UNIVERSAL GENERATION OF THE CYLINDER HOMOMORPHISM OF CUBIC HYPERSURFACES

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ABSTRACT. In this article, we prove that the Chow group of algebraic cycles of a smooth cubic hypersurface X over an arbitrary field k is generated, via the natural cylinder homomorphism, by the algebraic cycles of its Fano variety of lines $F(X)$, under an assumption on the 1-cycles of X/k. As an application, if $X/\mathbb{C}$ is a smooth complex cubic fourfold, our result provides a proof of the integral Hodge conjecture for curve classes on the polarized hyper-Kähler variety $F(X)$. In addition, when k is finitely generated over its prime subfield with $\mathrm{char}(k) \neq 2, 3$, we use our conclusion to prove the integral analog of the Tate conjecture for 1-cycles on $F(X)$ for a smooth cubic fourfold X/k.

Introduction

Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface of dimension $n \geq 3$ over a field k, and let $F := F(X)$ be the Fano variety of lines on X. The universal family of lines

$$P := \{([\ell], x) \in F(X) \times X \mid x \in \ell\}$$

is a $\mathbb{P}^1$-bundle over F. Denote by $p : P \to F$ and $q : P \to X$ the natural projections. As a correspondence of $X \times F$, P induces a group homomorphism on the Chow groups

$$P_* = q_* p^* : \mathrm{CH}_{r-1}(F) \longrightarrow \mathrm{CH}_r(X) \tag{0.0.1}$$

for any integer $r \geq 1$, which is called the cylinder homomorphism.

It is a classical result by Paranjape that any 1-cycle on a smooth cubic hypersurface Y over an algebraically closed field k with $\dim Y \geq 5$ is generated by lines. Equivalently, in these cases, the cylinder homomorphism (0.0.1) is surjective when $r = 1$. In [11] M. Shen proved, by more involved analysis on the geometry, that the Chow group $\mathrm{CH}_1(X)$ of a smooth cubic X with $\dim X \geq 3$ defined over an algerbaically closed field is generated by lines. His conclusion extends to the cases $\dim X = 3, 4$, where in general X does not contain a plane such that Paranjape's approach does not apply.

The same result is also obtained by Z. Tian and R. Zong in [15] using the deformation of rational curves, where they proved that the Chow groups of 1-cycles on rationally connected Fano complete intersections of index at least 2 are generated by lines.

The main ingredient of M. Shen's proof is to establish the following two crucial relations of 1-cycles on X:

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Proposition 0.1. [11] *Suppose that $X \subset \mathbb{P}_k^{n+1}$ is a smooth cubic hypersurface over an algebraically closed field k with $n \geq 3$, and let $\gamma_1, \gamma_2 \in \mathrm{CH}_1(X)$ be 1-cycles of degree e_1 and e_2 respectively. Denote by h the divisor class of the hyperplane on X.*

(i) There exists a 0-cycle $\gamma \in \mathrm{CH}_0(F)$ such that

$$(2e_1 - 3)\gamma_1 + q_* p^* \gamma = a h^{n-1}$$

in $\mathrm{CH}_1(X)$ for some integer a. If γ_1 is an irreducible curve C in general position, then γ is represented by the secant lines of C .

(ii) We have

$$2e_2\gamma_1 + 2e_2\gamma_2 + q_* p^* \gamma' = 3e_1 e_2 h^{n-1}$$

in $\mathrm{CH}_1(X)$, where $\gamma' = p_* q^* \gamma_1 \cdot p_* q^* \gamma_2 \in \mathrm{CH}_0(F)$.

It is not known whether the cylinder homomorphism is surjective for higher dimensional cycles and for smooth cubic hypersurfaces over a non-closed field. Nevertheless, according to the study in [10], based on a geometric interpretation by Voisin in [16], the proof of the above proposition could be adapted such that the cycle relations remain for curve classes on X defined over an arbitrary field.

