# Tower Defense is NP-complete

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**Abstract**

We study the decision problem of placing towers on a grid so that no enemy walking along a fixed road survives. For every fixed life $\lambda \geq 1$ and every fixed finite nonempty range invariant under rotation by 90° other than the wazir range, we prove that it is strongly NP-complete on well-formed roads when all enemies have life $\lambda$ and the reload time may grow with the instance. Exchanging the roles of life and reload time, the same reduction gives strong NP-completeness for every fixed reload time $c \geq 1$ when the lives may grow with the instance. The proofs reduce from Independent Set: one decision site per vertex, whose towers see the road at three separated singleton places, and one extra tower that must stand on a square that sees it four times. Packs of enemies preceded by blockers make two adjacent occupied sites lose. On the positive side the problem is polynomial for reload time 1 and bounded lives, for a common range with one square and bounded reload times and lives, and, on every well-formed road, for bounded reload times and lives when every range is a nonempty union of the five leaper ranges $R_{1,1}$, $R_{1,0}$, $R_{1,2}$, $R_{2,2}$, $R_{2,0}$ (ferz, wazir, knight, elephant, dabbabah) that contains another one of them whenever it contains $R_{2,2}$. We pose one open problem: is the problem polynomial on well-formed roads for every fixed nonempty range $R = -R$, life bound and reload time bound?

## 1 Introduction

Tower defense games ask the player to place towers so that enemies walking along a path are destroyed before they reach its end. Their computational complexity has hardly been studied. The one hardness result on a tower defense game we found is the NP-hardness of a placement problem in Desktop Tower Defense, in which the enemies walk through a maze that the towers build, and the question is whether $k$ turrets can maximize the shortest path from start to end [1]. That problem concerns the path length induced by blocking squares, a graph-theoretic question (the keywords of the paper are Hamiltonian path and graph theory). It does not involve targeting, damage or timing. Hardness proofs for other video games rely on general schemes in which a game mechanic is shown to make the game hard [2], or on constraint logic [3]. A poly-round version of the Fire Emblem tactical role-playing games has been shown NP-complete [4]. We found no earlier result for the classical variant where enemies follow a fixed road and towers shoot with a reload time.

**What is new.** We fix a discrete model with a fixed road, enemies with spawn times and lives, and towers with a range and reload time (Section 2), show that the problem is in NP (Proposition 5), and prove:

**Theorem 1** (Fixed lives, reload time part of the input)**.** *Let $R \subset \mathbb{Z}^2 \setminus \{0\}$ be any fixed finite nonempty range that is invariant under rotation by 90°, with $R \neq R_{1,0}$, and let $\lambda \geq 1$ be any fixed life. Then* Tower Defense *on well-formed roads, with all lives equal to $\lambda$, all towers of range $R$*

*and one common reload time that is part of the input, with all numbers in unary, is NP-complete in the strong sense.*

The reload time is constant inside an instance but unbounded overall (it is *locally constant*, see the conventions in Section 2). The wazir range $R_{1,0}$ is excluded. The hypothesis covers, for example, the ferz range $R_{1,1}$, the knight range $R_{1,2}$ and every box or disc centred at a tower. If the roles of life and reload time are exchanged, the same reduction gives:

**Theorem 2** (Fixed reload time, lives part of the input)**.** *Let $R$ be as in Theorem 1 and let $c \geq 1$ be any fixed reload time. Then* Tower Defense *on well-formed roads, with all towers of range $R$ and reload time $c$ and all lives equal to one common value that is part of the input (the lives are locally constant), with all numbers in unary, is NP-complete in the strong sense.*

**Polynomial cases.** In Section 7 we show that the problem is polynomial for bounded lives when (i) the reload time is 1 (Theorem 11), (ii) all ranges are the same singleton and reload times are bounded (Theorem 12), and (iii) every range is a nonempty union of the five leaper ranges $R_{1,1}, R_{1,0}, R_{1,2}, R_{2,2}, R_{2,0}$ (the presets) that contains another preset if it contains $R_{2,2}$, with bounded reload times and lives, on every well-formed road and under no other hypothesis on the road (Theorem 13). The proofs show that sufficiently many towers that see the road with pairwise disjoint seen sets always win. For (iii) such towers are found in many disjoint column strips of the road (Section 7.4).

**Proof idea.** Let $\varrho$ be the rotation by 90° and $p$ the lexicographic maximum of $R$. A tower on a square $T$ sees the road square $T + \varrho^j p$ and no other square of its quadrant, so the road can pass through the four squares $T + \varrho^j p$ as corners (*needles*), each a singleton place. A *decision site* is a square that sees three needles $A_x, B_x, C_x$, and a *great site* one that sees four. The road has one decision site per vertex $x$ of a graph, with $A_x, B_x$ at road distance $d_x$ and $C_x$ far behind, and one great site (Section 4). To find an anticlique of size $\beta$ we give the player $\beta + 1$ towers. For an edge $\{v, w\}$ a pack of enemies is preceded by blockers at distances $d_v$ and $d_w$: a tower on site $v$ shoots the blocker instead of the pack at its first place. A pack of total life $3(\beta + 1)$ receives $3\beta + 4$ shots from a good placement, loses one for each blocked site, and survives iff it loses two, which happens for two adjacent occupied sites or if the extra tower is not on the great site. Adding disjoint edges makes the total life divisible by $\lambda$, and a large reload time makes a tower fire at most once per pass of a pack (Section 5). If instead the reload time is a fixed constant and the lives may grow, the packs become convoys of enemies of life $3(\beta + 1)$ spaced by the fixed reload time, and the needles, being singletons, give a site the same shots as before (Section 6).

**Outline.** Section 2 contains the model, the conventions and the membership in NP. Section 3 introduces the extremal element and the needles, Section 4 builds the road, Section 5 proves Theorem 1, Section 6 proves Theorem 2, Section 7 contains the polynomial cases and Section 8 the one open problem. The leaper notation $R_{u,v}$ is introduced in Section 2.

# 2 Preliminaries: model and membership in NP

## 2.1 Instances

*Squares* are points of $\mathbb{Z}^2$. Two squares are *orthogonally adjacent* if they differ by one in exactly one coordinate, and *diagonally adjacent* if they differ by one in both.

**Definition 3** (Road)**.** A *road* is a sequence $\rho = (r_0, \ldots, r_{L-1})$ of squares, $L \geq 1$. Square $r_0$ is the entrance and $r_{L-1}$ the exit. The road is *well-formed* if

1. the $r_i$ are pairwise distinct;
2. $r_i$ and $r_{i+1}$ are orthogonally adjacent for all $i < L - 1$;
3. $r_i, r_j$ with $|i - j| \geq 2$ are not orthogonally adjacent;
4. $r_i, r_j$ with $|i - j| \geq 3$ are not diagonally adjacent.

Straight lines, L-turns and U-turns of width $\geq 2$ are well-formed. Squares at index distance 2 may be diagonal (an L-turn). By parity, diagonal squares are at even index distance, so (4) forbids index distance $4, 6, \ldots$.

**Definition 4** (Tower Defense instance)**.** An instance consists of

- a road $\rho$ of length $L$;
- a list of $n$ *enemies* $e_1, \ldots, e_n$, where $e_j = (s_j, \lambda_j)$ has spawn time $s_j \in \mathbb{N}_0$ and life $\lambda_j \geq 1$; the spawn times are pairwise distinct, so at most one enemy stands on a square at any time;
- a list of $k$ *towers* $t_1, \ldots, t_k$, where $t_i = (R_i, c_i)$ has a finite range $R_i \subset \mathbb{Z}^2 \setminus \{0\}$ and a reload time $c_i \geq 1$.

A *placement* is a map $P$ from towers to squares, injective, with no tower on a road square. A tower at $p$ *sees* the road square $q$ if $q - p \in R_i$. (Towers with one common range $R_i = R$ and one common reload time $c_i = c$ give the setting of Theorems 1 and 2.)

**Conventions.** A parameter (range, reload time, life) is *constant* if it is fixed independently of the instance. It is *locally constant* if it is the same for all towers (or enemies) inside one instance but may grow with the instance size. Locally constant values given in unary are polynomially bounded.

**Leaper ranges.** For integers $u \geq v \geq 0$ with $u \geq 1$ let

$$R_{u,v} = \{(\pm u, \pm v), (\pm v, \pm u)\} \qquad \text{(all sign choices)},$$

the range of a $(u, v)$-leaper [7]. For orientation, the chess names [8] are ferz $= R_{1,1}$, wazir $= R_{1,0}$, knight $= R_{1,2}$, elephant (alfil) $= R_{2,2}$ and dabbabah $= R_{2,0}$ (Figure 1). Every $R_{u,v}$ is invariant under the eight symmetries of the grid.

## 2.2 Dynamics

Time is discrete, steps $\tau = 0, 1, 2, \ldots$. Each enemy moves one square per step along the road, so the enemy $e_j$ is on road square $r_{\tau - s_j}$ during step $\tau$ for $s_j \leq \tau \leq s_j + L - 1$; we call $\tau - s_j$ its *position*. Each tower fires at most one shot per step, dealing one damage. Each step consists of four phases, **in this order**:

1. **Spawn.** Every enemy with $s_j = \tau$ appears on $r_0$ (so it can be shot in the step it spawns).
2. **Fire.** Every tower that is *ready* and sees a living enemy chooses a target and shoots, all towers simultaneously, on the state after the spawn phase. After a shot at step $\tau$ the tower is not ready in steps $\tau + 1, \ldots, \tau + c_i - 1$ and ready again at $\tau + c_i$. A ready tower with no enemy in range does not fire and stays ready.

