\succ$ cannot consume candidate x_j, because this would require that x_j is fully eaten after x_i. This proves that π satisfies Condition (1) of pair-priceability.

Condition (2): Next, we will show that $\sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) \leq R(\succ) \cdot \max(u_K^\lambda)$ for each ranking $\succ \in \mathcal{R}$. To this end, we observe that, for all $i \in \{1, \dots, m-1\}$, we have that $b_i(\succ) - b_{i+1}(\succ) = \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \sum_{x_j \in X_i \setminus \{x_i\}} \mu_i(\succ, x_j) = \sum_{x_j \in X_i \setminus \{x_i\}} \pi(\succ, (x_i, x_j))$. Hence, $\sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) = \sum_{i=1}^{m-1} b_i(\succ) - b_{i+1}(\succ) = b_1(\succ) - b_m(\succ)$. Further, we recall that $b_1(\succ) = R(\succ) \cdot \max(u_K^\lambda)$ by definition.

Hence, we only need to show that $b_m(\succ) \geq 0$. For this, we note that, in each round i, the eating process stops at time $\frac{|X_i|}{B_i}$ since each ranking continuously consumes candidates with speed proportional to its remaining budget. As a consequence, in each round, i, the total share consumed by a ranking $\succ$ is $\sum_{x_j \in X_i} \mu_i(\succ, x_j) = \frac{|X_i|}{B_i} \cdot b_i(\succ)$. Finally, the total payment of the ranking $\succ$ is

$$\begin{aligned}\sum_{x_j \in X_i \setminus \{x_i\}} \pi(\succ, (x_i, x_j)) &= \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \sum_{x_j \in X_i \setminus \{x_i\}} \mu_i(\succ, x_j) \\ &\leq \frac{B_i}{|X_i|} \cdot \frac{|X_i|}{B_i} \cdot b_i(\succ) \\ &= b_i(\succ).\end{aligned}$$

This proves that, in every round, a ranking never pays more than its remaining budget, so $b_m(\succ) \geq 0$. In turn, we derive that Condition (2) of pair-priceability is satisfied.

Condition (3): For the third condition of pair-priceability, we fix a pair of candidates $(x_i, x_j) \in K(\rhd, \rhd)$ and show that $\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) \leq \lambda_i$. To see this, we recall that, during the eating process, each candidate is consumed until $\sum_{\succ \in \mathcal{R}} \mu_i(\succ, x_j) = 1$. Hence, we derive that

$$\sum_{\succ \in \mathcal{R}} \pi(\succ, (x_i, x_j)) = \sum_{\succ \in \mathcal{R}} \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \mu_i(\succ, x_j) \leq \lambda_i,$$

which proves this condition.

Condition (4): Finally, we turn to the last condition of pair-priceability and show that $\sum_{\succ \in \mathcal{R}} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) > \max(u_K^\lambda) - 1$. To this end, we prove that, for all $i \in \{1, \dots, m\}$, it holds that the total budget B_i in round i, satisfies that

$$\sum_{j=i}^{m} \lambda_j \cdot (m - j) \leq B_i < 1 + \sum_{j=i}^{m} \lambda_j \cdot (m - j).$$

In particular, this means that $B_m < 1$. Further, just as for Condition (2), it holds that $\sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) = b_1(\succ) - b_m(\succ)$ for each ranking $\succ \in \mathcal{R}$. Since $\sum_{\succ \in \mathcal{R}} b_1(\succ) = B_1 = \max(u_K^\lambda)$ and $\sum_{\succ \in \mathcal{R}} b_m(\succ) = B_m$, this means that $\sum_{\succ \in \mathcal{R}} \sum_{(x_i,x_j) \in K(\rhd,\rhd)} \pi(\succ, (x_i, x_j)) = B_1 - B_m > \max(u_K^\lambda) - 1$, as desired.

It remains to prove out auxiliary claim, for which we will proceed by induction. First, the induction basis holds trivially since $B_1 = \max(u_K^\lambda) = \sum_{i=1}^{m} \lambda_i \cdot (m - i)$ by definition. Next, fix some $i \in \{1, \dots, m-1\}$ and assume that $\sum_{j=i}^{m} \lambda_j \cdot (m - j) \leq B_i < 1 + \sum_{j=i}^{m} \lambda_j \cdot (m - j)$. We will show that $\sum_{j=i+1}^{m} \lambda_j \cdot (m - j) \leq B_{i+1} < 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$. To this end, we recall that $b_{i+1}(\succ) = b_i(\succ) - \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot \sum_{x_j \in X_i \setminus \{x_i\}} \mu_i(\succ, x_j)$ for each ranking $\succ$. By summing over all rankings $\succ \in \mathcal{R}$, we obtain that

$$\begin{aligned} B_{i+1} &= \sum_{\succ \in \mathcal{R}} b_{i+1}(\succ) \\ &= \sum_{\succ \in \mathcal{R}} \left(b_i(\succ) - \min(\frac{B_i}{|X_i|}, \lambda_i) \sum_{x_j \in X_i \setminus \{x_i\}} \mu_i(\succ, x_j) \right) \\ &= B_i - \min(\frac{B_i}{|X_i|}, \lambda_i) \cdot (|X_i| - 1). \end{aligned}$$

Here, the last inequality uses that $\sum_{\succ \in \mathcal{R}} \sum_{x_j \in X_i \setminus \{x_i\}} \mu_i(\succ, x_j) = |X_i| - 1$ because each candidate is fully eaten by the input rankings during the eating process.

We next proceed with a case distinction regarding $\min(\frac{B_i}{|X_i|}, \lambda_i)$. First, suppose that this minimum is attained by λ_i. In this case, we have that $B_{i+1} = B_i - \lambda_i \cdot (|X_i| - 1)$. Now, in the i-th round, there are $m - i + 1$ candidates available, so it holds that $|X_i| - 1 = m - i$. Hence, we have that $B_{i+1} = B_i - \lambda_i \cdot (m - i)$. Further, by the induction hypothesis, it holds that $\sum_{j=i}^{m} \lambda_j \cdot (m - j) \leq B_i < 1 + \sum_{j=i}^{m} \lambda_j \cdot (m - j)$. Combining these three insights shows that $\sum_{j=i+1}^{m} \lambda_j \cdot (m - j) \leq B_{i+1} < 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$, thus proving our induction step in this case.