The main aim of this article is to generalize the above two formulas and to study the surjectivity of the cylinder homomorphisms for higher dimensional cycles. Precisely, we prove that

Theorem 0.2. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over a field k, and let $F := F(X)$ be its Fano variety of lines. Suppose that $\Gamma \in \mathrm{CH}_r(X)$ is an algebraic cycle of dimension r with degree $e := \Gamma \cdot h^r$, where h is the divisor class of hyperplane on X. Then there exists an algebraic cycle $\gamma_\Gamma \in \mathrm{CH}_{r-1}(F)$ such that*

$$(2e - 3)\Gamma + q_* p^* \gamma_\Gamma = a \cdot h^{n-r} \tag{0.2.1}$$

for some integer a. Moreover, if we assume that X/k admits a degree one 1-cycle defined over k, then for any r-dimensional algebraic cycle Ξ on X with $r > 1$ we have

$$2 \cdot \Xi + q_* p^* \gamma = c \cdot h^{n-r} \tag{0.2.2}$$

for some $c \in \mathbb{Z}$.

The formula (0.2.1) is inspired both by the equation (i) in Proposition 0.1 and the result in R. Mboro's paper [8], where he proved it by assuming $\Gamma \subset X$ is a subvariety in general position or a smooth subvariety. The argument we use in this article could circumvent that constraint.

Theorem 0.2 concludes that the cylinder homomorphism is surjective modulo the hyperplane section classes. Restricted on an algebraically closed field k, the assumption of the existence of a degree one 1-cycle on X/k is fulfilled. Under this situation we prove that the hyperplane section class is also generated by algebraic cycles from $F(X)$. Consequently, the cylinder homomorphism $\mathrm{CH}_{r-1}(F(X)) \to \mathrm{CH}_r(X)$ is surjective except for the group of divisor classes, see Corollary 2.5.

In Section 3, we turn to apply our results to the integral Hodge conjecture and the integral Tate conjecture.

Let X be a smooth cubic fourfold over the complex numbers $\mathbb{C}$. Beauville and Donagi showed that its Fano variety of lines $F(X)$ is a polarized hyper-Kähler variety of $K3^{[2]}$-type. As a result of the surjectivity of the cylinder homomorphism, we prove the integral Hodge conjecture for 1-cycles on $F(X)$ in Theorem 3.1, namely any integral Hodge class in $H^6(F(X),\mathbb{Z})$ is algebraic. Recently, G. Mongardi and J. C. Ottem [9] proved the integral Hodge conjecture for 1-cycles on hyper-Kähler varieties of $K3$ type and generalized Kummer type. Our conclusion provides a different proof as a particular case of them.

It is natural to consider the integral Tate conjecture of the Fano variety of lines $F := F(X)$ of a smooth cubic fourfold X over a field k. In other words, is the cycle class map

$$\mathrm{CH}^3(F_k)\otimes\mathbb{Z}_\ell \to H^6_{\mathrm{et}}(F_{\bar{k}},\mathbb{Z}_\ell(3))^G \tag{0.2.3}$$

surjective? where G denotes the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ and the prime number $\ell \neq \mathrm{char}(k)$. The answer to the Tate conjecture for divisors on F over a number field or a finite field is affirmative due to Y. André [1] and F. Charles [4]. By Deligne's hard Lefschetz theorem, it also holds for 1-cycles with rational coefficients on F. However, the integral Tate conjecture for 1-cycles fails in general, e.g. Kollár's examples of certain smooth 3-fold in $\mathbb{P}^4$, see [2]. Nevertheless, Schoen addressed a similar but weaker conjecture in [5]: let Y be a smooth proper k-variety, and the prime number $\ell \neq \mathrm{char}(k)$, the following cycle class map

$$\mathrm{CH}^r(Y_{\bar{k}})\otimes\mathbb{Z}_\ell \to \varinjlim_U H^{2r}_{\mathrm{et}}(Y_{\bar{k}},\mathbb{Z}_\ell(r))^U \tag{0.2.4}$$