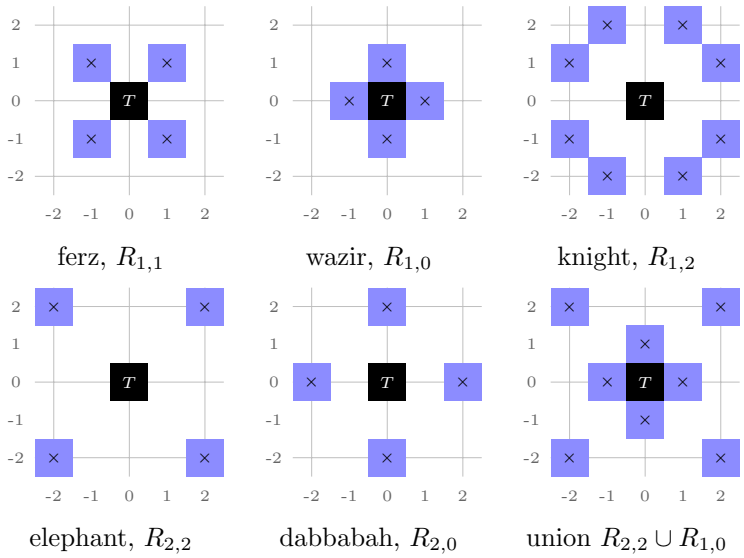



Figure 1: The five leaper ranges and one union. In each picture the tower $T$ is the black square at the centre, and the squares it can see are shaded and marked with a cross ($\times$). The axes show the offsets $-2, \ldots, 2$.

3. **Damage.** All shots are applied together: each enemy loses as much life as it was shot at. Enemies with life $\leq 0$ are removed. Shots at an enemy that is already doomed are wasted (no retargeting).

4. **Move.** Every surviving enemy advances one square. A surviving enemy on $r_{L-1}$ leaves the road.

**Simultaneous shots.** Several towers may shoot the same enemy in the same step, and all shots are applied together. The hardness proofs (Theorems 1 and 2) and Lemma 10 with its consequences (Theorem 12, and Theorem 13 except for its short-road step) do not use this: towers on different decision sites see different road squares, and the towers of the windows lemma have disjoint seen sets, so they never shoot one enemy in the same step, and all other estimates count shots fired, which are at least the shots that take effect. The proofs that do use it are Theorem 11, where $\Lambda$ towers may shoot $e$ in the same step, the short-road step of Theorem 13, where more than $\Lambda$ identical towers fire a common volley.

## 2.3 Targeting

A tower at $p$ chooses among the living enemies on squares it sees the one that is *furthest advanced*, i.e. of maximum position (equivalently, of minimum spawn time). All enemies move at the same speed and spawn times are distinct, so the furthest advanced enemy is unique. Towers that see the same enemies therefore shoot the same enemy.

## 2.4 Win condition

The towers *win* if no enemy leaves $r_{L-1}$ alive, i.e. if every enemy has been removed in a damage phase at some step $\leq s_j + L - 1$ (the step in which it stands on $r_{L-1}$, after which the move phase would let it escape). Otherwise the towers *lose*. The instance is a yes-instance of Tower Defense

if some placement wins. The decision problem asks, given the instance, whether such a placement exists. The time horizon is $T = \max_j s_j + L$ steps.

### 2.5 Encoding

All numbers are encoded in *unary*: $L$, the $s_j$, the $\lambda_j$, the $c_i$, the coordinates of road squares and range offsets. The input size is therefore at least $L + n + k + \sum_j (s_j + \lambda_j) + \sum_i (c_i + |R_i|)$. Hardness in this encoding is strong NP-hardness (the reduction of Theorem 1 has polynomially bounded numbers), and the play-out below is polynomial in the number of steps. The proposition below does not use the unary encoding of anything but the number of steps.

**Proposition 5** (Tower Defense is in NP)**.** Tower Defense *for general instances (arbitrary ranges $R_i$ and reload times $c_i$) is in NP.*

*Proof. Relevant squares.* Let $C_i = \{\, r - d : r \in \rho,\ d \in R_i \,\} \setminus \rho$ be the squares from which a tower of type $i$ sees at least one road square. Then $|C_i| \le L \cdot |R_i|$. A tower of type $i$ placed on a square outside $C_i$ never sees an enemy, so in every step it does not fire, whatever the other towers do (firing depends only on the state, and the tower's own absence of shots changes nothing). Hence we may assume that every tower either stands on $C_i$ or is *idle*, and an idle tower can be moved to a square far from the road that is not used by any other tower. Such squares exist, since only $k$ squares are occupied. A winning placement is thus equivalent to a winning assignment $t_i \mapsto C_i \cup \{\bot\}$, and each tower has at most $L|R_i| + 1$ choices. The certificate lists, for each tower, a square of $C_i$ or $\bot$; each entry is one pair of coordinates, so the certificate has polynomial size.

*Verification.* The verifier checks that the chosen squares are distinct, off the road and in $C_i$ (or $\bot$), then computes the play-out of the game for this placement, applying the four phases of Section 2.2 for $\tau = 0, \dots, T-1$ with $T = \max_j s_j + L$, since after step $\max_j s_j + L - 1$ no enemy is on the road. A step costs: spawn $O(n)$; fire $O(k\,(n + |R_i|))$ using a hash set for $R_i$ (each tower tests the road square of each enemy in range for membership of an offset, and picks the maximum position); damage $O(k + n)$; move $O(n)$. Total time $O(T \cdot (n + k(n + \max|R_i|)))$, polynomial in the input size since $T \le \max_j s_j + L$ is at most the input size in unary. The verifier accepts if no enemy survives on $r_{L-1}$ at the move phase. □

## 3 Sites that see the road three and four times

Throughout Sections 3–6 the range $R \subset \mathbb{Z}^2 \setminus \{0\}$ is a fixed finite nonempty set that is invariant under the rotation $\varrho(a, b) = (-b, a)$ by 90° and different from the wazir range $R_{1,0}$. Since $\varrho^2 = -1$, we have $R = -R$. Let

$$M = \max_{r \in R} \|r\|_\infty, \qquad h = 2M + 1, \qquad \omega = 6M.$$

Two squares seen by one tower differ by an element of $R - R$, so their Chebyshev distance is at most $2M < h$. For a tower on a square $q$ off the road, the road squares that it *sees* are those $r_i$ with $r_i - q \in R$.

### 3.1 Extremal elements and needles

Let $\preceq$ be the lexicographic order on $\mathbb{Z}^2$ ($(u, v) \prec (u', v')$ if $u < u'$, or $u = u'$ and $v < v'$). It is a total order and invariant under translations. Let $p = (p_1, p_2) = \max_{\preceq} R$ be the *extremal element* of $R$, and put $s = p_1 + p_2$, $t = p_1 - p_2$ and $u_j = \varrho^j p$ for $j \in \mathbb{Z}/4$. Let $\preceq_j$ be the lexicographic order turned by $\varrho^j$: $a \preceq_j b$ iff $\varrho^{-j} a \preceq \varrho^{-j} b$. It is translation invariant, and $u_j = \max_{\preceq_j} R$ because $\varrho^j R = R$.

**Lemma 6** (The extremal element)**.** *(a) $p_1 > 0$, $p_1 > -p_2$ and $p_1 \geq p_2$. So $s \geq 1$ and $t \geq 0$.*
*(b) $s = 1$ iff $p = (1,0)$ or $p = (m, 1-m)$ for some $m \geq 2$. In the first case $R = R_{1,0}$.*
*(c) If $p = (m, 1-m)$ with $m \geq 2$, then the mirror image $\tilde{R} = \{(a,-b) : (a,b) \in R\}$ is also invariant under $\varrho$, and its extremal element is $(m, m-1)$, with $s = 2m-1 \geq 3$.*

*Proof.* (a) $\varrho p = (-p_2, p_1)$ and $\varrho^{-1}p = (p_2, -p_1)$ lie in $R$, so they are $\preceq p$. If $p_1 = 0$ then $p_2 > 0$ and $\varrho^{-1}p \succ p$. So $p_1 > 0$ and the two inequalities follow, the case $-p_2 = p_1$ being excluded because then $\varrho p = (p_1, p_1) \succ p$. (b) $s = 1$ means $p_2 = 1 - p_1$ and $p_1 \geq 1$. If $p_1 = 1$ then $p = (1,0)$ and $R \subseteq \{a \leq 1\}$, so $R \subseteq [-1,1]^2$ by invariance, and $(1,1) \succ p$ is not in $R$, nor are its rotations. So $R = \{(\pm 1, 0), (0, \pm 1)\}$. (c) The column $\{b : (m,b) \in R\}$ at the largest abscissa $m$ has the maximum $1-m$. By invariance $|b| \leq m$, and $b = -m$ would give $\varrho(m,-m) = (m,m) \succ p$. So the column is $\{(m, 1-m)\}$. Reflecting gives the column $\{(m, m-1)\}$ of $\tilde{R}$, which is its extremal element. A reflection conjugates $\varrho$ to $\varrho^{-1}$, so $\tilde{R}$ is invariant. □

A road built for $\tilde{R}$ and then reflected in the horizontal axis serves for $R$, since a tower at $q$ sees $r$ iff the reflected tower sees the reflected road square. So by Lemma 6 we may and do assume from now on that

$$s \geq 2, \qquad t \geq 0.$$

(The case $p = (1,0)$, i.e. $R = R_{1,0}$, is excluded.)

For a square $T$ (the *centre*) the $j$-th *needle* is $T + u_j$ and the $j$-th *quadrant* is $Q_j(T) = T + u_j + \varrho^j\{(a,b) : a, b \geq 0\}$.

