UNIVERSAL GENERATION OF THE CYLINDER HOMOMORPHISM OF
CUBIC HYPERSURFACES

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ABSTRACT. Let $X$ be a smooth cubic hypersurface, and let $F(X)$ be its Fano variety of lines. Assume there exists a degree one 1-cycle on $X$, we prove that the cylinder homomorphism maps the Chow group of $F(X)$ surjectively onto the Chow group of $X$. As an application, if $X$ is a smooth complex cubic fourfold, we prove the integral Hodge conjecture for curve classes on the hyper-Kähler variety $F(X)$. In addition, when $X/k$ is a smooth cubic fourfold over a finitely generated field $k$ with char($k$) $\neq$ 2, 3, we prove the integral analog of the Tate conjecture for 1-cycles on $F(X)$.

Introduction

Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface over a field $k$, and let $F := F(X)$ be the Fano variety of lines on $X$. There is a incidence variety

$$ P := \{([\ell], x) \in F \times X \mid x \in \ell \}. $$

Denote by $p : P \to F$ and $q : P \to X$ the natural projections. The incidence variety $P$ induces a homomorphism between the Chow groups

$$ (1) \quad P_* = q_*p^* : \text{CH}_{r-1}(F) \to \text{CH}_r(X), $$

which is called the cylinder homomorphism of $X$.

Assume that the field $k$ is algebraically closed and the dimension of $X$ is at least five. A classical result of Paranjape[12] says that the cylinder homomorphism (1) of $X$ is surjective when $r = 1$. In [15], M. Shen used a different method to extend this conclusion to $X$ of dimensions 3 and 4. The main idea of his method is to establish two crucial relations of 1-cycles on $X$. It can be explained in the following manner.

Let $X$ be a smooth cubic hypersurface of dimension $n \geq 3$, let $C$ be a general smooth curve in $X$. Then $C$ has finitely many secant lines $E_i \subset X$, and there exists a cycle relation

$$ (2) \quad (2 \deg C - 3)[C] + \sum [E_i] \in \mathbb{Z} \cdot h^{n-1} $$

in $\text{CH}_1(X)$, where $h$ is the hyperplane section class of $X$. Another relation is about a pair of general smooth curves $C_1$ and $C_2$ in $X$. It is shown that there are finitely many lines $E_i \subset X$ meeting both $C_1$ and $C_2$. Then the second relation is

$$ (3) \quad 2 \deg C_2[C_1] + 2 \deg C_1[C_2] + \sum [E_i] \in \mathbb{Z} \cdot h^{n-1}. $$

These two relations deduce the surjectivity of the cylinder homomorphism for 1-cycles on cubic hypersurfaces. It is not known yet whether the homomorphisms are surjective for higher dimensional cycles or for cubic hypersurfaces defined over non-algebraically fields. The main goal of this article is to find analogous relations as above for higher dimensional cycles. To be precise, the following is our main result

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Theorem 0.1 (see Theorem 2.5). Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface over a field $k$, and let $F := F(X)$ be its Fano variety of lines. Denote by $h_X$ the hyperplane section class of $X$. Assume that $X/k$ admits a degree one $1$-cycle defined over $k$. Suppose that $\Gamma \in \text{CH}_r(X)$ is an algebraic cycle of dimension $r > 1$ with degree $e := \Gamma \cdot h_X^n$. We have the following two cycle relations. There exist an algebraic cycle $\gamma \in \text{CH}_{r-1}(F)$ such that

$$2\Gamma + q_*p^*\gamma \in \mathbb{Z} \cdot h_X^{n-r}.$$ 

There exists an algebraic cycle $\gamma' \in \text{CH}_{r-1}(F)$ such that

$$(2e - 3)\Gamma + q_*p^*\gamma' \in \mathbb{Z} \cdot h_X^{n-r}.$$ 

Under the assumption of the existence of a degree one $1$-cycle on $X$, Theorem 0.1 implies that the cylinder homomorphisms are surjective modulo the classes of intersection of hyperplanes. Moreover, if we specify this degree one $1$-cycle to be a $k$-line, it can be shown that the classes of intersection of hyperplanes are contained in the image of cylinder homomorphisms, see Lemma 2.8. Thus we immediately obtain the main corollary,

Corollary 0.2. Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface of dimension $n \geq 3$ over a field $k$, and let $F(X)$ be the Fano variety of lines. Assume that $X$ has a line $L$ defined over $k$. The cylinder homomorphism

$$P_* : \text{CH}_{r-1}(F) \longrightarrow \text{CH}_r(X)$$

is surjective except $r = \text{dim } X - 1$.

Of course all the results above unconditionally hold when $k$ is algebraically closed.

The proof of the main theorem 0.1 is carried out in Section 2. It is based on some new observations of the geometry of the second punctual Hilbert scheme of cubic hypersurfaces, recently established by S. Galkin, E. Shinder and C. Voisin in [8] and [20], which will be reviewed in Section 1. By using this geometric structure, the method of our proof is purely cycle-theoretic. Hence the result works for non-algebraically closed fields.

Our method is derived from M.Shen’s idea in [14] to reprove the $1$-cycle relations (2) and (3). Following that idea, R. Mboro proved the aforementioned relation (5) when $\Gamma \subset X$ is a subvariety in general position or a smooth subvariety, see [9]. The argument we use can drop this constraint such that the relations hold for algebraic cycles.

Let $X/\mathbb{C}$ be a smooth complex cubic fourfold. Beauville and Donagi showed that its Fano variety of lines $F(X)$ is a polarized hyper-Kähler variety of $K3$-[2] type. Using the integral Hodge conjecture for cubic fourfolds by C.Voisin [19] and our result of the surjectivity of the cylinder homomorphism , we can prove

Corollary 0.3 (see Corollary 3.1). The integral Hodge conjecture for curve classes on the hyper-Kähler variety $F(X)$ is true.

Recently, G. Mongardi and J. C. Ottem [11] showed that the integral Hodge conjecture for curve classes on hyper-Kähler varieties of $K3$-type and the generalized Kummer type is true. Our method provides a weaker proof of a particular case of their results.