As the second case, we suppose that $\min(\frac{B_i}{|X_i|}, \lambda_i) = \frac{B_i}{|X_i|} < \lambda_i$. Equivalently, this means that $\lambda_i \cdot (m - i + 1) > B_i$ because $|X_i| = m - i + 1$. Further, using the induction hypothesis, we infer that $\lambda_i \cdot (m - i + 1) > \sum_{j=i}^{m} \lambda_j \cdot (m - j)$ or, equivalently, that $\lambda_i > \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$. On the other hand, λ is tail-bounded by assumption, so $\lambda_i \leq 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$. Finally, since our weights are integers by definition, these two inequalities can only be true if $\lambda_i = 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$.

Next, let $\delta_i \in [0, 1)$ such that $B_i = \delta_i + \sum_{j=i}^{m} \lambda_j \cdot (m - j)$; such a δ_i exists by the induction hypothesis. Using our insights on λ_i, it holds that

$$\begin{aligned} B_i &= \delta_i + \sum_{j=i}^{m} \lambda_j \cdot (m - j) \\ &= \delta_i + \left(1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j) \right) \cdot (m - i) + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j) \\ &= \delta_i + (m - i) + (m - i + 1) \cdot \sum_{j=i+1}^{m} \lambda_j \cdot (m - j). \end{aligned}$$

Finally, by the assumption of our case analysis, we have that $B_{i+1} = B_i - \frac{B_i}{|X_i|} \cdot (|X_i| - 1) = \frac{B_i}{m-i+1}$. Combined with our previous equation, this shows that $B_{i+1} = \frac{\delta_i + m - i}{m-i+1} + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$. Since $\delta_i \in [0, 1)$, this means that $\sum_{j=i+1}^{m} \lambda_j \cdot (m - j) \leq B_{i+1} < 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$, so our induction step also holds in this case. We hence conclude that the ranking chosen by PSE satisfies all conditions of pair-priceability. □

A.2 Proof of Theorem 2

We next turn to the proof of Theorem 2 and show that tail-boundedness is necessary so that uPJR rankings are guaranteed to exist for weighted Kemeny utility functions. More specifically, we demonstrate that, for every weighted Kemeny utility function that is not tail-bounded, there is a profile where no output ranking satisfies uPJR.

Theorem 2. *For every weighted Kemeny utility function that is not tail-bounded, there is a profile for which no ranking satisfies uPJR.*

Proof. Fix a weight vector $\lambda \in \mathbb{N}_0^m$ that is not tail-bonded, which means that there is an index $i \in \{1, \dots, m-1\}$ with $\lambda_i > 1 + \sum_{j=i+1}^{m} \lambda_j \cdot (m - j)$. Further, let u_K^λ denote the induced weighted Kemeny utility function. Under u_K^λ, the maximal utility of a ranking is $\max(u_K^\lambda) = \sum_{i=1}^{m-1} \lambda_i \cdot (m - i)$. Next, let $\succ_1 = x_1 \dots x_m$ denote an arbitrary ranking and let $\succ_2 = x_m \dots x_1$ be its reverse ranking. We will show that no ranking satisfies uPJR for the profile R with $R(\succ_1) = \frac{\lambda_i - 1}{\max(u_K^\lambda)}$ and $R(\succ_2) = \frac{\max(u_K^\lambda) - \lambda_i + 1}{\max(u_K^\lambda)}$.

To this end, we first show that $u_K^\lambda(\succ_1, \rhd) + u_K^\lambda(\succ_2, \rhd) = \max(u_K^\lambda)$ for every $\rhd \in \mathcal{R}$. Thus, fix a ranking $\rhd = y_1 \dots y_m$ and let $Y_i = \{y_i, \dots, y_m\}$ for all $i \in \{1, \dots, m\}$. We recall that $\hat{u}_K(\succ, x, X) = |\{y \in X \setminus \{x\} \colon x \succ y\}|$ for all rankings $\succ \in \mathcal{R}$, Algorithm 1: Proportional Sequential Eating Rule (PSE)

Input: A profile $R : \mathcal{R} \to [0, 1]$ and a weight vector λ
Output: Output ranking $\rhd$

$b_1(\succ) \leftarrow R(\succ) \cdot \max(u_K^\lambda)$ for all $\succ \in \mathcal{R}$
$X_1 \leftarrow C$
for $i = 1, \dots, m$ **do**
  $Y \leftarrow X_i$
  $\mu(\succ, x) \leftarrow 0$ for all $\succ \in \mathcal{R}, x \in Y$
  while $Y \neq \emptyset$ **do**
    for all $\succ \in \mathcal{R}$ **in parallel do**
      increase $\mu(\succ, \mathtt{B}_Y(\succ))$ at rate $b_i(\succ)$
      if $\sum_{\succ \in \mathcal{R}} \mu(\succ, \mathtt{B}_Y(\succ)) = 1$ **then**
        $Y \leftarrow Y \setminus \{\mathtt{B}_Y(\succ)\}$
      end if
    end for
  end while
  $x_i \leftarrow$ last fully eaten candidate in X_i
  $X_{i+1} \leftarrow X_i \setminus \{x_i\}$
  for all $\succ \in \mathcal{R}$ **do**
    $b_{i+1}(\succ) \leftarrow b_i(\succ) - \min\left(\lambda_i, \frac{\sum_{\succ \in \mathcal{R}} b_i(\succ)}{|X_i|}\right) \cdot \sum_{x \in X_i \setminus \{x_i\}} \mu(\succ, x)$
  end for
end for
return $\rhd = x_1 x_2 \dots x_m$

sets $X \subseteq C$, and candidates $x \in X$. Since $\succ_1$ and $\succ_2$ are reverse to each other, this means that $\hat{u}_K(\succ_1, x, X) + \hat{u}_K(\succ_2, x, X) = |X| - 1$ for all $X \subseteq C$ and $x \in X$. Finally, since $u_K^\lambda(\succ, \rhd) = \sum_{i=1}^m \lambda_i \cdot \hat{u}_K(\succ, y_i, Y_i)$ for every input ranking $\succ$ and the output ranking $\rhd = y_1 \dots y_m$, it follows that

$$\begin{aligned} u_K^\lambda(\succ_1, \rhd) + u_K^\lambda(\succ_2, \rhd) &= \sum_{i=1}^m \lambda_i \cdot (\hat{u}_K(\succ_1, y_i, Y_i) + \hat{u}_K(\succ_2, y_i, Y_i)) \\ &= \sum_{i=1}^m \lambda_i \cdot (m - i). \end{aligned}$$

Here, the last equality holds because $|Y_i| - 1 = m - i$. This proves our auxiliary claim.