**Lemma 7** (Quadrants)**.** *Let $T$ be a square and $q$ a square that sees a square $v \in Q_j(T)$. Then $q - T \succeq_j 0$. Consequently:*

*(a) If $q$ sees a square of $Q_j(T)$ and a square of $Q_{j+2}(T)$, then $q = T$.*

*(b) $T$ sees a square $v \in Q_j(T)$ iff $v = T + u_j$.*

*Proof.* Write $v = T + u_j + \varrho^j\delta$ with $\delta \succeq 0$ and $w = q - T$. Then $v - q = \varrho^j(p + \delta) - w \in R$, so $\varrho^{-j}(v-q) = p + \delta - \varrho^{-j}w \preceq p$, i.e. $\varrho^{-j}w \succeq \delta \succeq 0$. For (a) put $w' = \varrho^{-j}(q-T)$. The two hypotheses give $w' \succeq 0$ and $-w' = \varrho^{-(j+2)}(q-T) \succeq 0$, so $w' = 0$. For (b), $q = T$ gives $0 \succeq \delta$, so $\delta = 0$; and $T$ sees $T + u_j$. □

**Needles as corners.** For $j \in \mathbb{Z}/4$ the squares $u_j + \varrho^j e_1$ and $u_j + \varrho^j e_2$ ($e_1 = (1,0)$, $e_2 = (0,1)$) are larger than $u_j = \max_{\preceq_j} R$ in the order $\preceq_j$, so they are not in $R$. Hence the road may pass through the needle $T + u_j$ as a corner whose two *arms* leave in the directions $\varrho^j e_1$ and $\varrho^j e_2$: east and north for $j = 0$, north and west for $j = 1$, west and south for $j = 2$, south and east for $j = 3$. Both arms lie in the quadrant $Q_j(T)$, and by Lemma 7(b) a tower on $T$ sees no arm square except the needle. Figure 3 shows the needles and their arms for a range with $p = (3,0)$, around a decision site (three needles) and around a great site (four).

## 4 The road

Put $\kappa = 3h$, $\pi = 11h + 2p_1 + 2|p_2| + 1$ (even) and $\Lambda = 4h + p_1 + 2|p_2|$. Let $n \geq 1$, let $d_1, \ldots, d_n$ be even integers $\geq d_0 := 8h + 2|p_2|$, and let $\Gamma \geq 0$. The road $\rho = \rho(d_1, \ldots, d_n; \Gamma)$ is the concatenation of three parts. Centres are $T_x = (-x\pi,\ xt)$ for $x = 1, \ldots, n$, with needles $A_x = T_x + u_0$, $B_x = T_x + u_1$, $C_x = T_x + u_2$, and the centre $T_\star$ of the great site is $(-n\pi - \pi_\star,\ nt)$ with $\pi_\star = \kappa + 12h + 3\pi$, with needles $G_j = T_\star + u_j$, $j = 0, \ldots, 3$. All turns are right angles. Put $H_x = d_x/2 - p_2$ ($\geq 4h$).

*Part I (westward through the sites).* The road starts $\kappa$ squares east of $A_1$ in the row of $A_1$. For $x = 1, \dots, n$ it

- runs west to $A_x$, turns north and runs $H_x$ squares up,
- turns west and runs $s$ squares (the *cap*), turns south and runs $H_x - t$ squares down, which ends at $B_x$,
- turns west and runs along the row of $B_x$ to $A_{x+1}$ (the row of $B_x$ is the row of $A_{x+1}$ because $T_{x+1} - T_x = (-\pi, t)$ and $p_2 + t = p_1$). After $B_n$ it runs $\kappa$ squares west.

The index distance from $A_x$ to $B_x$ is $H_x + s + (H_x - t) = d_x$. Then the road turns north and runs up to the row $3h + \Gamma_1$ above the highest cap, turns east and runs to the column $3h$ east of its start, turns south and runs down to the row $-\Lambda$, and turns west and runs to the column of $C_1$.

*Part II (the third places).* For $x = 1, \dots, n$ the road runs north in the column of $C_x$ up to $C_x$ (the *ascent*), turns west, runs $\kappa$ squares, turns south, runs down to the row $-\Lambda$ (the *descent*) and turns west towards the column of $C_{x+1}$ (the *lower corridor*). After $C_n$ the descent goes $\Gamma_2$ rows deeper (below $-\Lambda$), and the road runs west to the column $5h$ east of $G_0$ and north to the row of $G_0$, where Part III begins.

*Part III (the great site).* Turn west, run to $G_0$, turn north, run $\Omega$ squares up, west $s$ squares, down $\Omega - t$ squares to $G_1$, then west $\Omega$ squares, south $s$ squares, east $\Omega - t$ squares to $G_2$, south $\Omega$ squares, east $s$ squares, north $\Omega - t$ squares to $G_3$, and finally east $h$ squares, where the road ends. (This pinwheel visits the four needles of $T_\star$ through the caps in this order. The index distance between consecutive needles is $2\Omega + 2p_2$.)

The free lengths $\Gamma_1, \Gamma_2, \Omega \geq 0$ are chosen, each at most $\Gamma + O(h)$, so that $c_x - (a_n + d_n) \geq \Gamma$ for all $x$, $\gamma_0 - c_n \geq \Gamma$ and $\gamma_{j+1} - \gamma_j \geq \Gamma$, where $a_x$, $c_x$, $\gamma_j$ are the indices of $A_x, C_x, G_j$. So $A_x, B_x$ are visited in Part I, all third places $C_x$ only after Part I is over, and the great site last. Figure 2 shows the whole road for the ferz range and two decision sites, and Figure 3 the neighbourhood of a decision site and of the great site for a range with $p = (3, 0)$.

**Lemma 8** (Road with three- and four-fold sites)**.** *$\rho$ is well-formed, has size polynomial in $n$, $\sum_x d_x$ and $\Gamma$, and:*

1. *$T_x$ and $T_\star$ are off the road. $T_x$ sees exactly the three road squares $A_x, B_x, C_x$ of indices $a_x < a_x + d_x < c_x$, pairwise more than $\omega$ apart, and $T_\star$ sees exactly $G_0, \dots, G_3$, pairwise more than $\omega$ apart.*
2. *$a_x + d_x < a_{x+1}$, $a_n + d_n < c_1 < \cdots < c_n < \gamma_0$, and the gaps are at least $\Gamma$ as above.*
3. *Every other off-road square sees road squares whose indices lie in at most two intervals of length $\omega$.*

*Proof.* The index statements of (1) and (2) are read off the construction. We first give the tags. Every road square gets at most one *tag* $(K, j)$, where $K$ is a site (a centre $T_x$ or $T_\star$) and $j$ one of its needles, as follows; the legs include their end squares.

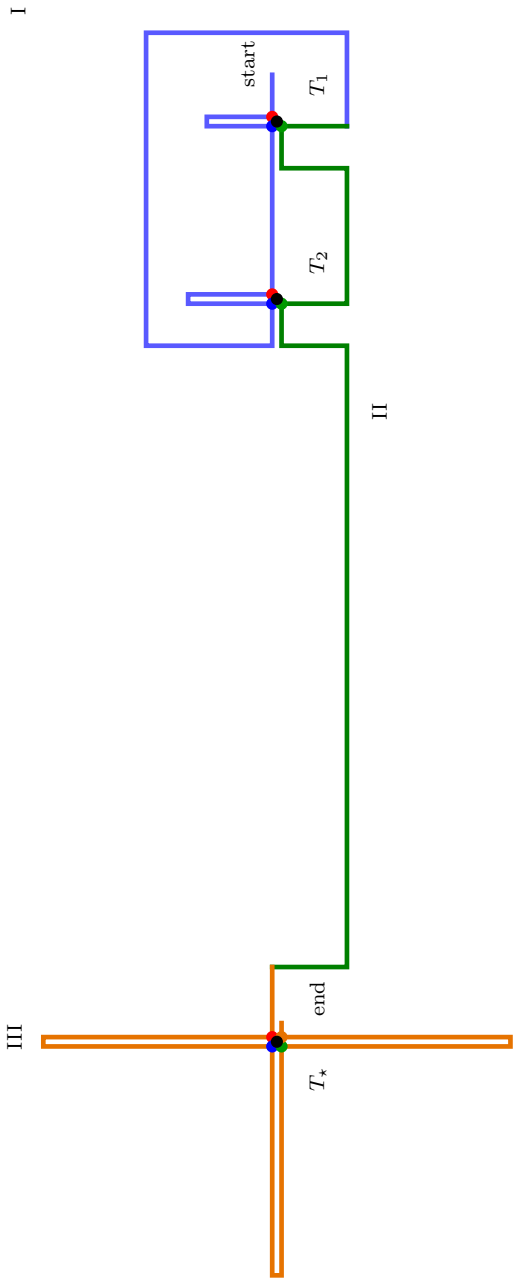



Figure 2: The whole road $\rho(d_1, d_2; \Gamma)$ for the ferz range $R_{1,1}$ and two decision sites ($p = (1, 1)$, $h = 3$, $s = 2$, $t = 0$, $\kappa = 9$, $\pi = 38$, $\Lambda = 15$, $\pi_\star = 159$), with $d_1 = 30$, $d_2 = 38$, $\Gamma = 100$, $\Gamma_1 = \Gamma_2 = 0$, $\Omega = 49$, drawn to scale and turned by 90° (the road starts at the top). Parts are labelled I, II, III. Part I (blue) passes the sites $T_1, T_2$, whose needles $A_x, B_x$ (red, blue) are joined by the caps, and returns around them; Part II (green) visits the third places $C_x$ (green dots) and runs to the great site; Part III (orange) is the pinwheel around $T_\star$ with the needles $G_0, \ldots, G_3$. The road has 850 squares.