is surjective, where the direct limit is over the open subgroups U of the absolute Galois group G. Our result also applies to the integral analog of the Tate conjecture for 1-cycles on $F(X)$. To be precise, let k be a field finitely generated over its prime subfield with characteritic different than 2 and 3, the cycle class map

$$cl:\mathrm{CH}_1(F(X_{\bar{k}}))\otimes\mathbb{Z}_\ell \to \varinjlim_U H^6_{\mathrm{et}}(F(X_{\bar{k}}),\mathbb{Z}_\ell(3))^U$$

is surjective for all prime number $\ell \neq \mathrm{char}\, k$, see Corollary 3.4. Actually, this statement is predicted to be true in general. As an evidence, Schoen proved that (0.2.4) is surjective for $r = \dim Y - 1$ if the Tate conjecture holds for divisors on surfaces over finite extensions of k.

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1. The Hilbert square of cubic hypersurfaces

The proof of Theorem 0.2(cf. Theorem 2.3) is based on the geometric constructions introduced by S. Galkin and E. Shinder in [7] and developed by C. Voisin. In this section, we first recall the geometric backgrounds and prove several necessary lemmata, then the main theorem follows in the Section 2.

Let X be a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}_k$, let $F := F(X)$ be the Fano variety of lines on X, and let P_X be a projective bundle over X whose fiber over $x \in X$ is the set of lines in $\mathbb{P}^{n+1}$. The Hilbert scheme $X^{[2]}$ of length 2 subschemes on X admits a rational map

$$\Phi : X^{[2]} \dashrightarrow P_X,$$

which to an unordered pair of points $x, y \in X$ that are not contained in a common line in X, one associates the pair $(\ell_{xy}, z) \in \mathbb{P}_X$, where ℓ_{xy} is the line in $\mathbb{P}^{n+1}$ generated by x, y, and $z \in X$ is the residue point of the intersection $\ell_{xy} \cap X$. Let $p : P := \mathbb{P}(\mathscr{E}) \to F$ be the universal family of lines, where $\mathscr{E}$ the tautological rank two vector bundle on F. Note that P is naturally contained in P_X. It induces a $\mathbb{P}^1$-bundle $p_2 : \mathbb{P}(\mathrm{Sym}^2 \mathscr{E}) := P_2 \to F(X)$ whose fiber is the length 2 subschemes supported on a line in X. It is easily to see that the rational map Φ is not well-defined on P_2 as a closed subscheme of $X^{[2]}$. In [16, Proposition 2.9], Voisin proved that Φ can be resolved by blowing up $X^{[2]}$ along the center P_2, and the induced morphism $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ from the blow-up $\widetilde{X^{[2]}}$ coincides with the blow-up of P_X along the closed subscheme P. Moreover, the exceptional divisors of the two blow-ups are the same, hence let us denote it by $\mathcal{E}$.

We use the following diagram to summarize the above geometric structures

(1.0.1)
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, hook, "\pi_1"] \arrow[dr, hook, "j"] \arrow[dd, "\pi_2"'] & & P \arrow[d, hook, "i_1"] \\
& \widetilde{X^{[2]}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\tau"] & P_X \\
P_2 \arrow[r, hook, "i_2"] & X^{[2]}. \arrow[ur, dashed, "\Phi"'] &
\end{tikzcd}$$

Moreover, the diagram

(1.0.2)
$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & P \arrow[d, "p"] \\
P_2 \arrow[r, "\pi_F"] & F
\end{tikzcd}$$

is a fiber square.

We have the following simple lemma on algebraic cycles:

Lemma 1.1. *Use the notations as above. For any algebraic cycle $\Xi \in \mathrm{CH}_k(X^{[2]})$ of dimension k, we have a $(k-2)$-dimensional algebraic cycle $\gamma \in \mathrm{CH}_{k-2}(F)$ such that*
$$\pi_* \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Xi) = q_* p^* \gamma$$
in $\mathrm{CH}_{k-1}(X)$. Indeed, $\gamma = \pi_{F*} {i_2}^* \Xi$.