Next, uPJR requires for the profile R that $u_K^\lambda(\succ_1, \rhd) \geq \lfloor R(\succ_1) \cdot \max(u_K^\lambda) \rfloor = \lambda_i - 1$ and $u_K^\lambda(\succ_2, \rhd) \geq \lfloor R(\succ_2) \cdot \max(u_K^\lambda) \rfloor = \max(u_K^\lambda) - \lambda_i + 1$. Since $u_K^\lambda(\succ_1, \rhd) + u_K^\lambda(\succ_1, \rhd) = \max(u_K^\lambda)$ for every $\rhd \in \mathcal{R}$, an output ranking $\rhd$ can only satisfy uPJR if $u_K^\lambda(\succ_1, \rhd) = \lambda_i - 1$ and $u_K^\lambda(\succ_2, \rhd) = \max(u_K^\lambda) - \lambda_i + 1$. This means that the output ranking $\rhd = y_1 \dots y_m$ has to agree with $\succ_2$ on the i top-ranked candidates. Indeed, if this was not the case, there is an index $j \leq i$ such that $\hat{u}_K(\succ_1, y_j, \{y_j, \dots, y_m\}) \geq 1$. However, this implies that $u_K^\lambda(\succ_1, \rhd) \geq \lambda_j \cdot \hat{u}_K(\succ_1, y_j, \{y_j, \dots, y_m\}) \geq \lambda_j \geq \lambda_i$, where the last inequality follows as weight vectors are non-increasing. On the other hand, if the output ranking orders the first i candidates according to $\succ_2$, $\succ_1$ does not obtain any utility from these candidates and the maximal utility of this ranking is $\sum_{j=i+1}^m \lambda_i(m - i)$. Indeed, this utility is obtained when ordering the remaining $m - i$ candidates according to $\succ_1$. However, we have by assumption that $\lambda_i - 1 > \sum_{j=i+1}^m \lambda_i(m - i)$. Hence, uPJR is violated for $\succ_1$ in this case. It follows that there is no ranking $\rhd$ with $u_K^\lambda(\succ_1, \rhd) = \lambda_i - 1$ and $u_K^\lambda(\succ_2, \rhd) = \max(u_K^\lambda) - \lambda_i + 1$, so no ranking satisfies uPJR for R and u_K^λ. $\square$

A.3 On the utilitarian social welfare of PSE

In this appendix, we will discuss a drawback of the Proportional Sequential Eating rule: compared to the optimal ranking, the output ranking of the PSE rule can have a very low utilitarian social welfare, even when only focusing on the unweighted Kemeny utility function. To make this claim formal, we define the *(utilitarian) social welfare* of a ranking $\rhd$ in a profile R under a utility function u by $\sum_{\succ \in \mathcal{R}} R(\succ) \cdot u(\succ, \rhd)$. Further, the *welfare ratio* of an SWF f for a utility function $u\colon \mathcal{R} \times \mathcal{R} \to \mathbb{N}_0$ is the worst-case ratio between the social welfare of the ranking $f(R)$ and the social welfare of the optimal ranking $OPT_u(R) = \arg\max_{\rhd \in \mathcal{R}} \sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u(\succ, \rhd)$ over all profiles R. Formally, the welfare ratio is given by $sw(f, u) = \inf_{R \in \mathcal{R}} \frac{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u(\succ, f(R))}{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u(\succ, OPT_u(R))}$, where the infimum is taken over all profiles with m candidates.

We next show that there is a family of profiles where the welfare ratio of PSE for the Kemeny utility function is only in $\Omega(\frac{1}{\sqrt{m}})$. We note that this sharply contrasts the behavior of the Proportional Sequential Borda rule for which a result by Lederer

(2026) shows that its welfare ratio is lower-bounded by roughly $\frac{1}{4}$. Moreover, our result also means that the average utility of a group of rankings can be far from its proportional share.

Proposition 1. *It holds for the Proportional Sequential Eating rule that $sw(\texttt{PSE}, u_K) \in \Omega(\frac{1}{\sqrt{m}})$.*

Proof. To prove this proposition, we will present a parameterized family of profiles for which the welfare ratio of PSE is in $\Omega(\frac{1}{\sqrt{m}})$. To this end, we fix an integer $k \geq 3$ and assume that $m = k^2$. Further, we partition our candidates into k sets $X_1, \dots, X_k$ such that $|X_i| = k$ for all $i \in \{1, \dots, k\}$. For ease of notation, we denote the candidates in a set X_i by $x_1^i, \dots, x_k^i$. Next, we let R^{X_i} denote the profile over the candidates in X_i where every ranking $\succ$ over X_i has the same weight of $R^{X_i}(\succ) = \frac{1}{|X_i|!}$. Based on these profiles, we will construct our final profile $R^k = R^{k,k}$ inductively. First, $R^{k,1}$ is simply given by R^{X_1}. Next, for every $i \in \{1, \dots, k-1\}$, the profile $R^{k,i+1}$ is derived from $R^{k,i}$ by "inserting" the profile $R^{X_{i+1}}$ into each ranking such that there are precisely i less preferred candidates than those in X_{i+1} in the new ranking. To make that more formal, let $\succ = y_1 \dots y_{i\cdot k}$ denote a ranking in $R^{k,i}$. For deriving R^{k+1}, we replace $\succ$ with $k!$ rankings $\succ'_1, \dots, \succ'_{k!}$, each of which obtains a weight of $R^{k,i+1}(\succ'_i) = \frac{1}{k!} \cdot R^{k,i}(\succ)$. Moreover, each of the rankings $\succ'_i$ satisfies that *(i)* all candidates in $\{y_1, \dots, y_{i\cdot(k-1)}\}$ are ranked ahead of those in X_{i+1} which are ranked ahead of those in $\{y_{1+i\cdot(k-1)}, \dots, y_{i\cdot k}\}$ and *(ii)* all rankings $\succ'_i$ report different orders of the candidates in X_{i+1}.