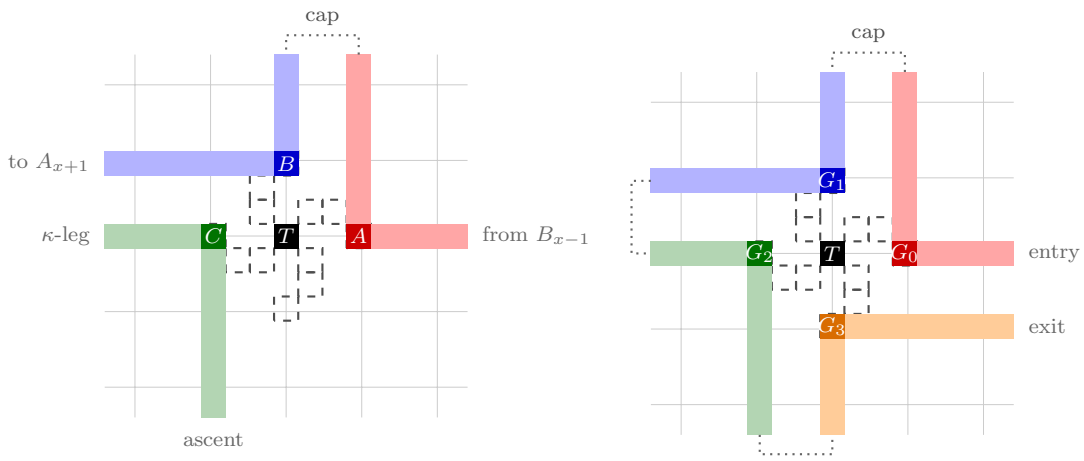



Figure 3: The range $R$ generated by $(1,1),(1,-2),(0,3)$ under rotations by 90° (twelve squares, dashed: $T+R$), with extremal element $p=(3,0)$, so $s=3$, $t=3$, $M=3$, $h=7$. Left, the neighbourhood of a decision site $T$ (black, centre of the window): the needles $A=T+(3,0)$ (index $a_x$, red), $B=T+(0,3)$ (blue, index $a_x+d_x$) and $C=T+(-3,0)$ (green, a third place visited much later), each a corner whose arms are the squares of its colour. The arms of $A$ and $B$ rise as two columns three apart and are joined by the cap far above the window, the corridor leaves along the row of $B$ (which is the row of $A_{x+1}$), the arm of $C$ comes from below and turns west. Right, the great site: four needles $G_0,\dots,G_3$ in the four quadrants, joined pairwise by caps (outside the window) in a pinwheel; the road enters along the row of $G_0$ and leaves along the row of $G_3$. The fourth position $T+(0,-3)$ is empty at a decision site.

| tag | squares |
|---|---|
| $(T_x,0)$ | the second half of the corridor entering $A_x$, and the leg of $H_x$ squares up from $A_x$ |
| $(T_x,1)$ | the leg of $H_x-t$ squares down to $B_x$, and the first half of the corridor leaving $B_x$; for $x=n$ the corridor leaving $B_n$ up to the $\kappa$-leg and the column up to the top row |
| $(T_x,2)$ | the ascent to $C_x$, the $\kappa$-leg, the descent and the lower corridor to the next ascent; for $x=n$ only up to the end of the deepened descent |
| $(T_\star,0)$ | the entry leg and the leg of $\Omega$ squares up from $G_0$ |
| $(T_\star,1)$ | the leg of $\Omega-t$ squares down to $G_1$ and the leg of $\Omega$ squares west of $G_1$ |
| $(T_\star,2)$ | the leg of $\Omega-t$ squares east into $G_2$ and the leg of $\Omega$ squares south of $G_2$ |
| $(T_\star,3)$ | the leg of $\Omega-t$ squares north into $G_3$ and the exit leg |

The squares inside the caps and the remaining legs (the top, east and bottom runs of the encircling part, the connections to the great site) have no tag. A square tagged $(K,j)$ lies in $Q_j(T_K)$: the arms of $A_x$ go east or north, those of $B_x$ north or west, those of $C_x$ west or south, and likewise for $G_0,\dots,G_3$ (a corner at the end of a cap turns from the directions of one needle to those of the next, and is tagged by the needle whose quadrant it lies in).

*Separation.* Call two road squares *related* if they lie on one leg or on consecutive legs, or if they carry tags of different needles of one site. We claim that unrelated squares are at Chebyshev distance $\ge h$, and that the centre of a site is at distance $\ge h$ from every road square without a tag of that site. This is a matter of the lengths. The legs of a site $T$ with abscissa $X$ lie in the columns $X+p_1$, $X-p_2$, $X-p_1$, $X-p_1-\kappa$ and the rows of the corridors and caps and $-\Lambda$. The columns of the arms of consecutive sites are $\pi-2p_1-\kappa=8h+2|p_2|+1\ge h$ apart. The caps are $\ge H_x-t\ge 3h$ above the rows of the corridors. The rows $-\Lambda$, $-\Lambda-\Gamma_2$ and the row $3h+\Gamma_1$ above the highest cap

are $\geq 3h$ from all other rows. The legs of one site that are not related (the ascent and the descent, the $\kappa$-leg and the lower corridor, the legs of one needle that are not consecutive) are $\kappa = 3h$ or $H_x \geq 4h$ apart. The great site is $\geq 12h$ west of the last site, and its caps are $\Omega - t \geq 3h$ long. Pairs of related squares: on one leg the index difference equals the distance, at most $2M$; on consecutive legs, with distances $a, b \leq 2M$ to the common corner, it is $a + b \leq 4M$.

So two squares seen by one square, which have distance $\leq 2M < h$, are related, and we get:

(∗) If a square sees two road squares whose indices differ by more than $\omega$, then these carry tags $(K, j)$ and $(K, j')$ with $j \neq j'$ for one site $K$.

*(1).* By the claim, the road squares at distance $\leq M$ from $T_x$ are tagged by $T_x$. Tagged squares lie in the quadrants $Q_j(T_x)$, and $T_x \notin Q_j(T_x)$ since $-p \not\geq 0$. So $T_x$ is off the road, and by Lemma 7(b) it sees exactly the needles $A_x, B_x, C_x$. The same holds for $T_\star$. The index differences are $d_x$ and at least $\Gamma$, which exceed $\omega$ (by $d_x \geq 8h > 6M$).

*(3).* Let $q$ be a square other than the centres, and let $i_1 < \cdots < i_r$ be the indices it sees. Suppose some seen index exceeds $i_1 + \omega$, and let $i^*$ be the smallest one. By (∗) the squares of $i_1$ and $i^*$ carry tags $(K, j)$, $(K, j')$ with $j \neq j'$. If some seen index $i > i^* + \omega$ existed, (∗) applied to the pairs $(i_1, i)$ and $(i^*, i)$ gives the tags $(K, j'')$ with $j'' \notin \{j, j'\}$ (the site is $K$ again, as every square has at most one tag). Among three needles of one site two are opposite, so Lemma 7(a) gives $q = T_K$, a contradiction. Hence all seen indices lie in $[i_1, i_1 + \omega] \cup [i^*, i^* + \omega]$. □

# 5 Proof of Theorem 1

**Idea.** We reduce from Independent Set with one tower per vertex and one extra tower. Every decision site $T_x$ sees the road at three singleton places, a pair $A_x, B_x$ at index distance $d_x$ and a third place $C_x$ far behind, so a tower on it shoots a pack of enemies at most three times, and loses one of these shots exactly when it is blocked. The extra tower on a *great site*, which sees the road four times, gives four shots. All other squares see at most two separated places. With $\beta$ vertex towers the extra tower brings the number of shots to $3\beta+4$, and a pack of total life $3(\beta+1)$ survives iff at least two shots are lost. That happens when two adjacent sites are occupied or the extra tower is not on the great site. Adding disjoint edges to the graph makes $3(\beta + 1)$ divisible by $\lambda$, so every constant life is reached. The shots at the third and fourth places come after all blocking, so no blocker is shot down before it is needed, and one pack per edge suffices.

## 5.1 The instance

Reduce from Independent Set on a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ without isolated vertices and with $k \geq 2$. Put $\lambda' = \lambda / \gcd(\lambda, 3)$ and let $j \geq 0$ be the least number with $\lambda' \mid k + j + 1$. Add $j$ disjoint edges on new vertices and replace $k$ by $\beta = k + j \geq 2$. This raises the independence number by exactly $j$, creates no isolated vertex and increases $n$ by $2j < 2\lambda'$. So we assume $\lambda' \mid \beta+1$, and then $\lambda \mid 3(\beta + 1)$. Put

$$L = 3(\beta + 1) = \lambda\mu.$$

**Numbers.** Let $\omega = 6M$ and

$$c = 2\mu + \omega, \qquad \ell = c + 2\mu, \qquad Q = 2\big(\ell(n + 1) + d_0\big),$$
$$d_x = Q + \ell x, \qquad d_x^+ = d_x + \mu, \qquad \Delta = 4d_n^+ + c + 2\mu.$$

All $d_x$ are even (since $\omega$ is even, so is $\ell$), increasing and $\geq d_0$.