Proof. Since $\mathcal{E} \cdot \tau^* \Xi = j_* j^* \tau^* \Xi$, the formula immediately follows from the commutativity of the diagram (1.0.1) and the commutativity of the flat-pullbacks and proper pushforwards on cycles of the diagram (1.0.2). □

Consider the natural proper morphism
$$\varphi : X^{[2]} \to G(2, n+2)$$
from $X^{[2]}$ to the Grassmannian $G(2, n+2)$ of lines in the projective space $\mathbb{P}^{n+1}$ by sending two points $x, y \in X$ to the generating line $[\ell_{xy}]$. For the tautological rank 2 vector bundle $\mathscr{E}$ over $G(2, n+2)$, the incidence variety
$$\mathbb{P}(\mathscr{E}) \cong \{(\ell, x) \in G(2, n+2) \times \mathbb{P}^{n+1} | x \in \ell\}.$$

is a $\mathbb{P}^1$-bundle over $G(2, n+2)$. Let $\pi_{X^{[2]}} : Q \to X^{[2]}$ be the pull-back of $\mathbb{P}(\mathscr{E})$ via φ, and let $\alpha : Q \to \mathbb{P}^{n+1}$ be the composition. Indeed, the blowing up $\tau : \widetilde{X^{[2]}} \to X^{[2]}$ can be interpreted as one component of the divisor $\alpha^{-1}(X)$ in Q which has degree one over $X^{[2]}$. Since X is a cubic hypersurface, another component of $\alpha^{-1}(X)$ having degree two over $X^{[2]}$, is the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal. The above descriptions are summarized by the following cartesian diagrams

$$\begin{tikzcd}
\widetilde{X \times X} \coprod \widetilde{X^{[2]}} \arrow[r] \arrow[d] & P_X \arrow[r, "\pi_X"] \arrow[d, hook, "i'"] & X \arrow[d, hook, "i_X"] \\
\mathcal{Q} \arrow[r, "\varphi'"] \arrow[d, "\pi_{X^{[2]}}"] & \mathbb{P}(\mathscr{E}) \arrow[r, "\pi_P"] \arrow[d, "\pi_G"] & \mathbb{P}^{n+1} \\
X^{[2]} \arrow[r, "\varphi"] & G(2, n+2). &
\end{tikzcd} \tag{1.1.1}$$

For the conventions, we also give the following diagram

$$\begin{tikzcd}
X^{[2]} & \widetilde{X \times X} \arrow[l, "\sigma"'] \arrow[r, "\Psi"] \arrow[d, "\rho"] & P_X \arrow[d, "\pi_X"] \\
& X \times X \arrow[r, "p_1"] & X,
\end{tikzcd} \tag{1.1.2}$$

where p_1 is the projection to the first factor.

Lemma 1.2. *With the notations as in (1.1.1) and (1.1.2), given any algebraic cycle $W \in \mathrm{CH}^i(X^{[2]})$, we have*

$$\pi_* \widetilde{\Phi}_* \tau^* W = i_X^* \pi_{P*} \pi_G{}^* \varphi_* W - \pi_* \Psi_* \sigma^* W.$$

Proof. As a correspondence of $X^{[2]} \times P_X$, the action induced by the algebraic cycle $[\widetilde{X \times X}] + [\widetilde{X^{[2]}}]$ on the Chow group of $X^{[2]}$, as seen from the commutative diagram (1.1.1), is the push-forward of φ then followed by the pull-backs of $\pi_G \circ i'$. Therefore, the action of the component $\widetilde{X^{[2]}}$ on the Chow group of $X^{[2]}$ is

$$[\widetilde{X^{[2]}}]_* = i'^* \circ \pi_G^* \circ \varphi_* - [\widetilde{X \times X}]_*.$$