For instance, when $k = 3$, we have in $R^{3,1}$ each of the 6 possible rankings over $X_1 = \{x_1^1, x_2^1, x_3^1\}$, each of which has weight $\frac{1}{6}$. Next, in $R^{3,2}$, we, e.g., replace $x_1^1 x_2^1 x_3^1$ with the six rankings

$$\begin{gathered} x_1^1 x_2^1 x_1^2 x_2^2 x_3^2 x_3^1, \quad x_1^1 x_2^1 x_1^2 x_3^2 x_2^2 x_3^1, \quad x_1^1 x_2^1 x_2^2 x_1^2 x_3^2 x_3^1, \\ x_1^1 x_2^1 x_2^2 x_3^2 x_1^2 x_3^1, \quad x_1^1 x_2^1 x_3^2 x_1^2 x_2^2 x_3^1, \quad x_1^1 x_2^1 x_3^2 x_2^2 x_1^2 x_3^1. \end{gathered}$$

Further, the other 5 rankings in $R^{3,1}$ are replaced analogously, and all 36 rankings in $R^{k,2}$ have a weight of $\frac{1}{36}$. Finally, to go from $R^{3,2}$ to $R^{3,3}$, we replace, for example, the ranking $x_1^1 x_2^1 x_1^2 x_2^2 x_3^2 x_3^1$ again with 6 rankings that insert the candidates x_1^3, x_2^3, x_3^3 at positions 5 to 7:

$$\begin{gathered} x_1^1 x_2^1 x_1^2 x_2^2 x_1^3 x_2^3 x_3^3 x_3^2 x_3^1, \quad x_1^1 x_2^1 x_1^2 x_2^2 x_1^3 x_3^3 x_2^3 x_3^2 x_3^1, \quad x_1^1 x_2^1 x_1^2 x_2^2 x_2^3 x_1^3 x_3^3 x_3^2 x_3^1, \\ x_1^1 x_2^1 x_1^2 x_2^2 x_2^3 x_3^3 x_1^3 x_3^2 x_3^1, \quad x_1^1 x_2^1 x_1^2 x_2^2 x_3^3 x_1^3 x_2^3 x_3^2 x_3^1, \quad x_1^1 x_2^1 x_1^2 x_2^2 x_3^3 x_2^3 x_1^3 x_3^2 x_3^1. \end{gathered}$$

The remaining 35 rankings are modified accordingly, and the resulting profile $R^{3,3}$ consists of 216 rankings, each of which has a weight of $\frac{1}{216}$.

Finally, from this construction, it follows that in $R^k = R^{k,k}$, each candidate $x_1^1, \dots, x_k^1$ is bottom-ranked by input rankings with a total weight of $\frac{1}{k}$. Hence, all these candidates will be eaten at the same point of time during the execution of PSE. Further, after these candidates are eaten, each candidate $x_1^2, \dots, x_k^2$ is bottom-ranked by rankings with a total weight of $\frac{1}{k}$. As a consequence, each of these candidates will again be eaten at the same time. This process repeats until the candidates but those in X_k are eaten, so a candidate $x \in X_k$ is top-ranked in our output ranking. Further, after removing one candidate $x \in X_k$ and updating the budgets, we arrive at a profile where all rankings again have the same weight. The reason for this is that for each ranking $\succ$ in R^k, every permutation of the candidates in X_k is also in R^k with the same weight. Hence, we can now repeat this process and will again deduce that a candidate in X_k is chosen next. By continuing this type of reasoning, we get for the output ranking $\rhd$ of PSE that $x_{i_k}^k \rhd x_{i_{k-1}}^{k-1} \rhd \dots \rhd x_{i_1}^1$ for all $i_1, \dots, i_k \in \{1, \dots, k\}$. Without loss of generality, we assume that the output ranking of PSE is given by $x_k^k x_{k-1}^k \dots x_1^k x_k^{k-1} x_{k-1}^{k-1} \dots x_1^{k-1} \dots x_k^1 x_{k-1}^1 \dots x_1^1$. This can be ensured by tie-breaking, but we note that our subsequent analysis is independent of this assumption

We next compute an upper bound for the total utility of $\rhd$ in R. To this end, fix an input ranking $\succ$ and a candidate x_i^j. Among the candidates in $\{x_{i'}^{j'} : i' \in \{1, \dots, k\}, j' \in \{1, \dots, j-1\}\}$, there are exactly $j-1$ candidates $x_{i'}^{j'}$ with $j' < j$ such that $x_i^j \succ x_{i'}^{j'}$. Further, there are $i-1$ candidates x_i^j such that $x_i^j \succ x_{i'}^j$ and $x_i^j \rhd x_{i'}^j$. Hence, the total contribution of x_i^j to the Kemeny utility of $\succ$ for $\rhd$ is at most $(j-1)+(i-1)$. Summing over all candidates yields that

$$\begin{aligned} u_K(\succ, \rhd) &\leq \sum_{i=1}^{k} \sum_{j=1}^{k} (i-1) + (j-1) \\ &= \sum_{i=1}^{k} \left(k \cdot (i-1) + \sum_{j=1}^{k} (j-1) \right) \\ &= \sum_{i=1}^{k} \left(k \cdot (i-1) + \frac{(k-1)k}{2} \right) \\ &= k^2(k-1). \end{aligned}$$

Since this holds for every input ranking, the social welfare of $\rhd$ in R^k under u_K is upper bounded by $k^2(k-1)$.