**Enemies.** A *pack* with first spawn time $s$ consists of the $\mu$ enemies spawned at $s, s+1, \ldots, s+\mu-1$, each of life $\lambda$. So every pack has total life $L$. Number the edges $e_1, \ldots, e_{|E|}$. For the $m$-th edge $\{v, w\}$, $v < w$, there is a *group* of three packs with first spawn times $t_m - o$, where

$$o \in \{0,\ d_v^+,\ d_w^+\}, \qquad t_m = 2d_n^+ + (m-1)\Delta.$$

Call them $Z_0, Z_v, Z_w$. Let $\Gamma =$ (last spawn time) $-$ (first spawn time) $+ c + \mu$, which is $O(|E|\Delta)$. Figure 4 shows one group on a straight piece of the road, with all numbers worked out for a small example.

**Road and towers.** The road is $\rho(d_1, \ldots, d_n; \Gamma)$ of Lemma 8 (Figure 2 shows it for $n = 2$). There are $\beta+1$ towers, all with range $R$ and reload time $c$. All numbers are polynomial in $n$ ($\mu, c, \ell = O(n)$, $Q, d_x, \Delta = O(n^2)$, and $\Gamma = O(|E|n^2)$).

### 5.2 Passes and shots

The *pass* of a pack at a road square is the set of the $\mu$ steps in which its members stand on it. A pack with first spawn time $s$ starts its pass at $A_x$ at step $s + a_x$, at $B_x$ at $s + a_x + d_x$, at $C_x$ at $s + c_x$ and at $G_j$ at $s + \gamma_j$. For a pack with offset $o$ in its group and a site $x$, the pack with offset $o + d_x^+$ in the same group, if present, is its *blocker at* $x$. Its pass at $B_x$ starts $\mu$ steps before the pass of the blocked pack at $A_x$, so it ends right before. Only $Z_0$ has blockers, namely $Z_v$ at $v$ and $Z_w$ at $w$: the offsets of the group differ by $d_v^+, d_w^+$ and $d_w^+ - d_v^+ = \ell(w - v)$, and $\ell(w - v) \neq d_x^+$ since $d_x^+ > \ell n$. Let $\mathrm{Bl}(Z) \subseteq V$ be the set of sites where $Z$ has a blocker. So $\mathrm{Bl}(Z_0) = \{v, w\}$ and $\mathrm{Bl}(Z_v) = \mathrm{Bl}(Z_w) = \emptyset$ (Figure 4).

**Lemma 9** (Passes)**.** *(a) All spawn times are distinct.*
*(b) At every site $x$, any two passes at $A_x$ or $B_x$ start at least $c+\mu$ apart, except the pass of a blocker at $B_x$ and the pass of the blocked pack at $A_x$, which start exactly $\mu$ steps apart.*
*(c) Two passes at one of the squares $C_x$, $G_j$ start at least $c + \mu$ apart. For a tower on $T_x$, every pass at $C_x$ starts at least $c + \mu$ after the start of every pass at $A_x$ and $B_x$. For a tower on $T_\star$, the passes at $G_{j+1}$ start at least $c + \mu$ after those at $G_j$.*

*Proof.* The passes of the packs of the $m$-th group at $A_x$ or $B_x$ start in an interval of length at most $3d_n^+$, and $t_{m+1} - t_m = \Delta$. So passes of different groups, and likewise the spawn times, are at least $\Delta - 3d_n^+ \geq c + \mu$ apart. Inside a group, two offsets differ by at least $\ell \geq 2\mu$, which gives (a) and shows that two passes of the same type (both at $A_x$, or both at $B_x$) start at least $\ell \geq c + \mu$ apart. The pass of the pack with offset $o'$ at $A_x$ starts $o - o' - d_x$ steps after the pass of the pack with offset $o$ at $B_x$. For $o = o'$ this is $-d_x$. For $o - o' = -d_y^+$ it is $-d_y^+ - d_x$. For $o - o' = \pm\ell(w - v)$ its absolute value is at least $Q - \ell n \geq c + \mu$. For $o - o' = d_y^+$, $y \in \{v, w\}$, it equals $\mu + \ell(y - x)$, which is $\mu$ for $y = x$ (blocker) and has absolute value at least $\ell - \mu \geq c + \mu$ otherwise. This is (b). For (c), the passes of two packs at one square start at least as far apart as their spawn times, so at least $c + \mu$ by the above. The last statements hold because the indices differ by at least $\Gamma$ (Lemma 8) and the spawn times by at most $\Gamma - c - \mu$. □

**Shots.**

(P1) Let a tower stand on $T_x$ or $T_\star$. By Lemma 8(1) it sees only the needles, one square each. A pass lasts $\mu < c$ steps, so the tower fires at most once in a pass. By Lemma 9 it is ready at the start of every pass except a pass of a blocked pack at $A_x$. So, in every other pass, it fires at the first living member of the pack if some member is alive at some step of the pass.

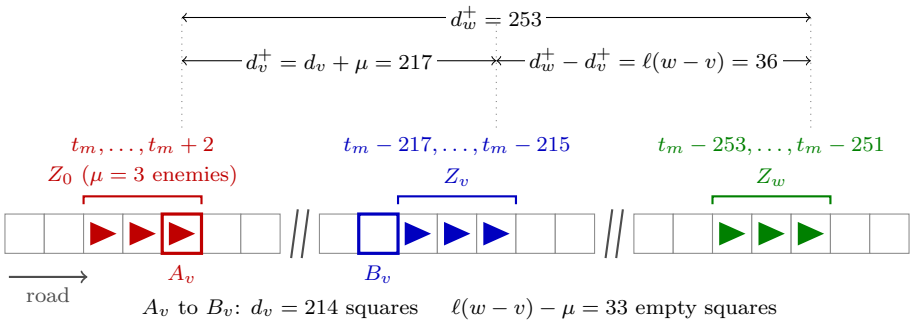



Figure 4: A group of three packs $Z_0, Z_v, Z_w$ for an edge $\{v, w\}$, on a straight piece of the road (the road is unfolded: it runs from left to right and the breaks stand for omitted squares, so the picture is not to scale). The labels above the packs are the spawn times of their enemies. Each enemy is a triangle pointing along the road; the front enemy of a pack is the one spawned first. The picture shows the step $\tau = t_m + a_v$ in which the first enemy of $Z_0$ stands on $A_v$. The pass of $Z_v$ at $B_v$ has ended in the step before, so the tower on $T_v$ has just shot the last enemy of $Z_v$ on $B_v$ and is not ready when $Z_0$ passes $A_v$. *Numbers.* Example: $R = R_{1,1}$, so $M = 1$, $\omega = 6M = 6$, $h = 2M + 1 = 3$, $p = (1, 1)$ and $d_0 = 8h + 2|p_2| = 26$; life $\lambda = 3$ and $\beta = 2$, so $L = 3(\beta + 1) = 9$ and the pack size is $\mu = L/\lambda = 3$ (an enemy needs $\lambda = 3$ shots, a pack $L = 9$); a graph with $n = 3$ vertices and the edge $\{v, w\} = \{1, 3\}$, so $w - v = 2$. Then $c = 2\mu + \omega = 12$, $\ell = c + 2\mu = 18$, $Q = 2(\ell(n + 1) + d_0) = 196$, $d_x = Q + \ell x$ gives $d_v = d_1 = 214$ and $d_w = d_3 = 250$, and $d_x^+ = d_x + \mu$ gives $d_v^+ = 217$, $d_w^+ = 253$. The enemies of a pack spawn in consecutive steps, so a pack covers $\mu = 3$ consecutive squares. The packs spawn at $t_m - o$ with $o \in \{0, d_v^+, d_w^+\}$, so $Z_v$ is $d_v^+ = 217$ squares ahead of $Z_0$ and $Z_w$ is $d_w^+ = 253$ squares ahead, that is $d_w^+ - d_v^+ = \ell(w - v) = 36$ ahead of $Z_v$. Hence there are $d_v^+ - \mu = d_v = 214$ empty squares between $Z_0$ and $Z_v$ (the last of them is $B_v$, at index distance $d_v$ from $A_v$) and $\ell(w - v) - \mu = 33$ between $Z_v$ and $Z_w$. The next group follows $\Delta = 4d_n^+ + c + 2\mu = 4 \cdot 253 + 12 + 6 = 1030$ steps later.

(P2) If a member of a blocker $Z'$ of $Z$ at $x$ is alive during the pass of $Z'$ at $B_x$, the tower on $T_x$ fires in that pass, at some step $\leq$ the end of the pass, so it is not ready again before the start of that pass plus $c = 2\mu + \omega$. The pass of $Z$ at $A_x$ ends $2\mu - 1$ steps after that start, so the tower gives $Z$ no shot at $A_x$.

(P3) A tower on any other square sees the indices of a pack in at most two intervals of length $\omega$ (Lemma 8(3)). A pack needs at most $\mu + \omega < c$ steps to pass through such an interval, so the tower fires at most once at the pack per interval: at most two shots in total.

(P4) If a pack has received $s'$ shots in total, its members have at least $L - s'$ life left in total. If every shot hit a living member and no two shots hit the same member in the same step, equality holds.

(P5) A pack that stands on $B_x$ has been shot only at squares of index at most $a_n + d_n < c_1$. A tower on $T_y$ sees two of these $(A_y, B_y)$, a tower on $T_\star$ none, and any other tower at most two intervals, so it has shot the pack at most twice. Hence a pack has received at most $2(\beta+1) < L$ shots by then.

### 5.3 Correctness

*Proof. Membership* is Proposition 5. For hardness map $(G, k)$ to the instance of Section 5.1. The road is well-formed by Lemma 8, all lives are $\lambda$, and the reload time $c$ is the same for all towers. The instance has polynomial size in unary, since $n, \mu, c, \ell, d_x, \Delta, \Gamma$ and the road length are polynomial in $n$ (Lemma 8) and the number of enemies is $3\mu|E|$.