It follows from the diagram (1.1.2) that

$$[\widetilde{X \times X}]_* = \Psi_* \circ \sigma^*.$$

After pushing-forward to X, we get the equality. $\square$

Let $X \subset \mathbb{P}^N$ be a smooth projective variety. There is a natural rational map from $X \times X$ to the Hilbert scheme of two points $X^{[2]}$ with the indeterminacy along the diagonal Δ_X. The Blow-up of $X \times X$ along the diagonal Δ_X resolves the rational map. We thus have a diagram as follows:

$$\begin{tikzcd}
E_\Delta \arrow[r, hook, "j_E"] \arrow[d, "\pi_\Delta"] & \widetilde{X \times X} \arrow[r, "\sigma"] \arrow[d, "\rho"] & X^{[2]} \\
\Delta_X \arrow[r, hook, "\iota_\Delta"] & X \times X \arrow[ur, dashed, "\mu"] &
\end{tikzcd} \tag{1.2.1}$$

Lemma 1.3. *Assume that Γ is a symmetric codimension r algebraic cycle in $X\times X$ that is not contained in the diagonal Δ_X, i.e. no irreducible component of Γ is entirely supported in Δ_X. Then there exists a codimension r algebraic cycle Σ in $X^{[2]}$ such that $\mu^*\Sigma := \rho_*\sigma^*\Sigma = \Gamma$.*

Proof. Voisin presented the proof for $r = n$ in [16, Corollary 2.4]. Our argument here is the same. Assume $\Gamma = Z_1 + Z_2$ where $Z_1 = \sum n_i Z_{1,i}$ is a linear combination of irreducible subvarieties $Z_{1,i}$ of $X \times X$ and invariant under the involution i, and Z_2 is of the form $Z_2' + i(Z_2')$. Since $Z_{1,i}$ does not lie in Δ_X, we take $\widetilde{Z_{1,i}}$ be the proper transform of $Z_{1,i}$ in $\widetilde{X\times X}$. Define the algebraic cycle Σ to be $\sum_i n_i[\sigma(\widetilde{Z_{1,i}})] + \sigma_*\rho^* Z_2'$. Since σ is a flat morphism, we have

$$\begin{aligned}\mu^*\Sigma &= \sum_i n_i\rho_*\sigma^*\sigma[(\widetilde{Z_{1,i}})] + \rho_*\sigma^*\sigma_*\rho_* Z_2' \\ &= \sum_i n_i\rho_*[\widetilde{Z_{1,i}}] + \rho_*\rho^* Z_2 + \rho_*\rho^* i(Z_2) \\ &= \sum n_i Z_{1,i} + Z_2' + i(Z_2'),\end{aligned}$$

Which verifies that Σ is the desired algebraic cycle.

∎

With the notations as in (1.1.1) and (1.1.2), given two algebraic cycles $\alpha, \beta \in \mathrm{CH}^*(X)$, we write

$$\alpha\hat{\otimes}\beta := \sigma_*\rho^*(\alpha\otimes\beta) \in \mathrm{CH}^*(X^{[2]}). \tag{1.3.1}$$

Lemma 1.4. *Recall that $\mathcal{E}$ is the exceptional divisor of the blowing up $\tau : \widetilde{X^{[2]}} \to X^{[2]}$. Let h_X be the restriction of the hyperplane class h of $\mathbb{P}^{n+1}$ to the cubic hypersurface X, let $h_{\mathcal{Q}} \in \mathrm{Pic}(\mathcal{Q})$ the pullback of h to $\mathcal{Q}$ via the morphism $\alpha : \mathcal{Q} \to \mathbb{P}^{n+1}$, see (1.1.1), and let $\delta \in \mathrm{Pic}(X^{[2]})$ the half diagonal divisor of $X^{[2]}$ such that $\sigma^*\delta = j_{E*}(E_\Delta)$ in $\mathrm{Pic}(\widetilde{X\times X})$. Then We have*

$$\mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}} - \tau^*(2h_X\hat{\otimes}1 - 3\delta),$$

and

$$g := c_1(\varphi^*\mathscr{E}) = -h_X\hat{\otimes}1 + \delta$$

in $\mathrm{Pic}(\widetilde{X^{[2]}})$.