Next, we turn to the optimal ranking for our profile R^k. For this, we consider the ranking $\rhd^* = x_1^1 \dots x_k^1 x_1^2 \dots x_k^2 \dots x_1^k \dots x_k^k$. Further, we fix again an input ranking $\succ$ and an arbitrary candidate x_i^j such that $j < k$ and x_i^j is not the sole candidate in X_j that is ranked below the candidates in X_k by $\succ$. By this choice, it holds that x_i^j is ranked ahead of every candidate in $X_{j+1} \cup \dots \cup X_k$ in $\succ$. Hence, the contribution of this candidate to the Kemeny utility $u_K(\succ, \rhd)$ is at least $(k-j) \cdot k$. Further, for each set X_j with $j < k$, there are $k-1$ such candidates. Hence, each set X_j (with $j < k$) contributes a Kemeny utility of at least $(k-j) \cdot k \cdot (k-1)$. Finally, by summing over all such sets, we obtain that

$$u(\succ, \rhd^*) \geq \sum_{j=1}^{k-1} (k-j) \cdot k \cdot (k-1) = k \cdot (k-1) \cdot \frac{k(k-1)}{2}.$$

This analysis is again valid for every input ranking with positive weight, so it holds that $\sum_{\succ \in \mathcal{R}} R^k(\succ) \cdot u_K(\succ, \rhd^*) \geq \frac{k^2(k-1)^2}{2}$. Clearly, this also a lower bound for the total utility of the optimum ranking. Hence, by considering the welfare ratio for R^k, we now obtain that

$$sw(\texttt{PSE}, u_K) \leq \frac{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u_K(\succ, \texttt{PSE}(R^k))}{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot u_K(\succ, OPT_{u_K}(R^k))} \leq \frac{k^2 \cdot (k-1)}{k^2(k-1)^2/2} = \frac{2}{(k-1)}.$$

Since $k = \sqrt{m}$, this proves our proposition. $\square$

A.4 Proof of Theorem 3

We now turn to our analysis of $\texttt{PSS}$ rules and show that, if the underlying score system s of such a rule is ratio-bounded, then its output ranking is guaranteed to satisfy uPJR for the corresponding scoring Kemeny utility u_K^s. Further, we present a pseudocode description of the class of $\texttt{PSS}$ rules in Algorithm 2.

Theorem 3. *For every ratio-bounded score system s, the ranking returned by the Proportional Sequential Scoring rule induced by s satisfies uPJR for every profile and the scoring Kemeny utility function u_K^s.*

As sketched in the main body, we will prove this theorem with on an approach analogous to that of Theorem 1. Specifically, we will next introduce an auxiliary notion that we call rank-priceability for scoring Kemeny utilities and show that this condition implies uPJR for every profile R and scoring Kemeny utility function u_K^s. Similar to pair-priceability, this axiom distributes a budget of $\max(u_K^s)$ to the rankings proportional to their weights. However, unlike pair-priceability, rankings buy candidates rather than pairs of candidates. In particular, each ranking is allowed to pay at most its marginal utility gain for a candidate, no ranking is allowed to spend more than its entire budget, there is an upper bound on how much we can spend on each candidate, and we again need to spend almost the entire budget. Formalizing these concepts results in the following axiom.

Definition 4 (Rank-priceability for scoring Kemeny utilities)**.** A ranking $\rhd = x_1 \dots x_m$ satisfies rank-priceability for a profile R and a scoring Kemeny utility function u_K^s if there is a payment scheme $\pi : \mathcal{R} \times C \to \mathbb{R}_{\geq 0}$ such that

(1) $\pi(\succ, x_i) \leq s_i(\hat{u}(\succ, x_i, \{x_i, \dots, x_m\}))$ for all $\succ \in \mathcal{R}$ and $x_i \in C$,
(2) $\sum_{x_i \in C} \pi(\succ, x_i) \leq R(\succ) \cdot \max(u_K^s)$ for all $\succ \in \mathcal{R}$,
(3) $\sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) \leq s_i(m-i)$ for all $x_i \in C$, and
(4) $\sum_{\succ \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ, x_i) > \max(u_K^s) - 1$.

We note that this axiom lifts Lederer's notion of rank-priceability to scoring Kemeny utilities (Lederer 2026). Again generalizing the insights of this author, we next show that rank-priceability still implies uPJR for scoring Kemeny utility functions.

Lemma 3. *If a ranking satisfies rank-priceability for a profile and a scoring Kemeny utility function, it also satisfies uPJR.*

Proof. Fix a ranking $\rhd = x_1 \dots x_m$, a profile R, and a scoring Kemeny utility function u_K^s induced by a score system $(s_j)_{j \in \{1,\dots,m\}}$. We assume for contraction that $\rhd$ satisfies rank-priceability for R and u_K^s but fails uPJR. By the latter condition, there is a ranking $\succ$ such that $u_K^s(\succ, \rhd) < \lfloor R(\succ) \cdot \max(u_K^s) \rfloor$. Since the quantities of both sides of this inequality are integers, this means that $u_K^s(\succ, \rhd) \leq R(\succ) \cdot \max(u_K^s) - 1$. On the other hand, since $\rhd$ is rank-priceable for R and u_K^s, there is a payment scheme π that satisfies the four conditions of rank-priceability. In particular, since $\sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} R(\succ') = 1 - R(\succ)$, we obtain from Condition (2) of rank-priceability that

$$\sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} \sum_{x_i \in C} \pi(\succ, x_i) \leq \sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} R(\succ') \cdot \max(u_K^s) = (1 - R(\succ)) \cdot \max(u_K^s).$$

On the other hand, from Condition (1), we get that

$$\sum_{x_i \in C} \pi(\succ, x_i) \leq \sum_{x_i \in C} s_i(\hat{u}(\succ, x_i, \{x_i, \dots, x_m\})) = u_K^s(\succ, \rhd).$$

Algorithm 2: Proportional Sequential Scoring Rule (PSS)

Input: A profile $R : \mathcal{R} \to [0,1]$ and a score system $s = (s_j)_{j \in \{1,\dots,m\}}$
Output: Output ranking $\rhd$