*If $G$ has an anticlique $X$ of size $\beta$,* place towers on $T_x$ for $x \in X$ and on $T_\star$. Suppose a member $e$ of a pack $Y$ leaves the road alive. Then $e$ is alive in every pass of $Y$. By (P1) the tower on $T_\star$ fires in each of its four passes, and each tower on $T_x$ in its passes at $B_x$ and $C_x$, and in the pass at $A_x$ unless $x \in \mathrm{Bl}(Y)$. Since $X$ is an anticlique, $|X \cap \mathrm{Bl}(Y)| \le 1$. So $Y$ receives at least $4 + 3\beta - 1 = L$ shots. Towers on different sites see different squares, so no two shots hit a member in the same step, and all shots hit living members. By (P4) all members of $Y$ are dead, a contradiction.

*Conversely let a placement win.* (Each of the three streams of a group is a pack because an enemy of the constant life $\lambda$ dies after $\lambda$ shots, whereas $Z_0$ has to absorb up to $3\beta + 4$ shots and a blocker up to $2\beta + 2$ shots before it is used, so only the total life of a stream can grow with $\beta$.) Let $g \in \{0, 1\}$ be the number of towers on $T_\star$, $a$ the number on decision sites and $r$ the number on other squares, so $g + a + r = \beta + 1$. A pack receives at most $4g + 3a + 2r$ shots (P1, P3), and it needs $L = 3(\beta + 1)$ shots to die (P4). So

$$3(\beta + 1) \le 4g + 3a + 2r = 3(\beta + 1) + g - r, \qquad \text{hence } r \le g.$$

*Loss at a site.* Let a tower stand on $T_x$ and let $\{x, w\} \in E$ (there is one, as $G$ has no isolated vertex). In the group of this edge, the blocker of $Z_0$ at $x$ is alive when it reaches $B_x$ by (P5), so by (P2) the tower on $T_x$ gives $Z_0$ no shot at $A_x$: at most two shots.

If $g = 0$, then $r = 0$, $a = \beta + 1 \ge 1$, and by the loss $Z_0$ receives at most $3a - 1 < L$ shots, a contradiction. If $g = 1$ and $r = 1$, then $a = \beta - 1 \ge 1$ and $Z_0$ receives at most $4 + 3(\beta - 1) + 2 - 1 < L$, a contradiction. So $g = 1$, $r = 0$, $a = \beta$: all towers but one stand on decision sites $T_x$, $x \in X'$, with $|X'| = \beta$. Suppose $v, w \in X'$ are adjacent. In the group of $\{v, w\}$ the pack $Z_0$ loses a shot at $v$ and at $w$ by (P5) and (P2), so it receives at most $4 + 3\beta - 2 < L$ shots, a contradiction. So $X'$ is an anticlique of size $\beta$, and removing the $j$ added edges leaves an anticlique of size $k$ in $G$, since it contains at most one endpoint of each added edge. □

## 6 Proof of Theorem 2

Now the reload time $c \ge 1$ is a fixed constant and the lives are locally constant. The range, the road of Lemma 8 and the reduction from Independent Set are those of Section 5. Only the enemies and the numbers change, and the following remarks say why the same proof works.

**Instance.** Let $G = (V, E)$ be as in Section 5, with $k \ge 2$ and no isolated vertex. Add $j = \max\{0, \max\{2, |R| - 4\} - k\}$ disjoint edges on new vertices, so that $\beta = k + j \ge \max\{2, |R| - 4\}$, with $2j$ new vertices. There are $\beta + 1$ towers of range $R$ and reload time $c$. Every enemy has life $\Lambda = 3(\beta + 1)$, which is the total life of a pack in Section 5. Put

$$N = 2(\beta + 1)\omega + 1, \qquad \ell = 2Nc, \qquad Q = 2c\big(\ell(n + 1) + d_0\big), \qquad d_x = Q + \ell x, \qquad \Delta = 4d_n + Nc.$$

All $d_x$ are even multiples of $c$. A *convoy* with first spawn time $s$ is the set of the $N$ enemies spawned at $s, s + c, \ldots, s + (N - 1)c$. For the $m$-th edge $\{v, w\}$, $v < w$, there is a group of three convoys $Z_0, Z_v, Z_w$ with first spawn times $t_m - o$, $o \in \{0, d_v, d_w\}$, $t_m = 2d_n + (m - 1)\Delta$. The road is $\rho(d_1, \ldots, d_n; \Gamma)$ with $\Gamma$ the difference of the last and the first spawn time plus $c$. All numbers are polynomial in $n$ and $|E|$, since $N = O(n)$ for fixed $R$.

**Remarks.**

(V1) *Needles gain nothing from a short reload time.* A tower on a site sees only needles, which are singleton clusters (Lemma 8(1)). An enemy stands on a needle during exactly one step, so a tower shoots a given enemy at most once at each needle, that is at most three times on a decision site and four times on the great site, whatever $c$ is. In Section 5 the same bound comes from $c > \mu$.

(V2) *Readiness.* The spawn times and the $d_x$ are multiples of $c$. So the steps in which an enemy stands on $A_x$ or $B_x$ are congruent modulo $c$, two of them are equal or at least $c$ apart, and a tower on $T_x$ is ready whenever an enemy is there. The steps at $C_x$ and $G_j$ come at least $\Gamma \geq c$ after all of them, and are again congruent modulo $c$ among themselves. Hence a tower on a site fires at every needle visit of an enemy that has not died, except when it fires at another enemy in that step. As $\ell > (N-1)c$ and $Q > \ell n + Nc$, the member $j$ of $Z_0$ stands on $A_x$ in the same step as the member $j$ of $Z_x$ ($x \in \{v, w\}$) on $B_x$, and these are the only two enemies of the instance that stand on $A_x$ and $B_x$ in the same step. The tower shoots the one on $B_x$, which is further advanced. This is the blocking of Section 5, member by member, and $\mathrm{Bl}(Z_0) = \{v, w\}$, $\mathrm{Bl}(Z_v) = \mathrm{Bl}(Z_w) = \emptyset$.

(V3) *Convoys.* A tower on a square that sees at most two windows of length $\omega$ (Lemma 8(3)) may shoot one enemy once for each road square that it sees in a window, so with a single enemy the count of at most two shots per pack is lost. Over a convoy it is recovered: a convoy needs $(N-1)c + \omega + 1$ steps to pass through a window, and the tower fires at most once per $c$ steps, so a tower on such a square fires at most $N + \omega$ shots at the enemies of one convoy per window, at most $2(N + \omega)$ in total.

*Proof of Theorem 2.* Membership is Proposition 5, and the instance has polynomial size in unary. If $X$ is an anticlique of size $\beta$, put towers on the sites $T_x$, $x \in X$, and on $T_\star$. By (V1) and (V2) every enemy $e$ that is alive at all its needle visits receives a shot at every needle of the great site and at $B_x$ and $C_x$ of each occupied site, and at $A_x$ unless $x \in \mathrm{Bl}(e)$. Towers on different sites see different squares, so no two of these shots hit $e$ in the same step. Since $|X \cap \mathrm{Bl}(e)| \leq 1$, the enemy receives at least $4 + 3\beta - 1 = \Lambda$ shots and dies.

Conversely let a placement win, with $g, a, r$ towers on the great site, on decision sites and on other squares. We count the shots at the $N$ enemies of a convoy $Z_0$, which need $N\Lambda = 3N(\beta + 1)$ shots in total. By (V1) and (V3) they receive at most $4Ng + 3Na + 2r(N + \omega)$, so

$$3N(g + a + r) \leq 4Ng + 3Na + 2r(N + \omega), \qquad \text{i.e.} \qquad N(r - g) \leq 2r\omega.$$

If $r > g$ this gives $N \leq 2r\omega \leq 2(\beta + 1)\omega$, which is false. So $r \leq g \leq 1$. Let a tower stand on $T_x$ and let $\{x, w\} \in E$. A member of the blocker $Z_x$ of $Z_0$ standing on $B_x$ has received before at most $2a$ shots from sites and at most $|R|$ from the tower off the sites, in total at most $2(\beta - 1) + |R| < \Lambda$ in the case $r = 1$ and at most $2(\beta + 1) < \Lambda$ otherwise. So it is alive, and by (V2) the tower on $T_x$ gives $Z_0$ no shot at $A_x$. In the group of $\{x, w\}$ every member of $Z_0$ therefore loses the shot at $A_x$, and $Z_0$ receives at most $N$ shots fewer from this tower. If $g = 0$, then $r = 0$, $a = \beta + 1$, and $Z_0$ receives at most $3N(\beta + 1) - N < N\Lambda$ shots. If $g = r = 1$, then $a = \beta - 1 \geq 1$ and $Z_0$ receives at most $4N + 3N(\beta - 1) - N + 2(N + \omega) = N(3\beta + 2) + 2\omega < N\Lambda$, since $N > 2\omega$. So $g = 1$, $r = 0$, $a = \beta$. If two adjacent vertices $v, w$ are occupied, then in the group of $\{v, w\}$ the convoy $Z_0$ loses $N$ shots at $A_v$ and $N$ at $A_w$, and receives at most $4N + 3N\beta - 2N < N\Lambda$ shots. So the occupied sites form an anticlique of size $\beta$, and removing the added edges leaves an anticlique of size $k$. □

# 7 Polynomial cases

In this section the towers $t_i = (R_i, c_i)$ may have different ranges and reload times, given in unary. Let $\Lambda$ be an upper bound for the lives and $C$ for the reload times. In a placement $P$ let $S_i = \{\, j : r_j - P(i) \in R_i \,\}$ be the set of indices that tower $i$ *sees*. The tower *sees the road* if $S_i \neq \emptyset$. The set $C_i$ of squares on which tower $i$ sees the road has $|C_i| \leq L|R_i|$ (proof of Proposition 5). A tower that does not see the road never fires, so it may be moved far away.