Proof. See [10, Lemma 4.3] ∎

2. Main results

Proposition 2.1. *Let $\Gamma \subset X$ be an closed subvariety of dimension $1 < r$ with degree $e := \Gamma\cdot h_X^{n-r}$, where h_X is the restriction of the hyperplane class of $\mathbb{P}^{n+1}$ to X. Assume that there exists a degree one 1-cycle $\mathfrak{l}$ in $\mathrm{CH}_1(X)$. With the notations as in the diagrams (1.1.2) and (1.2.1),*

(1) there exists an algebraic cycle Γ' of dimension $r+1$ on $X^{[2]}$ such that

$$\pi_*\Psi_*\sigma^*\Gamma' = 0;$$
$$\pi_*\Psi_*\sigma^*(h_X\hat{\otimes}1\cdot\Gamma') = \Gamma;$$
$$\pi_*\Psi_*\sigma^*(\delta\cdot\Gamma') = 0,$$

(2) and there exits an algebraic cycle $\Gamma^{''}$ of dimension $2r$ on $X^{[2]}$ such that

$$\pi_*\Psi_*\sigma^*(\Gamma^{''}\cdot(h_X\hat{\otimes}1)^k\cdot\delta^l)=\begin{cases}0, & \forall\ 0\leq k,l\leq r-1\\ e\cdot\Gamma, & k=r,l=0\\ (-1)^{r+1}\Gamma, & k=0,l=r\end{cases}$$

Proof. For the assertions in (1), we have a symmetric algebraic cycle $\Gamma\times\mathfrak{l}+\mathfrak{l}\times\Gamma$ on $X\times X$ of codimension $n+r-1$. It follows from the Lemma 1.3 that there exists a cycle Γ' on $X^{[2]}$ such that $\rho_*\sigma^*\Gamma'=(\Gamma\times\mathfrak{l}+\mathfrak{l}\times\Gamma)$. Then we have

$$\pi_*\Psi_*\sigma^*\Gamma'=p_{1*}\rho_*\sigma^*\Gamma'=p_{1*}(\Gamma\times\mathfrak{l}+\mathfrak{l}\times\Gamma)=0.$$

It also induces the second formula

$$\begin{aligned}\pi_*\Psi_*\sigma^*(h_X\hat{\otimes}1\cdot\Gamma')&=p_{1*}\rho_*(\sigma^*(h_X\hat{\otimes}1)\cdot\sigma^*\Gamma')=p_{1*}\rho_*(\rho^*(h_X\times1+1\times h_X)\cdot\sigma^*\Gamma')\\&=p_{1*}((h_X\times1+1\times h_X)\cdot\rho_*\sigma^*\Gamma')\\&=p_{1*}((h_X\times1+1\times h_X)\cdot(\Gamma\times\mathfrak{l}+\mathfrak{l}\times\Gamma))\\&=\deg(h_X\cdot\mathfrak{l})\cdot\Gamma=\Gamma,\ (\dim h_X\cdot\Gamma>0\text{ since }\dim\Gamma>1)\end{aligned}$$

and similarly for the third one

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\delta\cdot\Gamma')&=\pi_*\Psi_*(E_\Delta\cdot\sigma^*\Gamma')=p_{1*}\rho_*(j_{E*}{j_E}^*\sigma^*\Gamma')\\&=\pi_{\Delta*}({j_E}^*\sigma^*\Gamma')=\pi_{\Delta*}({j_E}^*(\rho^*(\Gamma\times\mathfrak{l}+\mathfrak{l}\times\Gamma))\\&=2\pi_{\Delta*}{\pi_\Delta}^*(\Gamma\cdot\mathfrak{l})=0.\end{aligned}$$

We thus proved the assertions in (1). By the linear extension, the formulas hold for algebraic cycles.