$b_1(\succ) \leftarrow R(\succ) \cdot \max(u_K^s)$ for all $\succ \in \mathcal{R}$
$X_1 \leftarrow C$
for $i = 1, \dots, m$ **do**
  $x_i \leftarrow \arg\max_{x \in X_i} \sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ, x, X_i))$
  $X_{i+1} \leftarrow X_i \setminus \{x_i\}$
  $b_{i+1}(\succ) \leftarrow b_i(\succ) - \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right)$ for all $\succ \in \mathcal{R}$
end for
return $\rhd = x_1 x_2 \dots x_m$

Further, by our contradiction assumption, we have that $u_K^s(\succ, \rhd) \leq R(\succ) \cdot \max(u_K^s) - 1$ and thus $\sum_{x_i \in C} \pi(\succ, x_i) \leq \max(u_K^s) - 1$. Hence, we now derive a contradiction to Condition (4) of rank-priceability, because

$$\begin{aligned}\sum_{\succ' \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ', x_i) &= \sum_{x_i \in C} \pi(\succ, x_i) + \sum_{\succ' \in \mathcal{R} \setminus \{\succ\}} \sum_{x_i \in C} \pi(\succ', x_i) \\ &\leq R(\succ) \cdot \max(u_K^s) - 1 + (1 - R(\succ)) \cdot \max(u_K^s) \\ &= \max(u_K^s) - 1.\end{aligned}$$

This is the desired contradiction, which shows that rank-priceability implies uPJR. □

Motivated by Lemma 3, we will next show that, for each ratio-bounded score system s, the ranking chosen by the PSS rule induced by s satisfies rank-priceability for u_K^s for every profile R. By combining our two lemmas, it thus follows that these rules are guaranteed to return rankings that satisfy uPJR.

Lemma 4. *The output ranking of a sequential proportional scoring rule induced by a ratio-bounded score system s satisfies rank-priceability for every profile R and the scoring Kemeny utility function u_K^s.*

Proof. Fix a profile R, score systems $s = (s_j)_{j \in \{1,\dots,m\}}$, and let $\rhd = x_1 \dots x_m$ denote the ranking chosen by the proportional sequential score rule induced by s. Further, we $b_i(\succ)$ denote the budget of a ranking $\succ$ in the i-th round of the execution of our PSS rule and let $X_i = \{x_i, \dots, x_m\}$ denote the set of available candidates in this round. We will show that the payment scheme π given by $\pi(\succ, x_i) = \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right)$ satisfies all conditions of rank-priceability, thus proving our lemma.

Condition (1): Fix a ranking $\succ$ and a candidate x_i. We need to show that $\pi(\succ, x_i) \leq s_i(\hat{u}(\succ, x_i, X_i))$. To this end, we let y denote the candidate with $\hat{u}(\succ, y, X_i) = m - i$, i.e., y is the top-ranked candidate of $\succ$ among the candidates in X_i. Now, a lower bound for the total score of y is $b_i(\succ) \cdot s_i(m-i)$, which is the score that ranking $\succ$ contributes by itself to y. Further, since x_i maximizes the total score in this round, we get that $\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ, x_i, X_i)) \geq b_i(\succ) \cdot s_i(m-i)$. Using this insight, it follows that

$$\begin{aligned}\pi(\succ, x_i) &= \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right) \\ &\leq \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \\ &\leq \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{s_i(m-i) \cdot b_i(\succ)} \\ &= s_i(\hat{u}(\succ, x_i, X_i)).\end{aligned}$$

This proves Condition (1) of rank-priceability.

Condition (2): Next, we will show $\sum_{x_i \in C} \pi(\succ, x_i) \leq R(\succ) \cdot \max(u_K^s)$ for all rankings $\succ$. To this end, we fix a ranking $\succ$ and note that $b_1(\succ) = R(\succ) \cdot \max(u_K^s)$. Further, by definition of our payment scheme, it holds that $\pi(\succ, x_i) = b_i(\succ) - b_{i+1}(\succ)$. Hence, $\sum_{x_i \in C} \pi(\succ, x_i) = \sum_{i=1}^{m-1} b_i(\succ) - b_{i+1}(\succ) = b_1(\succ) - b_m(\succ)$. Finally, we observe that $b_m(\succ) \geq 0$ because it is not possible that a ranking pays more than its current budget by the definition of Proportional Sequential Scoring rules. Hence, combining our insights shows that our payment scheme satisfies also Condition (2) of rank-priceability.

Condition (3): For the third condition, we need to show that, for every candidate $x_i \in C$, it holds that $\sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) \leq s_i(m-i)$. This follows directly from the definition of our payment scheme since

$$\begin{aligned}
\sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) &= \sum_{\succ \in \mathcal{R}} \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right) \\
&\leq \sum_{\succ \in \mathcal{R}} \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \\
&= s_i(m-i) \cdot \frac{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))} \\
&= s_i(m-i).
\end{aligned}$$

Condition (4): Finally, we turn to the last condition of pair-priceability, which requires us to show that $\sum_{\succ \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ, x_i) > \max(u^s_K) - 1$. Similar to the proof Condition (2), we observe that $\sum_{\succ \in \mathcal{R}} \sum_{x_i \in C} \pi(\succ, x_i) = \sum_{\succ \in \mathcal{R}} \sum_{x_i \in C} b_i(\succ) - b_{i+1}(\succ) = \sum_{\succ \in \mathcal{R}} b_1(\succ) - b_m(\succ)$. Since the total initial budget is $\sum_{\succ \in \mathcal{R}} b_1(\succ) = \max(u^s_K)$, we focus on showing that the total remaining budget is strictly less than 1. To this end, we will inductively prove that the total budget in the i-th round, $B_i = \sum_{\succ \in \mathcal{R}} b_i(\succ)$, satisfies that

$$\sum_{j=i}^{m} s_j(m-j)) \leq B_i < 1 + \sum_{j=i}^{m} s_j(m-j)).$$

First, the induction basis holds trivially since $B_1 = \max(u^s_K) = \sum_{j=1}^m s_j(m-j)$. Next, we inductively assume that $\sum_{j=i}^m s_j(m-j)) \leq B_i < 1 + \sum_{j=i}^m s_j(m-j)$ for some $i \in \{1, \dots, m-1\}$. To ease notation, we let $\delta_i \in [0,1)$ such that $B_i = \delta_i + \sum_{j=i}^m s_j(m-j)$. Now, it holds that

$$B_{i+1} = B_i - \sum_{\succ \in \mathcal{R}} \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right).$$