## 7.1 A counting lemma

**Lemma 10** (Windows)**.** *Let $W \geq 0$ and $N_0 = \Lambda\,(C+W)$. Suppose a placement contains $N_0$ towers whose seen sets are non-empty, have span $\max S_i - \min S_i \leq W$, and are pairwise disjoint. Then no enemy leaves the road alive in this placement, whatever the other towers do.*

*Proof.* Let $\Xi$ be the set of the $N_0$ towers. Suppose an enemy $e$ leaves the road alive, and put $b_i = \max S_i$ and $t_i = s_e + b_i$ for $i \in \Xi$, the step in which $e$ stands on $r_{b_i}$. The enemy $e$ is alive then. Every other enemy in range of $i$ at step $t_i$ stands on an index of $S_i$ that is not $b_i$ (spawn times are distinct), hence smaller than $b_i$. So $e$ is the furthest advanced enemy in range, and a tower $i \in \Xi$ that is ready at $t_i$ shoots $e$.

Let $\Xi_1$ be the towers of $\Xi$ that shoot $e$ at some step. Their shots are applied while $e$ is alive and $e$ survives, so $|\Xi_1| \leq \lambda_e - 1 \leq \Lambda - 1$. Let $i \in \Xi \setminus \Xi_1$. It is not ready at $t_i$, so it fired at some step $t'$ with $t_i - c_i < t' < t_i$ at an enemy $e' \neq e$ standing on an index $j' \in S_i$, so $b_i - W \leq j' \leq b_i$. The spawn time of $e'$ is $t' - j'$, which lies in $[\, s_e - C + 1,\ s_e + W - 1 \,]$. This interval contains at most $C + W - 1$ enemies. Fix such an enemy $e'$. Two towers of $\Xi$ never have $e'$ in range in the same step, since their seen sets are disjoint. In every step in which a tower of $\Xi$ shoots $e'$, the enemy $e'$ is alive and loses at least one life, so there are at most $\lambda_{e'} \leq \Lambda$ such steps. Hence at most $\Lambda$ towers of $\Xi$ fired at $e'$. Altogether

$$N_0 = |\Xi| \leq (\Lambda - 1) + (C + W - 1)\Lambda < \Lambda(C + W) = N_0,$$

a contradiction. □

## 7.2 Reload time 1 and bounded lives

**Theorem 11** (Reload time 1)**.** *Let all towers have reload time 1 (arbitrary ranges $R_i$) and all lives be $\leq \Lambda$. Then every placement in which at least $\Lambda$ towers see the road wins. Consequently, for every fixed $\Lambda$ the problem is solved in polynomial time: let $m$ be the size of a maximum matching between the towers and the squares, where tower $i$ is joined to the squares of $C_i$. If $m \geq \Lambda$ the answer is yes. Otherwise try all placements of at most $m < \Lambda$ towers on squares of their sets $C_i$, at most $(1 + \sum_i |C_i|)^{\Lambda - 1}$ placements, and play out each.*

*Proof.* Let $\Xi$ be $\Lambda$ towers that see the road and let $e$ leave the road alive. For $i \in \Xi$ the enemy $e$ is the furthest advanced enemy in range at $t_i = s_e + \max S_i$, as in the proof of Lemma 10, and the tower is ready because its reload time is 1. So each tower of $\Xi$ shoots $e$ at $t_i$. All shots are applied, they deal $\Lambda \geq \lambda_e$ damage, and $e$ dies, a contradiction.

A matching of size $\geq \Lambda$ places $\Lambda$ towers on distinct squares of their sets $C_i$ and the other towers far away, and the placement wins. If $m < \Lambda$, every placement has fewer than $\Lambda$ towers that see the road. Moving the other towers far away does not change the play, so a winning placement exists iff one exists with at most $m$ towers, each on a square of its $C_i$. A play-out takes time polynomial in $L$, the number of enemies and $\max_j s_j$. □

### 7.3 Ranges of cardinality one

**Theorem 12** (Ranges of one square)**.** *Let every range have exactly one square, every reload time be $\le C$ and every life $\le \Lambda$. Then every placement in which at least $C\Lambda$ towers see pairwise distinct road squares wins. If all towers have the same range $R = \{r\}$, the problem is polynomial for fixed $C$ and $\Lambda$: with $\mathcal{C} = \{r_j - r : j\} \setminus \rho$, the answer is yes if $\min(\beta, |\mathcal{C}|) \ge C\Lambda$, and otherwise one tries all placements of at most $C\Lambda - 1$ towers on squares of $\mathcal{C}$, at most $(1 + \beta|\mathcal{C}|)^{C\Lambda-1}$ placements.*

*Proof.* A tower with a one-square range sees at most one road square, so $S_i = \{b_i\}$ has span 0, and seeing pairwise distinct squares means that the sets $S_i$ are pairwise disjoint. Apply Lemma 10 with $W = 0$ and $N_0 = C\Lambda$. If all ranges equal $\{r\}$, towers on distinct squares see distinct squares, since $q \mapsto q + r$ is injective. So any $C\Lambda$ towers on distinct squares of $\mathcal{C}$ win, and the rest is as in Theorem 11. □

### 7.4 Unions of leaper ranges

The five *presets* are the leaper ranges $R_{1,1}$, $R_{1,0}$, $R_{1,2}$, $R_{2,2}$ and $R_{2,0}$ (Figure 1).

A range is *admissible* if it is a nonempty union of presets and, when it contains $R_{2,2}$, it contains at least one other preset. Of the 31 nonempty unions, 30 are admissible; only $R_{2,2}$ alone is not. Every admissible range is invariant under the eight symmetries of the grid. Different towers may have different admissible ranges.

**Theorem 13** (Unions of presets)**.** *Fix $\Lambda, C \ge 1$.* Tower Defense *is solvable in polynomial time on instances with a well-formed road, reload times $\le C$, lives $\le \Lambda$ and ranges that are admissible. The road is assumed to be well-formed and nothing else.*

**Constants.** Let $W = 6$, $N_0 = \Lambda(C + W)$, $h = 5$, $\beta = 4$ (the buffer), $P = 2h + 1 + \beta = 15$. By Lemma 10, $N_0$ towers whose seen sets are nonempty, pairwise disjoint and of span $\le 6$ win. The proof produces them from *strips*.

*Proof. Reduction to long roads.* Let $X$ and $Y$ be the numbers of distinct $x$- and $y$-coordinates of road squares. The set of admissible ranges is closed under $x \leftrightarrow y$, so if $Y > X$ we exchange the coordinates. Put $Z = (N_0 + 2)P$. If $X < Z$ and $Y < Z$, the road has fewer than $Z^2$ squares, a constant. A tower sees the road from at most $24L$ squares (at most 24 squares per road square, 24 is the size of the union of all presets). Towers of the same type (range and reload time) standing on squares with the same seen set behave identically, and more than $\Lambda$ of them change nothing, since the target dies at the first volley. So only a constant number of classes with multiplicity $\le \Lambda$ matter, and a play-out of each choice takes polynomial time. Otherwise assume $X \ge Z$. If fewer than $N_0$ towers are given, try all placements on the squares that see the road, $O((24L)^{N_0})$ of them. If at least $N_0$ towers are given, we construct $N_0$ towers as in the windows lemma and answer yes.

*Strips.* (Figure 5 shows a strip in the two cases below.) Cut the columns $x_{\min}, \ldots, x_{\max}$ into blocks of $P = 15$ consecutive columns, at least $\lfloor X/P \rfloor \ge N_0 + 2$ of them. In a block, the *core* are the 11 middle columns, with centre column $c$. The other 4 columns are the buffer, two on each side. *Discard every block that contains the entrance or the exit*, at most two blocks. Every road square in a remaining block has exactly two orthogonal road neighbours (conditions (2) and (3) of the road definition). In a kept block the *origin* $O$ is the lowest road square of the core, ties broken towards $c$, then to the west. We use coordinates relative to $O$. Let $\delta = |x_O - c| \le 5$.

*Case I: $\delta \ge 3$.* Say $O$ lies west of $c$, the other case is the mirror image (the presets are invariant under $x \mapsto -x$), and let $+x$ point towards $c$. No road square is below $O$ in the core, and no square

$(k, 0)$ with $1 \le k \le 5$ is on the road, since it would be at least as close to $c$. $O$ has two road neighbours, neither is south or east, so $(-1, 0)$ and $(0, 1)$ are road squares.

*Case II:* $\delta \le 2$. Now $O$ is at distance $\ge 3$ from both borders of the core, and nothing is below the row of $O$ within 2 columns of $O$. The two road neighbours of $O$ are $\{(-1, 0), (1, 0)\}$, $\{(-1, 0), (0, 1)\}$ or $\{(1, 0), (0, 1)\}$. In the second pattern, the square $(-1, 0)$ has a second road neighbour that is neither $(-1, 1)$ (it would form a $2 \times 2$ block with $O$, contradicting (3)) nor $(-1, -1)$ (below the row of $O$), so $(-2, 0)$ is a road square. Symmetrically, $(2, 0)$ is a road square in the third pattern. Move the origin to $(-1, 0)$ resp. $(1, 0)$ in these two patterns. Afterwards $(-1, 0), (0, 0), (1, 0)$ are road squares and no road square is below row 0 in the columns $-2, \ldots, 2$.