The arithmetic aspect of the Hodge conjecture is the Tate conjecture. Let $k$ be a field finitely generated over its prime subfield, let $\overline{k}$ be the separable closure of $k$ and let $G$ be the absolute Galois group $\text{Gal}(\overline{k}/k)$. Suppose that $X$ is a smooth cubic fourfold over $k$. Let $\ell$ be a prime number different from the characteristic of
The integral analog of the Tate conjecture for 1-cycles on its Fano variety of lines $F := F(X)$ asks if the cycle class map,

$$\text{CH}^3(F_k) \otimes \mathbb{Z}_\ell \to \lim_{\to} \mathbb{H}^6_{\text{et}}(F_k, \mathbb{Z}_\ell(3))^U$$

is surjective, where the direct limit is over the open subgroups $U$ of $G$. This cycle class map was originally addressed by C. Schoen in [13] to replace the naive integral Tate conjecture, which in general fails for smooth proper $k$-varieties. We will imitate the proof of Corollary 3.1 to show that

**Corollary 0.4** (see Corollary 3.4). If the characteristic of $k$ is different from 2 and 3, the integral analog of the Tate conjecture for 1-cycles on the Fano variety of lines of a smooth cubic fourfold $X/k$ is true.

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1. The Hilbert square of cubic hypersurfaces

Let $X/k$ be a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}_k$, let $F := F(X)$ be the Fano variety of lines on $X$. We denote by $P_X := \mathbb{P}(T_{\mathbb{P}^{n+1}_k}|_X)$ the projectivization of the tangent bundle $T_{\mathbb{P}^{n+1}_k}$ over $X$. The Hilbert scheme of length two subschemes of $X$ admits a birational map

$$\Phi : X[2] \dashrightarrow P_X.$$

To a point $\tau \in X[2]$, one associates the unique $k$-rational line $\ell_{\tau} \subset \mathbb{P}^{n+1}$ passing through $\tau$. For generic $\tau$, the line $\ell_{\tau}$ is not contained in $X$. Then the intersection $\ell_{\tau} \cap X$ arises a unique residue $k$-point $z \in X$. We define $\Phi(\tau) = (\ell_{\tau}, z)$.

The universal family of lines $P := \{((\ell, x) \in F \times X \mid x \in \ell)\}$ is a $\mathbb{P}^1$-bundle $p : P \simeq \mathbb{P}(\mathcal{E}) \to F$ of the tautological rank two vector bundle $\mathcal{E}$ on $F$. It induces a $\mathbb{P}^2$-bundle $p_2 : P_2 := \mathbb{P}(	ext{Sym}^2 \mathcal{E}) \to F(X)$ whose fiber over a line $\ell \subset X$ is the symmetric product $\ell[2]$. The birational map $\Phi$ is not well-defined on the subvariety $P_2$ of $X[2]$. In [20, Proposition 2.9], Voisin showed that $\Phi$ admits a resolution of singularities

$$X[2] \xleftarrow{\tau} \overline{X[2]} \xrightarrow{\Phi} P_X$$

Here the morphism $\tau : \overline{X[2]} \to X[2]$ the blowing up along the smooth center $i_2 : P_2 \to X[2]$ of codimension 2. The morphism $\Phi : X[2] \to P_X$ is the blow-up of the of the smooth center $i_1 : P \subset P_X$ of codimension 3. Moreover, the exceptional divisors of the two blow-ups are the same, hence let us denote it by $E$.

We use the following diagram to summarize the above descriptions.
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The way we define the birational map $\Phi$ arises an actual proper morphism $\varphi$ from $X^{[2]}$ towards the Grassmannian of lines $G(2,n+2)$. We abusively use $\mathcal{E}$ to denote the tautological rank 2 vector bundle on $G(2,n+2)$. The $\mathbb{P}^1$-bundle $P(\mathcal{E})$ over $G(2,n+2)$ is naturally isomorphic to the projective tangent bundle $\pi : \mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(T_{p_{n+1}}) \to \mathbb{P}^{n+1}$. Let $\pi_{X^{[2]} : Q \to X^{[2]}}$ be the pull-back of $P(\mathcal{E})$ under the map $\varphi$, and let $\alpha : Q \to P(\mathcal{E}) \to \mathbb{P}^{n+1}$ be the composition. In fact, the blow-up $\widetilde{X}^{[2]}$ can be interpreted as one component of the divisor $\alpha^{-1}(X)$ on $Q$ which has degree one over $X^{[2]}$. Since $X$ is a cubic hypersurface, another component of $\alpha^{-1}(X)$ having degree two over $X^{[2]}$, is the blow-up $\widetilde{X} \times X$ of $X \times X$ along the diagonal. We obtain the following cartesian diagrams

For the conventions, we also give the following diagram

Lemma 1.1. Use the notations in in diagrams (7), (8). For any algebraic cycle $\Xi \in \text{CH}_k(X^{[2]})$ of dimension $k$, we have a $(k-2)$-dimensional algebraic cycle $\gamma \in \text{CH}_{k-2}(F)$ such that

$$\pi_X^* \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Xi) = q_* p^* \gamma$$
in $\text{CH}_{k-1}(X)$. As a matter of fact, $\gamma = \pi_F^* i_2^* \Xi$.

Proof. Since $\mathcal{E} \cdot \tau^* \Xi = j_* j^* \tau^* \Xi$, the formula immediately follows from the commutativity of the diagrams (7) and (8).
Lemma 1.2. With the notations in diagrams (7), (9) and (10), given any algebraic cycle \( W \in CH^r(X^{[2]}) \), we have
\[
\pi_X^* \Phi_* \tau^* W = i_X^* \pi_P^* \pi_G^* \varphi_* W - \pi_X^* \Psi_* \sigma^* W.
\]

Proof. As seen from the commutative diagram (9), the action induced by the algebraic cycle \([X \times X] + [X^{[2]}] \) on the Chow group of \( X^{[2]} \), is equal to \((\pi_G \circ i')^* \circ \varphi_* \). Therefore, the action of the component \( X^{[2]} \) on the Chow group of \( X^{[2]} \) is
\[
[X^{[2]}]_* = i'^* \circ \pi_G^* \circ \varphi_* - [X \times X]_*.
\]
It follows from the diagram (10) that
\[
[X \times X]_* = \Psi_* \circ \sigma^*.
\]
Hence we obtain the equality after pushing-forward to \( X \).

For simplicity of notations, for any given cycle classes \( \alpha, \beta \in CH^r(X) \), we denote the hat tensor \( \alpha \hat{\otimes} \beta \) to be the algebraic cycle \( \sigma^r \rho^s (\alpha \otimes \beta) \) in \( CH^r(X^{[2]}) \).