Now, we first note that

$$\sum_{\succ \in \mathcal{R}} \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right) \leq \sum_{\succ \in \mathcal{R}} \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} = s_i(m-i).$$

Using our induction hypothesis, the lower bound on B_{i+1} follows now since

$$\begin{aligned}
B_{i+1} &= B_i - \sum_{\succ \in \mathcal{R}} \pi(\succ, x_i) \\
&= B_i - \sum_{\succ \in \mathcal{R}} \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right) \\
&\geq \delta_i + \sum_{j=i}^{m} s_j(m-j) - s_i(m-i) \\
&\geq \sum_{j=i+1}^{m} s_j(m-j).
\end{aligned}$$

Next, for the lower bound, we will show for all $\succ \in \mathcal{R}$ that

$$\frac{B_i}{1 + \sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \leq b_i(\succ).$$

To this end, we recall that x_i maximizes $\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))$. This score is lower-bounded by the average score, so we compute this next. In particular, since X_i contains $m-i+1$ candidates, the average score of a candidate is

$$\begin{aligned}
\frac{1}{m-i+1} \sum_{x \in X_i} \sum_{\succ \in \mathcal{R}} b_i(\succ) \cdot s_i(\hat{u}(\succ, x, X_i)) &= \frac{1}{m-i+1} \sum_{\succ \in \mathcal{R}} b_i(\succ) \sum_{j=0}^{m-i} s_i(j) \\
&= \frac{B_i}{m-i+1} \sum_{j=0}^{m-i} s_i(j).
\end{aligned}$$

Hence, we conclude that $\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i)) \geq \frac{B_i}{m-i+1} \sum_{j=0}^{m-i} s_i(j)$. Next, we recall that $s_i(m-i) \geq s_i(\hat{u}(\succ, x_i, X_i))$ because s_i is weakly increasing. Thus, it follows that

$$\frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \leq \frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(m-i)}{B_i \cdot \frac{1}{m-i+1} \cdot \sum_{j=0}^{m-i} s_i(j)}$$
$$= \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(m-i)}{(1+\sum_{j=i}^m s_j(m-j)) \cdot \frac{1}{m-i+1} \cdot \sum_{j=0}^{m-i} s_i(j)}.$$

Finally, we have by ratio-boundedness that $\frac{s_i(m-i)}{1+\sum_{j=i}^m s_j(m-j)} \leq \frac{\frac{1}{m-i+1} \cdot \sum_{j=0}^{m-i} s_i(j)}{s_i(m-i)}$, which equivalently means that $\frac{s_i(m-i) \cdot s_i(m-i)}{(1+\sum_{j=i}^m s_j(m-j)) \cdot \frac{1}{m-i+1} \cdot \sum_{j=0}^{m-i} s_i(j)} \leq 1$. Substituting this in our previous inequality shows that

$$\frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \leq b_i(\succ).$$

Since $B_i < 1 + \sum_{j=i}^m s_j(m-j)$ by our induction hypothesis, we derive from our analysis so far that

$$\pi(\succ, x_i) = \min\left(b_i(\succ), \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}\right)$$
$$\geq \frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))}.$$

By summing over all rankings $\succ$, we finally conclude that

$$\begin{aligned} B_{i+1} &\leq B_i - \sum_{\succ \in \mathcal{R}} \frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot \frac{s_i(m-i) \cdot b_i(\succ) \cdot s_i(\hat{u}(\succ, x_i, X_i))}{\sum_{\succ' \in \mathcal{R}} b_i(\succ') \cdot s_i(\hat{u}(\succ', x_i, X_i))} \\ &= B_i - \frac{B_i}{1+\sum_{j=i}^m s_j(m-j)} \cdot s_i(m-i) \\ &= \delta_i + \sum_{j=i}^m s_j(m-j) - \frac{\delta_i + \sum_{j=i}^m s_j(m-j)}{1+\sum_{j=i}^m s_j(m-j)} \cdot s_i(m-i) \\ &= \delta_i + \sum_{j=i}^m s_j(m-j) - s_i(m-i) + (1-\delta_i) \cdot \frac{s_i(m-i)}{1+\sum_{j=i}^m s_j(m-j)} \\ &= \sum_{j=i+1}^m s_j(m-j) + \delta_i + (1-\delta_i) \cdot \frac{s_i(m-i)}{1+\sum_{j=i}^m s_i(m-i)}. \end{aligned}$$

Finally, we note that $\delta_i + (1-\delta_i) \cdot \frac{s_i(m-i)}{1+\sum_{j=i}^m s_i(m-i)} < 1$ since $\delta_i \in [0,1)$ and $\frac{s_i(m-i)}{1+\sum_{j=i}^m s_j(m-j)} < 1$. This proves our induction step. In particular, this means that $B_m < s_m(0) + 1 = 1$, thus proving our result. $\square$

A.5 Proof of Theorem 4

Finally, we prove that for balanced score systems, ratio-boundedness is both necessary and sufficient for PSS rules to always select rankings that satisfy uPJR. Put differently, this result shows that our ratio-boundedness condition is tight for this class of score systems. That said, it may, of course, be possible to find stronger formulations of this condition that turn out equivalent for balanced score systems. Also, it might be possible to give stronger constraints for other classes of score systems.