In the second assertion, we consider the closed subscheme $\Gamma\times\Gamma$ on $X\times X$. Recall Lemma 1.3 that it exists a dimension $2r$ algebraic cycle $\Gamma^{''}\in\mathrm{CH}_{2r}(X^{[2]})$ such that $\rho_*\sigma^*\Gamma^{''}=[\Gamma\times\Gamma]$. Moreover, according to the explicit construction of $\Gamma^{''}$, the pullback $\sigma^*\Gamma^{''}$ is represented by the strict transformation $[\widetilde{\Gamma\times\Gamma}]$.

By the commutative diagram (1.2.1), we have the following

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma^{''}\cdot(h_X\hat{\otimes}1)^k\cdot\delta^l)&=p_{1*}(\sum_{s=0}^k\binom{k}{s}h_X^s\times h_X^{k-s}\cdot\rho_*(\sigma^*\Gamma^{''}\cdot E_\Delta^l))\\&=p_{1*}(\sum_{s=0}^k\binom{k}{s}h_X^s\times h_X^{k-s}\cdot\rho_*([\widetilde{\Gamma\times\Gamma}]\cdot E_\Delta^l)).\end{aligned}$$

If $l=0$ and $k<r$, we have

$$\begin{aligned}p_{1*}(\sum_{s=0}^k\binom{k}{s}h_X^s\times h_X^{k-s}\cdot\rho_*\sigma^*\Gamma^{''})&=\sum_{s=0}^k\binom{k}{s}p_{1*}(h_X^s\cdot\Gamma)(h_X^{k-s}\cdot\Gamma)\\&=0.\end{aligned}$$

When $k=r$ and $l=0$,

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma^{''}\cdot(h_X\hat{\otimes}1)^r) &= p_{1*}\rho_*(\sigma^*\Gamma^{''}\cdot\rho^*(\sum_{s=0}^{r}\binom{r}{s}h_X^s\times h_X^{r-s})) \\ &= \sum_{s=0}^{r}\binom{r}{s}p_{1*}(h_X^s\cdot\Gamma)(h_X^{r-s}\cdot\Gamma) \\ &= \deg\Gamma\cdot\Gamma.\end{aligned}$$

Now assume $1\le l\le r-1$. Recall the excess intersection formula

$$[\widetilde{\Gamma\times\Gamma}]=\rho^*([\Gamma\times\Gamma])-j_{E*}\{c(Q)\cap\pi_\Delta^*s(\Gamma,\Gamma\times\Gamma)\}_{2r},$$

where Q is tautological quotient bundle $\pi_\Delta^*N_{\Delta_X/X\times X}/\mathcal{O}(-1)$ on the exceptional divisor E_Δ, and $s(\Gamma,\Gamma\times\Gamma)$ is the Segre class of the closed subvariety Γ in $\Gamma\times\Gamma$. We simply write the i-th Segre class $s_i(\Gamma,\Gamma\times\Gamma)$ as s_i.

$$\begin{aligned}\rho_*([\widetilde{\Gamma\times\Gamma}]\cdot E_\Delta^l) &= [\Gamma\times\Gamma]\cdot\rho_*E_\Delta^l-\iota_{\Delta*}\pi_{\Delta*}(\{c(Q)\cap\pi_\Delta^*s(\Gamma,\Gamma\times\Gamma)\}_{2r}\cdot j_E^*E_\Delta^l) \\ &= 0-\sum_{i+t=n-r-1}\iota_{\Delta*}(s_i\cdot\pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}(-1))^l)).\end{aligned}$$

However, the $\pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}(-1))^l))$ is equal to zero because $t+l\le n-2$.