Lemma 1.3. Recall that \( \mathcal{E} \) is the exceptional divisor of the blowing up \( \tau : X^{[2]} \to X^{[2]} \). Let \( h_X \) be hyperplane section class of \( X \), let \( h_X \in Pic(\mathcal{Q}) \) polarization through the morphism \( \alpha : \mathcal{Q} \to \mathbb{P}^{n+1} \), see (7), and let \( \delta \in Pic(X^{[2]}) \) the half diagonal divisor of \( X^{[2]} \) such that \( \sigma^* \delta \) is the canonical exceptional divisor \( E_\Delta \) on \( X \times X \). Then we have
\[
\mathcal{E} = -h_X [X^{[2]}] - \tau^* (2h_X \otimes 1 - 3\delta),
\]
and
\[
g := c_1(\varphi^* \mathcal{E}) = -h_X \otimes 1 + \delta
\]
in \( Pic(\mathcal{X}^{[2]}) \).

Proof. See [14] Lemma 4.3.

2. The proof of the main theorem

Let \( X \) be a smooth cubic hypersurface variety. The natural 2 to 1 rational map \( \mu : X \times X \mapsto X^{[2]} \) can be resolved by blowing up \( X \times X \) along the diagonal \( \Delta_X \). We thus have a diagram as follows:

\[
\begin{array}{ccc}
E_\Delta & \xrightarrow{\mu} & X \times X \\
\downarrow \pi_\Delta & & \downarrow \rho \\
\Delta_X & \xrightarrow{i_\Delta} & X \times X,
\end{array}
\]
where \( E_\Delta \) is the exceptional divisor.

Lemma 2.1. Assume that \( \Gamma \) is a symmetric codimension \( r \) algebraic cycle in \( X \times X \) that is not contained in the diagonal \( \Delta_X \), i.e. no irreducible component of \( \Gamma \) is entirely supported in \( \Delta_X \). Then there exists a codimension \( r \) algebraic cycle \( \Sigma \) in \( X^{[2]} \) such that \( \mu^* \Sigma := \rho_* \sigma^* \Sigma = \Gamma \).

Proof. The algebraic cycle \( \Gamma \) is symmetric means that \( \Gamma \) can be presented by the sum of \( Z_1 + Z_2 \). Here \( Z_1 = \sum n_i Z_{1,i} \) is a linear combination of irreducible subvarieties \( Z_{1,i} \) of \( X \times X \) invariant under the involution \( i \), and \( Z_2 \) is of the form \( Z_2^i + i(Z_2^i) \). The proof for \( r = n \) is presented by Voisin in [20] Corollary 2.4. Our argument is similar.
Since $Z_{1,i}$ does not lie in $\Delta_X$, we take $\tilde{Z}_{1,i}$ to be the proper transform of $Z_{1,i}$ in $\tilde{X} \times X$. Define the algebraic cycle $\Sigma$ to be $\sum_i n_i[\sigma(\tilde{Z}_{1,i})] + \sigma_{s'}^*Z'_2$. Since the morphism $\sigma$ is flat and proper, we have

$$
\mu^*\Sigma = \sum_i n_i\rho_*\sigma^*[\sigma(\tilde{Z}_{1,i})] + \rho_*\sigma_{s'}^*\rho_{s'}Z'_2
$$

$$
= \sum_i n_i\rho_*[\tilde{Z}_{1,i}] + \rho_*\rho^*Z_2 + \rho_*\rho^*i(Z_2)
$$

$$
= \sum_i n_iZ_{1,i} + Z'_2 + i(Z'_2),
$$

Remark 2.2. The statement and the proof hold for any smooth projective varieties. But we will only apply it to cubic hypersurfaces. In addition, notice that Proposition 2.3. $X$ has a degree one $\varphi$-isomorphism. Assume that there exists a degree one $1$-cycle $\Gamma'$ on $X$. Define the algebraic cycle $\Sigma = (1)$ there exists an algebraic cycle $\Gamma$ such that $\rho_\sigma^*\sigma'[\sigma_{s'}^*\rho_{s'}^*Z'_2] = 2[\Sigma]$. Otherwise we have $\sigma_{s'}\sigma'[\Sigma] = 2[\Sigma]$. 

Proposition 2.3. Let $X$ be a smooth cubic hypersurface, let $\Gamma \subset X$ be an open subvariety of dimension $r > 1$ with degree $e := \Gamma \cdot H_X^{n-r}$, where $H_X$ is the hyperplane section class of $X$. Assume that there exists a degree one $1$-cycle $\Gamma'$ in $\text{CH}_1(X)$. With the notations in the diagrams (10) and (11),

1. there exists an algebraic cycle $\Gamma''$ of dimension $r + 1$ on $X^{[2]}$ such that

$$
\pi_*\Psi_\sigma^*\Gamma'' = 0;
$$

$$
\pi_*\Psi_\sigma^*(h_X \cdot 1 \cdot \Gamma') = \Gamma;
$$

$$
\pi_*\Psi_\sigma^*(\delta \cdot \Gamma') = 0,
$$

2. and there exists an algebraic cycle $\Gamma''$ of dimension $2r$ on $X^{[2]}$ such that

$$
\pi_*\Psi_\sigma^*(\Gamma'') \cdot (h_X \cdot 1)^k \cdot \delta^l = \begin{cases} 0, & \forall 0 \leq k, l \leq r - 1; \\
 e \cdot \Gamma, & k = r, l = 0; \\
 (-1)^{r+1} \Gamma, & k = 0, l = r. 
\end{cases}
$$

Proof. For the assertion in (1), we have a symmetric algebraic cycle $\Gamma \times \Gamma \times \Gamma$ on $X \times X$ of codimension $n + r - 1$. It follows from the Lemma 2.1 that there exists a cycle $\Gamma''$ on $X^{[2]}$ such that $\rho_\sigma^*\Gamma'' = (\Gamma \times \Gamma \times \Gamma)$. Then we have

$$
\pi_*\Psi_\sigma^*\Gamma'' = p_{1*}\rho_{s'}^*\sigma^*\Gamma'' = p_{1*}(\Gamma \times 1 \times 1).
$$

It also induces the second formula

$$
\pi_*\Psi_\sigma^*(h_X \cdot 1 \cdot \Gamma') = p_{1*}\rho_{s'}(\sigma^*(h_X \cdot 1) \cdot \sigma^*\Gamma') = p_{1*}\rho_{s'}(\rho^*(h_X \times 1 + 1 \times h_X) \cdot \sigma^*\Gamma')
$$

$$
= p_{1*}((h_X \times 1 + 1 \times h_X) \cdot \rho_*\sigma^*\Gamma')
$$

$$
= p_{1*}((h_X \times 1 + 1 \times h_X) \cdot (\Gamma \times 1 + 1 \times \Gamma))
$$

$$
= \deg(h_X \cdot 1 \cdot \Gamma) = \Gamma, \ (\dim h_X \cdot \Gamma > 0 \text{ since } \dim \Gamma > 1)
$$

and as the same as the third one

$$
\pi_*\Psi_\sigma^*(\delta \cdot \Gamma') = \pi_*\Psi_\sigma^*(E_{\Delta} \cdot \sigma^*\Gamma') = p_{1*}\rho_{s'}(j_{E*}^\Gamma j_{E}^\sigma \sigma^*\Gamma')
$$

$$
= \pi_{\Delta*}(j_{E}^\Gamma \sigma^*\Gamma') = \pi_{\Delta*}(j_{E}^\Gamma \rho^*(\Gamma \times 1 + 1 \times \Gamma))
$$

$$
= 2\pi_{\Delta*}(\Gamma \cdot 1) = 0.
$$

It concludes the assertions in (1).