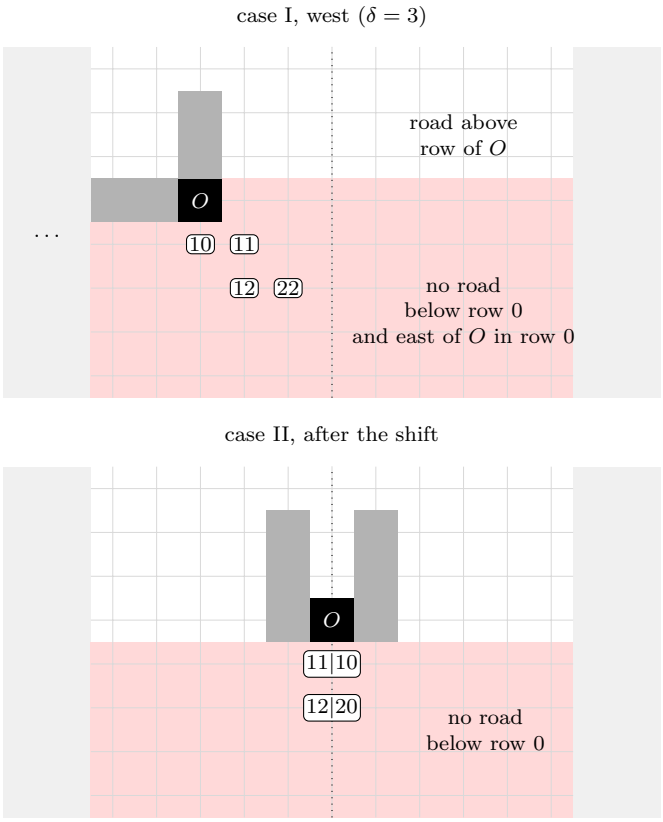



Figure 5: A strip core (white, 11 columns, dotted centre column) with a 2-column margin of the buffer on each side (light gray), and the origin $O$ (black). Dark gray squares are road squares forced by the case hypotheses, the rest of the road is arbitrary above row 0 (in case I also west of $O$ in the row of $O$). The red shaded areas contain no road by construction: in case I the squares below the row of $O$ together with the squares east of $O$ in its row (a quadrant at $O$ extended over the core), in case II the half-plane below the row of $O$ (within the core). The boxed labels are the prescribed squares, where a label $uv$ stands for the range $R_{u,v}$ and $uv|u'v'$ for a range that contains one of the two: in case I the tower square of a range with $R_{2,2}$, with $R_{1,2}$ but not $R_{2,2}$ (first choice, the fallback is $(1, -3)$), with $R_{1,1}$ only, and with $R_{1,0}$ and/or $R_{2,0}$ only. In case II the square $(0, -1)$ for ranges with $R_{1,1}$ or $R_{1,0}$ and $(0, -2)$ otherwise. The squares are off the road. The extent of the seen sets is $x \in [-2, 4]$ in case I and $[-2, 2]$ in case II.

*Placement.* In each kept block put one tower, as in Table 1, with the range of that tower (the ranges in different blocks may differ and the towers are matched to blocks arbitrarily). In case I the square is given relative to the origin in west normal form.

| range | case I square | case II square | span of seen set |
|---|---|---|---|
| contains $R_{2,2}$ | $(2,-2)$ | $(0,-1)$ if it has $R_{1,1}$ or $R_{1,0}$, else $(0,-2)$ | 0 in I, $\leq 6$ in II |
| no $R_{2,2}$, contains $R_{1,2}$ | $(1,-2)$ if its seen set has span $\leq 6$, else $(1,-3)$ | $(0,-1)$ if it has $R_{1,1}$ or $R_{1,0}$, else $(0,-2)$ | $\leq 6$ in I, $\leq 4$ in II |
| no $R_{2,2}, R_{1,2}$, contains $R_{1,1}$ | $(1,-1)$ | $(0,-1)$ | $\leq 4$ in I, $\leq 2$ in II |
| only $R_{1,0}$ and/or $R_{2,0}$ | $(0,-1)$ | $(0,-1)$ if it has $R_{1,0}$, else $(0,-2)$ | $\leq 4$ in I, $\leq 2$ in II |

Table 1: The square of the tower in a strip, relative to the origin. The last column is the index span of the seen set of the chosen square, for the unions of that row. The proof only uses that the chosen square has span $\leq 6$.

*Claim.* The tower square of Table 1 is off the road, its seen set is nonempty, and any two seen squares are joined by a path of road squares of length $\leq 6$ inside the window $x \in [-2,4]$, $y \in [-6,2]$ (case I) or $x \in [-2,2]$, $y \in [-4,2]$ (case II). Since orthogonally adjacent road squares have consecutive indices, the seen set has index span $\leq 6$. The claim depends only on the range and on the case hypotheses above, namely the forced road squares and the forced empty squares. It is a finite statement, about the squares of a bounded window, the finitely many admissible ranges and the finitely many road patterns in the window consistent with the hypotheses and with well-formedness, and is proved by a case distinction that we do not reproduce. The companion preset is needed for the elephant. Figure 6 shows a road of parallel U turns that are as close together as a well-formed road allows. An elephant alone sees two separated U turns from every square that sees the road at all (tower 2, in the gap between $U_3$ and $U_4$, sees both), whereas an elephant together with the wazir or the dabbabah range has a square that sees only one U turn (towers 1 and 3), so only the latter can be used as a single place. For a range with $R_{2,2}$ in case I the seen set is the origin alone: the squares of $(2,-2)+R_{2,2}$ are $(0,0)=O$, $(4,0)$, $(0,-4)$, $(4,-4)$, and the last two lie below row 0 in the empty part of the core. The square $(4,0)$ is empty because $\delta \geq 3$ (it is closer to $c$ than $O$ and in its row).

*Different blocks.* The tower squares lie in the core of their block, so they differ between blocks. Seen squares of a block lie at most 2 columns outside the core, because every element of a preset has $\max(|u|,|v|) \leq 2$, and the cores of two kept blocks are separated by the buffer of 4 columns, so the seen sets of towers in different blocks are disjoint.

*Counting.* At least $N_0 + 2 - 2 = N_0$ blocks are kept. Their towers have nonempty, pairwise disjoint seen sets of span $\leq 6$. Lemma 10 with $W = 6$ shows that the placement, with all other towers far away, wins. This also proves that for $\geq N_0$ towers the answer is yes, completing the algorithm. □

# 8 Open problem

Theorem 1 needs reload times that grow with $\beta$, and Theorem 2 needs lives that grow with $\beta$. For bounded lives and reload times the problem is polynomial in the cases of Section 7. We ask whether this holds in general.

**Open problem.** Fix a nonempty range $R = -R$, a bound $\Lambda$ on the lives and a bound $C$ on the reload times. Is Tower Defense always solvable in polynomial time on well-formed roads?

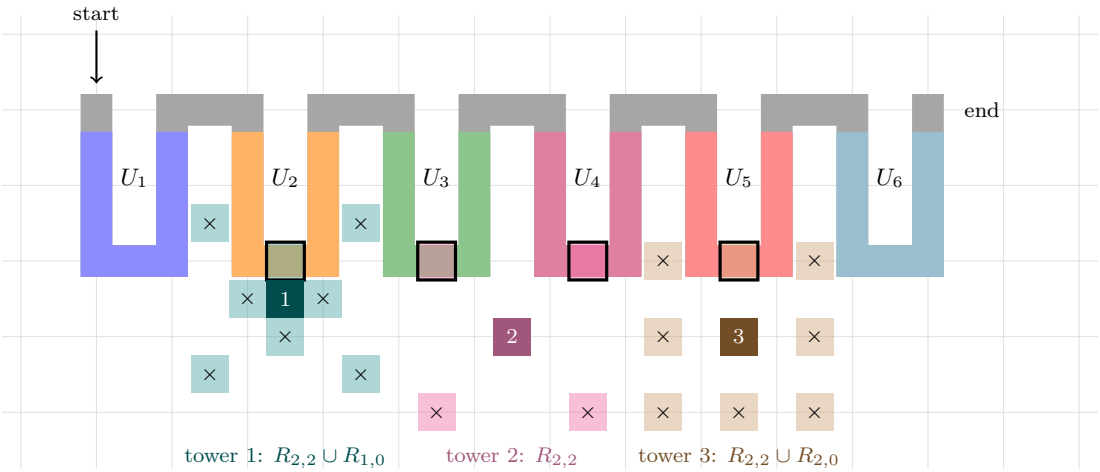



Figure 6: Six parallel U turns $U_1, \ldots, U_6$ (colored) at the smallest spacing of a well-formed road: all arms are two columns apart, so the bottoms of the U turns are 4 columns apart. Gray squares are the links on top. Each tower (numbered square, off the road) is drawn with its entire range $T + R$, shaded in its color. Squares of the range on the road have a bold outline, the others a cross ($\times$). Tower 1 has range $R_{2,2} \cup R_{1,0}$, is one row below the southernmost road row, and sees only $(5, 0)$, on $U_2$ (the wazir square above it; its four elephant squares are empty). Tower 2 has the elephant range $R_{2,2}$ alone and stands in the gap between $U_3$ and $U_4$, two rows below the road: it sees $(9, 0)$ on $U_3$ and $(13, 0)$ on $U_4$. Tower 3 has range $R_{2,2} \cup R_{2,0}$, is two rows below, and sees only $(17, 0)$, on $U_5$ (the dabbabah square above it). An elephant alone sees two separated U turns from every square that sees $U_2$, $U_3$, $U_4$ or $U_5$, but not with the wazir or the dabbabah.

## 9 Acknowledgement

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