Theorem 4. *For every balanced score system s that is not ratio-bounded, there is a profile R such that the ranking chosen by the corresponding Proportional Sequential Scoring rule fails uPJR for R and u_K^s.*

Proof. Fix a balanced score system $s = (s_j)_{j \in \{1,\dots,m\}}$ that fails ratio-boundedness. First, since s is balanced, there is for every $j \in \{1, \dots, m\}$ an integer $c_j \in \mathbb{N}_0$ such that $s_j(i) + s_j(m-j-i)$ for all $i \in \{0, \dots, m-j\}$. This implies for all $j \in \{1, \dots, m-1\}$ that

$$2\sum_{i=0}^{m-j} s_j(i) = \sum_{i=0}^{m-j} s_j(i) + s_j(m-j-i) = (m-j+1)(s_j(m-j) + s_j(0)).$$

Since $s_j(0) = 0$ by assumption, this implies that $\frac{\sum_{i=0}^{m-j} s_j(i)}{(m-j+1)\cdot s_j(m-j)} = \frac{1}{2}$. We will next use this insight to reformulate ratio-boundedness. To this end, we recall that this condition requires of s that

$$\frac{s_j(m-j)}{1+\sum_{i=j}^{m} s_i(m-i)} \leq \frac{\frac{1}{m-j+1}\sum_{j=0}^{m-j} s_j(i)}{s_j(m-j)} \qquad \text{for all } j \in \{1,\dots,m-1\}.$$

Now, we have reasoned that the right-hand side of this inequality is always $\frac{1}{2}$. Using this insight, ratio-boundedness can be reduced to the much simpler condition

$$s_j(m-j) \leq 1 + \sum_{i=j+1}^{m} s_i(m-i) \qquad \text{for all } j \in \{1,\dots,m-1\}.$$

Next, since s fails ratio-boundedness, there is an index $j^* \in \{1,\dots,m-1\}$ such that $s_{j^*}(m-j^*) > 1+\sum_{i=j^*+1}^{m} s_i(m-i)$. Since our score systems are integer-valued, this means that $s_{j^*}(m-j^*) \geq 2+\sum_{i=j^*+1}^{m} s_i(m-i)$.

Now, we fix a ranking $\succ_1 = x_1 \dots x_m$ and its reverse $\succ_2 = x_m \dots x_1$. Further, we assume that the tie-breaking order $>$ agrees with $\succ_1$, i.e., in case of a tie, we will always choose the candidate that is preferred by $\succ_1$. Finally, we note that $\max(u_K^s) = \sum_{i=1}^{m} s_i(m-i)$ and we consider the profile R given by $R(\succ_1) = \frac{-1+\sum_{i=1}^{j^*} s_i(m-i)}{\max(u_K^s)}$ and $R(\succ_2) = \frac{1+\sum_{i=j^*+1}^{m} s_i(m-i)}{\max(u_K^s)}$. We will show that, for this profile, the Proportional Sequential Scoring rule induced by s selects the ranking $x_1 \dots x_{j^*} x_m \dots x_{j^*+1}$. However, this ranking fails uPJR because

$$u_K^s(\succ_2, \rhd) = \sum_{i=j^*+1}^{m} s_i(m-i) < 1 + \sum_{i=j^*+1}^{m} s_i(m-i) = \lfloor R(\succ_2)\cdot \max(u_K^s)\rfloor.$$

Now, to show that our $\texttt{PSS}$ rule indeed selects $\rhd$, we first note that the initial budgets are given by $b_1(\succ_1) = -1+\sum_{i=1}^{j^*} s_i(m-i)$ and $b_1(\succ_2) = 1+\sum_{i=j^*+1}^{m} s_i(m-i)$. Since $s_{j^*}(m-j) \geq 2+\sum_{i=j^*+1}^{m} s_i(m-i)$ and all scores are non-negative, this means that $b_1(\succ_1) \geq b_1(\succ_2)$. We claim that, as long as this condition is true, our $\texttt{PSS}$ rule will choose the top-ranked candidate of $\succ_1$. To see this, we fix a step $j \leq j^*$ and assume that, in all previous steps, we only picked the top-ranked candidate of $\succ_1$ in the prior rounds. Since $\succ_2$ bottom-ranks these candidates in each step, it obtains utility 0 from these candidates. Thus, we will not reduce the budget of $\succ_2$, which, in turn, means that $\succ_1$ has to cover the full cost of all previous candidates. This shows that the remaining budgets of our rankings in round j are $b_j(\succ_1) = -1+\sum_{i=j}^{j^*} s_i(m-i)$ and $b_j(\succ_2) = 1+\sum_{i=j^*+1}^{m} s_i(m-i)$. Further, since $s_{j^*}(m-j^*) \geq 2+\sum_{i=j^*+1}^{m} s_i(m-i)$, we know that $b_j(\succ_1) \geq b_j(\succ_2)$.

Now, for each remaining candidate $x \in \{x_j,\dots,x_m\}$, the total score in this round is $b_j(\succ_1)\cdot s_j(\hat{u}(\succ_1, x, \{x_j,\dots,x_m\}) + b_j(\succ_2)\cdot s_j(\hat{u}(\succ_2, x, \{x_j,\dots,x_m\})$. Because $\succ_1$ and $\succ_2$ are reverse to each other, it follows that that $\hat{u}(\succ_2, x, \{x_j,\dots,x_m\}) = (m-j) - \hat{u}(\succ_1, x, \{x_j,\dots,x_m\})$. Further, by balancedness, we have that $s_j(\hat{u}(\succ_1, x, \{x_j,\dots,x_m\}) + s_j(m-j-\hat{u}(\succ_1, x, \{x_j,\dots,x_m\}) = c_j$. Hence, the total score of each candidate x is $b_j(\succ_2)\cdot c_j + (b_j(\succ_1)-b_j(\succ_2))s_j(\hat{u}(\succ, x, \{x_j,\dots,x_m\}))$. Since $b_j(\succ_1) - b_j(\succ_2) \geq 0$ and s_j is weakly increasing, the top-ranked candidate in $\succ_1$ has the maximal score (possibly among other candidates). By our tie-breaking assumption, we will thus choose the top-ranked candidate of $\succ_1$ in this round, which is x_j, and put it on the next position of the output ranking.

We note that this reasoning remains valid until $j = j^*$. After this step, it can be checked that $\succ_1$ is out of budget, whereas $\succ_2$ has not obtained any utility so far. Thus, we will now pick the candidates according to $\succ_2$, so the ranking $\rhd$ will be chosen. This completes our proof because, as reasoned before, this ranking fails uPJR for R and u_K^s. $\square$