For the second assertion, we consider the closed subscheme $\Gamma \times \Gamma \subset X \times X$. Again by Lemma 2.1, there is a $2r$-dimensional algebraic cycle $\Gamma'' \in \text{CH}_{2r}(X^{[2]})$.
such that $\rho_*\sigma^*\Gamma'' = [\Gamma \times \Gamma]$. In fact, the pullback $\sigma^*\Gamma''$ can be represented by the strict transform $[\Gamma \times \Gamma]$. By the commutative diagram (11), we have

$$
\pi_*\Psi_*\sigma^*(\Gamma'': (h_X \otimes 1)^k \cdot \delta') = p_{1*}(\sum_{s=0}^{k} \binom{k}{s} h_X^s \times h_X^{k-s} \cdot \rho_*\sigma^*\Gamma''. E_{\Delta})
$$

$$
= p_{1*}(\sum_{s=0}^{k} \binom{k}{s} h_X^s \times h_X^{k-s} \cdot \rho_*[\Gamma \times \Gamma] \cdot E_{\Delta})
$$

- If $l = 0$, it follows that

$$
\pi_*\Psi_*\sigma^*(\Gamma'': (h_X \otimes 1)^k) = p_{1*}(\sum_{s=0}^{k} \binom{k}{s} h_X^s \times h_X^{k-s} \cdot \rho_*[\Gamma \times \Gamma])
$$

$$
= \sum_{s=0}^{k} \binom{k}{s} p_{1*}(h_X^s \cdot \Gamma)(h_X^{k-s} \cdot \Gamma) = \begin{cases} 0, & k < r; \\ \deg \Gamma \cdot \Gamma, & k = r. \end{cases}
$$

- Assume $1 \leq l \leq r - 1$. Let $Q$ be the excess normal bundle $\pi^*_\Delta N_{\Delta X/X \times X}/\mathcal{O}(1)$ on $E_{\Delta}$. Recall the Blow-up formula in [7 Theorem 6.7]

$$
[\Gamma \times \Gamma] = \rho^*([\Gamma \times \Gamma]) - j_{E_*}(e(Q) \cap \pi^*_\Delta s([\Gamma, \Gamma \times \Gamma]_{\mathcal{O}(1)}))_{2r},
$$

where $s([\Gamma, \Gamma \times \Gamma])$ is the Segre class of the closed subvariety $\Gamma$ in $\Gamma \times \Gamma$. We simply write the $i$-th Segre class $s_i([\Gamma, \Gamma \times \Gamma])$ as $s_i$. The term $\rho_*([\Gamma \times \Gamma] \cdot E_{\Delta})$ is equal to

$$
[\Gamma \times \Gamma] \cdot \rho_* E_{\Delta}^r = i_{\Delta_*} \pi^*_\Delta_* \{ \{ c(Q) \cap \pi^*_\Delta s([\Gamma, \Gamma \times \Gamma]) \}_{2r} \cdot j_{E} E_{\Delta}^r \}
$$

$$
= 0 - \sum_{i+t=n-r-1} i_{\Delta_*}(s_i \cdot \pi^*_\Delta_* (c_i Q) c_1 (O(-1)^l)) = 0.
$$

since $\pi^*_\Delta_* (c_i Q) c_1 (O(-1)^l))$ is equal to zero because $t + l \leq n - 2$.

- If $k = 0$, $l = r$ and $r < n$, we have

$$
\pi_*\Psi_*\sigma^*(\Gamma'': \delta') = p_{1*}\rho_* (\sigma^*\Gamma'' \cdot E_{\Delta}^n)
$$

$$
= p_{1*}([\Gamma \times \Gamma] \cdot \rho_* E_{\Delta}^n - i_{\Delta_*} \pi^*_\Delta_* \{ \{ c(Q) \cap \pi^*_\Delta s([\Gamma, \Gamma \times \Gamma]) \}_{2r} \cdot j_{E} E_{\Delta}^r \})
$$

$$
= (-1)^{r+1} \sum_{i+t=n-r-1} i_{\Delta_*}(s_i \cdot \pi^*_\Delta_* (c_i Q) c_1 (O(1))^r))
$$

$$
= (-1)^{r+1} i_{\Delta_*}(s_0 \cdot c_0 (X) \cdot \pi^*_\Delta_* c_1 (O(1)^{n-1})) = (-1)^{r+1} [\Gamma].
$$

If $k = 0$, $l = r = n$, in other words, $\Gamma = [X]$, it turns out that

$$
p_{1*}\rho_* (\sigma^*\Gamma'' \cdot E_{\Delta}^n) = p_{1*}([X \times X] \cdot \rho_* E_{\Delta}^n) = (-1)^{n-1} [X] = (-1)^{n+1} [X].
$$

\[\square\]

**Remark 2.4.** The proof of the assertion (1) needs the existence of the degree one 1-cycle, which is not guaranteed in general. For algebraically closed fields, we could take this 1-cycle to be a line in $X$. Over non-closed fields, there exists cubic hypersurfaces containing no lines, e.g. number fields or finite field with small orders, see [9].

The above proposition implies the following main theorem.

**Theorem 2.5.** Let $X$ be a smooth cubic hypersurface, let $h_X$ be the hyperplane section class. Let $\Gamma \in \text{CH}_1(X)$ be an algebraic cycle of dimension $r > 1$ with degree $e$. Suppose that there exists a degree one 1-cycle $l$ in $\text{CH}_1(X)$. Then we have
We first prove the above two cycle relations by assuming $\Gamma$ is an irreducible closed subvariety, then the conclusion of algebraic cycles are deduced. The algebraic cycle $\Gamma'$ is the same as in (1) of Proposition 2.3. Recall the Lemma 1.3 to obtain:

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* \Gamma') = q_* p^* \gamma, \]

where $\gamma = \pi_F(i_2^* \Gamma' \in CH_{r-1}(F))$. Then we use the relation

\[ E = -h_{\partial |X|} - \tau^*(2h_X \otimes 1 - 3\delta) \]

in the Lemma 1.3 to obtain:

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* \Gamma') = -\pi_* \tilde{\Phi}_*(h_{\partial |X|} \cdot \tau^* \Gamma') - \pi_* \tilde{\Phi}_*(\tau^*(2h_X \otimes 1 - 3\delta) \cdot \tau^* \Gamma') \]

\[ = -h_X \cdot \pi_* \tilde{\Phi}_* \tau^* \Gamma' - \pi_* \tilde{\Phi}_*(\tau^*(2h_X \otimes 1 - 3\delta) \cdot \Gamma'). \]

The first term

\[ \pi_* \tilde{\Phi}_* \tau^* \Gamma' = i_X \pi_* \pi_* \pi_* \varphi \cdot \Gamma' - \pi_* \Psi \cdot \sigma^* \Gamma' \]

\[ = i_X \pi_* \pi_* \pi_* \varphi \cdot \Gamma' \in \mathbb{Z} \cdot h_X^{-r-1}. \]

is a hyperplane section class. The second term follows from Lemma 1.2 and Proposition 2.3:

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* (\Gamma'' \cdot g^{r-1})) = \tau^* (\Gamma'' \cdot g^{r-1}) \cdot \Gamma' \]

\[ = i_X \pi_* \pi_* \pi_* \varphi \cdot (\tau^* (\Gamma'' \cdot g^{r-1}) \cdot \Gamma') \equiv -\pi_* \Psi \cdot \sigma^* (\tau^* (\Gamma'' \cdot g^{r-1}) \cdot \Gamma') \mod \mathbb{Z} \cdot h_X^{-r-1} \]

\[ \equiv -2 \Gamma \mod \mathbb{Z} \cdot h_X^{-r-1}. \]

Therefore, we obtain the required relation

\[ 2\Gamma + q_* p^* \gamma \equiv 0 \mod \mathbb{Z} \cdot h_X^{-r-1}. \]

To get the formula (ii), we study the cycle class

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* (\Gamma'' \cdot g^{r-1})), \]

where $g = c_1(\varphi^* \delta') = -h_X \otimes 1 + \delta$ (see Lemma 1.3) and $\Gamma''$ is the algebraic cycle introduced in (2) of Proposition 2.3. Notice that the cycle $\Gamma'' \cdot g^{r-1}$ has dimension $r + 1$. We have

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* (\Gamma'' \cdot g^{r-1})) = q_* p_* \gamma', \]

where $\gamma' = \pi_F(i_2^* (\Gamma'' \cdot g^{r-1}) \in CH_{r-1}(F))$. Again use the same argument as above

\[ \pi_* \tilde{\Phi}_*(E \cdot \tau^* (\Gamma'' \cdot g^{r-1})) = -h_X \cdot \pi_* \tilde{\Phi}_* \tau^* (\Gamma'' \cdot g^{r-1}) - \pi_* \tilde{\Phi}_* \tau^* (2h_X \otimes 1 - 3\delta) \cdot \Gamma'' \cdot g^{r-1}). \]

Now use the assertion (2) of the Proposition 2.3 we obtain

\[ \pi_* \tilde{\Phi}_* \tau^* (\Gamma'' \cdot g^{r-1}) = i_X \pi_* \pi_* \pi_* \varphi \cdot (\Gamma'' \cdot g^{r-1}) - \pi_* \Psi \cdot \sigma^* (\Gamma'' \cdot g^{r-1}) \]

\[ \equiv -\pi_* \Psi \cdot \sigma^* (\Gamma'' \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} (-h_X \otimes 1)^k \cdot \delta^{r-k-1}) \mod \mathbb{Z} \cdot h_X^{-r-1} \]

\[ \equiv 0 \mod \mathbb{Z} \cdot h_X^{-r-1}. \]
and
\[ \pi_*\Phi^*((2h_X \otimes 1 - 3\delta ) \cdot \Gamma'' \cdot g^{-1}) \]
\[ = \psi_X \varphi \Gamma'' \cdot g^{-1} - \pi_*\sigma^*((2h_X \otimes 1 - 3\delta ) \cdot \Gamma'' \cdot g^{-1}) \]
\[ \equiv -\pi_*\sigma^*((2h_X \otimes 1 - 3\delta ) \cdot \Gamma'' \cdot \sum_{k=0}^{r-1} \binom{r-1}{k}(-h_X \otimes 1)^k \cdot \delta^{r-k-1}) \bmod \mathbb{Z} \cdot h_X^{n-r} \]
\[ \equiv (-1)^{2 \deg \Gamma} \cdot 3 \Gamma \bmod h_X^{n-r}. \]

We conclude that
\[ (13) \quad 2 \deg \Gamma - 3 \Gamma \equiv q_*p^*\gamma' \bmod \mathbb{Z} \cdot h_X^{n-r} \]
for an irreducible closed subvariety \( \Gamma \).

To extend the result to algebraic cycles, assume that a \( r \)-dimensional algebraic cycle \( \Gamma \in CH_1(X) \) can be written as \( \sum_i n_i \Gamma_i \), for distinct irreducible components \( \Gamma_i \) with degrees \( e_i \). The relation \( (12) \) remains after a multiple scalar, namely we have
\[ 2(\sum_j n_j e_j - e_i) \cdot \Gamma_1 \in q_*p^* CH_{r-1}(F) + \mathbb{Z} \cdot h_X^{n-r} \]
for each \( i \).

Together with \( (13) \), we obtain
\[ (2 \deg \Gamma - 3)(\sum_i n_i \Gamma_i) \in q_*p^* CH_{r-1}(F) + \mathbb{Z} \cdot h_X^{n-r}. \]

\[ \square \]

**Corollary 2.6.** Let \( X \subset \mathbb{P}^{n+1}_k \) be a smooth cubic hypersurface of dimension \( n \geq 3 \) over a field \( k \), and let \( F(X) \) be the Fano variety of lines. Assume that there exists a degree one 1-cycle in \( CH_1(X) \). Then the cylinder homomorphism
\[ p_* : CH_{r-1}(F(X)) \rightarrow CH_r(X) \]
is surjective for any degree \( r \geq 1 \) up to classes of intersection of hyperplanes.

**Proof.** The assertion for 1-cycles is due to M. Shen, see [14, Proposition 4.2]. The preceding two formulas in the main theorem deduce our corollary for higher dimensional cases.

\[ \square \]

**The surjectivity of cylinder homomorphisms** In the following, we show that the classes of intersection of hyperplanes of a smooth cubic hypersurface is the image of some algebraic cycles on its Fano variety of lines. In Lemma 2.8, we first prove the statement for the cubic hypersurface defined over an algebraically closed field. For non-algebraically closed field, using the following lemma, we prove that the conclusion still holds under a mild assumption.

**Lemma 2.7.** [5, p.599] Let \( V \) be a smooth \( k \)-variety of dimension \( d \). For any non-empty Zariski open subset \( U \subset V \), any zero-cycle on \( V \) is rationally equivalent to a zero-cycle supported on \( U \).

**Proof.** The proof is almost a line-by-line translation of the proof in [5]. It suffices to prove the case of a closed point \( p \in V \). Let \( \mathcal{O}_{V,p} \) be the local ring at the point \( p \). The local ring is Cohen-Macaulay since \( V \) is smooth. Hence, we can find a regular sequence \( (g, f_1, \ldots, f_{d-1}) \) in \( \mathcal{O}_p \) such that \( g \) vanishes on \( F \). Then the integral curve \( C \) defined by the equations \( f_1, \ldots, f_{d-1} \) is regular at \( p \) and not contained in \( F \). Let \( D \rightarrow C \) be the normalization of \( C \). The inverse image of the point \( p \in C \) is a single closed point \( q \in D \). The composition of maps \( \pi : D \rightarrow V \) is proper, which induces a homomorphism on Chow groups of zero-cycles. Hence, the 0-cycle \( p \) is rationally equivalent to \( \pi_*(q) \). The inverse image \( F_1 = \pi^{-1}(F) \) is a finite subset of closed points of \( D \). We take \( R := \mathcal{O}_{F_1,D} \) to be the localization at \( F_1, \)
which is a semi-local ring. It is known that the Picard group of a semi-local ring is trivial. Then the divisor \( q \) of \( D \) is a principal divisor, i.e. there exists an element \( f \) in the fraction field \( K(R) = K(D) \) such that \( \text{div}(f) = q \) on \( \text{Spec}(R) \). It implies that \( \text{div}(f) = z + q \) on \( D \) where \( z \) is a zero-cycle supported on \( D \setminus F_1 \). Therefore, \( p \) is rationally equivalent to a zero-cycle on \( V \) away from \( F \).

**Lemma 2.8.** Let \( X \) be a smooth hypersurface of dimension \( n \geq 3 \) over a field \( k \), let \( q_*p^* : \text{CH}^{n+i-3}(F(X)) \to \text{CH}^i(X) \) be the cylinder homomorphism. Assume that \( X \) contains a line \( L \) defined over \( k \). Then for any \( i > 1 \), the class \( H^X_{-i} \) of intersection of hyperplanes is equal to \( p_*z \) for some algebraic cycle \( z \in \text{CH}^{n+i-3}(F(X)) \).

**Proof.** Suppose the field \( k \) is algebraically closed. Then the assumption in this lemma is redundant. Let \( x \) be a closed point of \( X \). We denote

\[
C_x := \{ [\ell] \in F(X) | x \in \ell \}
\]

the closed subvariety of \( F(X) \) parameterizing lines in \( X \) passing through \( x \). Let \( \ell \) be a line on \( X \). We denote

\[
S_\ell := \{ \ell' \in F(X) | \ell' \cap \ell \neq \emptyset \text{ for a generic } \ell \subset X \}
\]

the closed subscheme of \( F(X) \) parameterizing lines in \( X \) meeting with \( \ell \). We claim that

\[ q_*p^*[C_x] = 2H_X^2, \quad \text{and } q_*p^*[S_\ell] = 5H_X. \]

We may assume \( x = [1, 0, \ldots, 0] \in X \). Then the cubic equation \( G \) defining \( X \) has the form

\[ G = X_0^2L(X_1, \ldots, X_{n+1}) + X_0Q(X_1, \ldots, X_{n+1}) + C(X_1, \ldots, X_{n+1}) \]

with homogeneous polynomial \( L \) (resp. \( Q, C \)) of degree 1 (resp. 2, 3). So the lines in \( X \) passing through \( x \) is parametrized by the set of points in \( \mathbb{P}^n \) cut out by the equations \( L = Q = C = 0 \). If \( x \) is a generic point, the dimension of \( C_x \) is \( n - 3 \). It is easy to see that the closed subscheme \( q_*p^*[C_x] \) of \( X \) is cut out by the equations \( L = Q = 0 \). Therefore we obtain \( q_*p^*[C_x] = 2H_X^2 \in \text{CH}^2(X) \).

If \( \ell \) is a general line, the dimension of \( S_\ell \) is \( n - 2 \). Therefore the algebraic cycle \( q_*p^*[S_\ell] \) is equal to \( m \cdot H_X \) for some integer \( m \). The integer \( m \) can be determined by M. Shen’s result of 1-cycle relations on cubic hypersurfaces (cf. [15, Lemma 3.10])

\[ m = q_*p^*[S_\ell] \cdot \ell = [S_\ell] \cdot p_*q^*\ell = \# \{ \text{lines meeting two general lines } \ell \text{ and } \ell \} = 5. \]

Therefore we have \( q_*p^*[S_\ell] = 5H_X. \)

To obtain the results for classes of higher codimensions, we take the above closed subvarieties in complete intersections of \( X \). Let \( X_{n-i} \) be the \( i \)-th complete intersection of hyperplanes of \( X \). We define

\[
C_{x, X_{n-i}} := \{ [\ell] \in F(X) | x \in \ell \subset X_{n-i} \}
\]

to be the closed subvarieties parametrizing the lines in \( X_{n-i} \) passing through a generic point \( x \in X_{n-i} \). Since \( C_{x, X_{n-i}} \) is indeed contained in \( F(X_{n-i}) \) we have

\[ q_*p^*[C_{x, X_{n-i}}] = 2j_*H_{X_{n-i}}^2 = 2H_X^{i+2}, \quad j : X_{n-i} \hookrightarrow X. \]

Similarly we define

\[
S_{\ell, X_{n-j}} := \{ [\ell'] \in F(X) | \ell' \subset X_{n-j}, \ell' \cap \ell \neq \emptyset \text{ for a generic } \ell \subset X_{n-j} \}
\]

as the closed subvarieties parametrizing the lines in the \( j \)-th complete intersection \( X_{n-j} \) meeting a generic line \( \ell \) in \( X_{n-j} \). Hence, we have

\[ q_*p^*[S_{\ell, X_{n-j}}] = 5j_*H_{X_{n-j}} = 5H_X^{j+1}, \quad \text{for } n - j \geq 3. \]

To close the proof, for any \( i > 1 \), we can take \( z \) to be \( 3[C_{x, X_{n+i-1}}] - [S_{\ell, X_{n+i-1}}] \in \text{CH}^{n+i-3}(F(X)) \).
Now let us prove it for arbitrary field. In fact, through the above argument, we can see that if \( x \) is a generic point in \( X \), the subvariety \( C_x \) has dimension \( n - 3 \) and the cycle \( q_\ast p_\ast [C_x] \) remains equal to \( 2H_X^2 \) since the equations \( L, Q, C \) intersect properly. Similarly, if \( \ell \) is a general line in \( X \), the intersection number \( q_\ast p_\ast [S_\ell] \cdot \ell \) is invariant under base change. Hence we still have \( q_\ast p_\ast [S_\ell] = 5H_X \).

By the assumption, let \( L \) be the \( k \)-line on \( X \). Choose any \( k \)-rational point \( p \in L \). It follows from Lemma 2.7 that \( p \) is rationally equivalent to a generic zero-cycle of degree one. We define the algebraic cycle \( [C_p] := [C_\ast] = \alpha \cdot q_\ast z \in CH_{n-3}(X) \). Since the support of \( z \) consists of generic points in \( X \) and degree \( z = 1 \) we obtain \( q_\ast p_\ast [C_p] = 2H_X^2 \).

We regard \( L \) as a closed point of \( F(X) \). Use Lemma 2.7 again, \( L \) is rationally equivalent to \( \sum n_\ast \ell_\ast \), a sum of general lines. Let us denote the cycle \([S_L] \) to be \( \sum n_\ast \ell_\ast \). Then \( p_\ast q_\ast [S_L] = 5H_X \).

**Corollary 2.9.** Let \( X \subset \mathbb{P}_{k}^{n+1} \) be a smooth cubic hypersurface of dimension \( n \geq 3 \) over a field \( k \), and let \( F(X) \) be the Fano variety of lines. Assume that \( X \) has a line \( L \) defined over \( k \). Then Chow group \( CH_1(X) \) is generated by the image of Chow group \( CH_{r-1}(F(X)) \) via the cylinder homomorphism for all integer \( r < n - 1 \).

**Proof.** It is a direct consequence of Corollary 2.6 and Lemma 2.8. \( \square \)

**Remark 2.10.** For cubic 4-folds, the divisor class \( h_X \) is also covered by the cylinder homomorphism. It is because we have \( q_\ast p_\ast g^2 = 21h_X \), see [16, Lemma A.4], where \( g \) is the first Chern class of the canonical polarization on \( F(X) \).

### 3. Integral Hodge conjectures and Tate conjectures

**Corollary 3.1.** The integral Hodge conjecture holds for 1-cycles on the Fano variety of lines of a smooth complex cubic fourfold \( X/\mathbb{C} \).

**Proof.** Let \( \alpha \in H^0 d^6(F(X), \mathbb{Z}) \) be an integral Hodge class of degree 6. The cylinder homomorphism on the cohomology is a morphism of Hodge structure, thus we have \( q_\ast p_\ast \alpha \in H^4 d^1(X, \mathbb{Z}) \). The integral Hodge conjecture for 2-cycles on a smooth cubic 4-fold is affirmed by Voisin, see [19]. Hence there exists a 2-cycle \( \gamma \in CH^2(X) \) such that the cohomology class \([\gamma]\) is equal to \( q_\ast p_\ast \alpha \). By Corollary 2.9 there exists an algebraic cycle \( \Gamma \in CH_1(F(X)) \) such that \( q_\ast p_\ast \Gamma = \gamma \). Then we have

\[
[\Gamma] - \alpha \in \text{Ker}(q_\ast p_\ast).
\]

By the Poincaré duality, the cylinder homomorphism

\[
q_\ast p_\ast : H^6(F, \mathbb{Z}) \to H^4(X, \mathbb{Z})
\]

is dual to the Abel-Jacobi map

\[
p_\ast q_\ast : H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z}).
\]

In [2], Beauville and Donagi proved that the Abel-Jacobi map is an isomorphism. Therefore, the map \( q_\ast p_\ast \) is an isomorphism of Hodge structure and the integral Hodge class \( \alpha \) is algebraic. \( \square \)

Let \( X/\mathbb{C} \) be a complex smooth cubic hypersurfaces of dimension \( n \), the associated Fano variety of lines \( F(X) \) is a smooth projective variety of dimension \( 2n - 4 \). Let \( n \) be even. If the kernel of the cylinder homomorphism on cohomology

\[
\Psi : H^{3n-6} (F(X), \mathbb{Z}) \to H^n(X, \mathbb{Z})
\]

is algebraic, then the surjectivity of the cylinder homomorphism between the Chow groups will show that the integral Hodge conjecture of \( F(X) \) follows from the integral Hodge conjecture of \( X \). In [17, §1], Ichiro Shimada proved that the kernel of
\(\psi \otimes \mathbb{Q}\) is generated by algebraic classes. In fact, let \(G\) be the Grassmannian of lines on \(\mathbb{P}^{n+1}\). \(\text{Ker } \psi \otimes \mathbb{Q}\) is contained in the image of the natural restriction map
\[
\text{im} : (H^{3n-6}(G, \mathbb{Q})) \to H^{3n-6}(F(X), \mathbb{Q})).
\]

It is well-known that the rational Hodge conjecture holds for 4-cycles of a smooth cubic 8-fold \(X\), see [19]. Hence, the rational Hodge conjecture for dimension 3-cycles on the Fano variety \(F(X)\) is true.

**Tate Conjecture.** Let \(F := F(X)\) be the Fano variety of lines on a smooth cubic fourfold \(X/k\). If \(k\) is a number field or a finite field, the Tate conjecture for \(1\)-cycles on \(F\) holds by Deligne's hard Lefschetz theorem and the Tate conjecture for divisors on \(F\), see [1] and [19]. We want to generalize this conclusion to the integral analog of Tate conjecture, see (6). Inspired by the proof of Corollary (3.1), we first show that the associated Abel-Jacobi mapping remains an isomorphism on \(\acute{\text{e}}\text{tale}\) cohomology.

**Lemma 3.2.** Let \(X/k\) be a smooth cubic fourfold in \(\mathbb{P}^5\) over either a finite field or a number field, and let \(F := F(X)\) be the Fano variety of lines on \(X\). Then the cylinder homomorphism on cohomology
\[
q_*p^*: H^6_{\acute{\text{e}}\text{t}}(F_k, \mathbb{Z}_\ell(1)) \to H^4_{\acute{\text{e}}\text{t}}(X_k, \mathbb{Z}_\ell(2))
\]
is an isomorphism, where \(U\) is any open subgroup of the absolute Galois group \(G := \text{Gal}(\bar{k}/k)\).

**Proof.** By the Poincaré duality, it suffices to prove that the Abel-Jacobi map
\[
q_*p^*: H^6_{\acute{\text{e}}\text{t}}(X_k, \mathbb{Z}_\ell(2)) \to H^4_{\acute{\text{e}}\text{t}}(F_k, \mathbb{Z}_\ell(1))
\]
is a \(U\)-equivariant isomorphism. Suppose \(k\) is a finite field. The proof is carried out by a smooth proper lifting. We denote by \(W(k)\) the ring of the Witt vectors of \(k\), and let \(\mathfrak{X}\) be a smooth proper lifting of \(X\) over \(W(k)\). One could embed \(W(k)\) into the complex numbers \(\mathbb{C}\). Therefore, we have the following fiber products diagrams
\[
\begin{array}{ccc}
X_{\mathbb{C}} & \longrightarrow & \mathfrak{X} \\
\downarrow & & \downarrow \\
\text{Spec } \mathbb{C} & \longrightarrow & \text{Spec } W(k) \leftarrow \text{Spec } k.
\end{array}
\]

By the proper and smooth base change theorem and comparison theorem, for any prime number \(\ell\) invertible in \(k\), there is the canonical isomorphisms
\[
(16) \quad H^2_{\acute{\text{e}}\text{t}}(X_k, \mathbb{Z}_\ell(i)) \simeq H^2_{\acute{\text{e}}\text{t}}(X_{\mathbb{C}}, \mathbb{Z}_\ell(i)) \simeq H^2_{\text{sing}}(X(\mathbb{C})^{an}, \mathbb{Z}(i)) \otimes \mathbb{Z}_\ell.
\]

This isomorphisms commute with Abel-Jacobi maps since both are induced by the universal family of lines. Therefore,
\[
p_*q^*: H^4_{\acute{\text{e}}\text{t}}(X_k, \mathbb{Z}_\ell(2)) \to H^2_{\acute{\text{e}}\text{t}}(F_k, \mathbb{Z}_\ell(1)),
\]
is a \(U\)-equivariant isomorphism.

When \(k\) is a number field, the fields \(\bar{\mathbb{Q}} \subset \mathbb{C}\) are separably closed. It follows particularly from [10] Cor. 4.3 that
\[
H^2_{\acute{\text{e}}\text{t}}(X_k, \mathbb{Z}_\ell(i)) = H^2_{\acute{\text{e}}\text{t}}(X_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell(i)) \simeq H^2_{\acute{\text{e}}\text{t}}(X_{\mathbb{C}}, \mathbb{Z}_\ell(i)).
\]
Then by comparison theorem and the same arguments as above we obtain the same conclusion. \(\square\)

The following integral analog of the Tate conjecture for codimension 2-cycles of cubic fourfolds is proved by F. Charles and A. Pirutka.
Theorem 3.3. [4 Theorem 1.1] Let $k$ be a field finitely generated over its prime subfield with the characteristic different from 2 and 3, and let $\overline{k}$ be the algebraic closure of $k$. If $X \subset \mathbb{P}^5_k$ is a smooth cubic fourfold, then for any prime number $\ell \neq \text{char} k$, the cycle class map

$$
\text{CH}^2(X_\overline{k}) \otimes \mathbb{Z}_\ell \to \lim_{U} H^4_{\text{et}}(X_\overline{k}, \mathbb{Z}_\ell(2))^U
$$

is surjective, where the direct limit is over the open subgroups $U$ of the absolute Galois group $G := \text{Gal}(\overline{k}/k)$.

Corollary 3.4. Let $X$ be a smooth cubic fourfold in $\mathbb{P}^5_k$ over a field $k$ finitely generated over its prime subfield with the characteristic different from 2 and 3. Let $F := F(X)$ be the Fano variety of lines on $X$. Then the integral analog of the Tate conjecture for 1-cycles of $F$ is true. In the other words, for all prime number $\ell \neq \text{char} k$, the cycle class map

$$
\text{cl} : \text{CH}^3(F_\overline{k}) \otimes \mathbb{Z}_\ell \to \lim_{U} H^6_{\text{et}}(F_\overline{k}, \mathbb{Z}_\ell(3))^U
$$

is surjective, where $U$ runs over the open subgroups of the absolute Galois group $G := \text{Gal}(\overline{k}/k)$.

Proof. As pointed out in [4 Page 3], it suffices to prove the result for the finite extension of the prime subfield of $k$. The general case is deduced. Let $[\alpha] \in \lim_{U} H^0_{\text{et}}(F_\overline{k}, \mathbb{Z}_\ell(3))^U$ be any cohomology class. Suppose that $\alpha \in H^0_{\text{et}}(F_\overline{k}, \mathbb{Z}_\ell(3))^U$ for some open subgroup $U$ is a representative of $[\alpha]$. It follows from Theorem 3.3 that the class $q_* p^* \alpha \in H^0_{\text{et}}(X_\overline{k}, \mathbb{Z}_\ell(2))^U$ lifts to an algebraic cycle $\Gamma \in \text{CH}^1(X_\overline{k}) \otimes \mathbb{Z}_\ell$. By the surjectivity of the cylinder homomorphism, it gives an algebraic 1-cycle $\gamma \in \text{CH}^1(F(X_\overline{k})) \otimes \mathbb{Z}_\ell$ such that $q_* p^* \gamma = \Gamma$. By Lemma 3.2 the map $q_* p^* : H^0_{et}(X_\overline{k}, \mathbb{Z}_\ell(1))^U \to H^0_{et}(X_\overline{k}, \mathbb{Z}_\ell(2))^U$ is an isomorphism for all open subgroups $U$. Thus we have $\text{cl}(\gamma) = \alpha$.

The integral analog of the Tate conjecture for 1-cycles is predicted to be true for any smooth projective $k$-variety. As an evidence, if $k$ is a finite field, C. Schoen proved that this conjecture is true if the Tate conjecture holds for surfaces over finite extensions of $k$, see [13 Theorem 0.5]

References

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