At the end, for $k=0, l=r$, if $r<n$, we have

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma^{''}\cdot\delta^r) &= p_{1*}\rho_*(\sigma^*\Gamma^{''}\cdot E_\Delta^r) \\ &= p_{1*}([\Gamma\times\Gamma]\cdot\rho_*E_\Delta^r-\iota_{\Delta*}\pi_{\Delta*}(\{c(Q)\cap\pi_\Delta^*s(\Gamma,\Gamma\times\Gamma)\}_{2r}\cdot j_E^*E_\Delta^r)) \\ &= (-1)^{r+1}\sum_{i+t=n-r-1}\iota_{\Delta*}(s_i\cdot\pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}(1))^r)) \\ &= (-1)^{r+1}\iota_{\Delta*}(s_0\cdot c_0(X)\cdot\pi_{\Delta*}c_1(\mathcal{O}(1))^{n-1})) \\ &= (-1)^{r+1}[\Gamma].\end{aligned}$$

If $r=n$, in other words, $\Gamma=[X]$, it turns out that

$$p_{1*}\rho_*(\sigma^*\Gamma^{''}\cdot E_\Delta^r)=p_{1*}([X\times X]\cdot\rho_*E_\Delta^n)=(-1)^{n-1}[X]=(-1)^{n+1}[X].$$

□

Remark 2.2. *(a) The proof of the assertion* (1) *involves the existence of the degree one* 1*-cycle, which is not guaranteed in general. For algebraically closed fields, we could take the cycle* $\mathfrak{l}$ *to be a line in* X*. Over non-closed fields, the case that a cubic hypersufrace containing no lines occurs, e.g. number fields or finite field with small orders, see* [6].

(b) Unlike the assertion (1)*, the formulas of the assertion* (2) *can not be extend to algebraic cycles since they are not linear.*

The above proposition implies the following main theorem.

Theorem 2.3. *Let* $\Gamma\in\mathrm{CH}_r(X)$ *be an algebraic cycle of dimension* $r>1$ *with degree* e*. Suppose that there exists a degree one* 1*-cycle* $\mathfrak{l}$ *in* $\mathrm{CH}_1(X)$*. Then we have*

(i)

$$2\Gamma+q_*p^*\gamma\equiv 0 \bmod \mathbb{Z}\cdot h_X^{n-r},$$

in $\mathrm{CH}_r(X)$*, where* $\gamma=p_*q^*\mathfrak{l}\cdot p_*q^*\Gamma\in\mathrm{CH}_{r-1}(F)$*;*

(ii) There exists a cycle $\gamma_\Gamma \in \mathrm{CH}_{r-1}(F)$ such that

$$(2e-3)\Gamma + q_*p^*\gamma_\Gamma \equiv 0 \bmod \mathbb{Z}\cdot h_X^{n-r},$$

in $\mathrm{CH}_r(X)$.

Proof. We first prove the above two cycle relations by assuming Γ is an irreducible closed subvariety, then present that the results of algebraic cycles are deduced from them. The notation Γ' is the algebraic cycle as same as in (1) of Proposition 2.1. Recall the Lemma 1.1 that

$$\pi_*\widetilde{\Phi}_*(\mathcal{E}\cdot\tau^*\Gamma') = q_*p^*\gamma,$$

where $\gamma = \pi_{F*}{i_2}^*\Gamma' \in \mathrm{CH}_{r-1}(F)$. Then we use the relation

$$\mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}} - \tau^*(2h_X\hat{\otimes}1 - 3\delta)$$

in the Lemma 1.4 to obtain:

$$\begin{aligned}\pi_*\widetilde{\Phi}_*(\mathcal{E}\cdot\tau^*\Gamma') &= -\pi_*\widetilde{\Phi}_*(h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}}\cdot\tau^*\Gamma') - \pi_*\widetilde{\Phi}_*(\tau^*(2h_X\hat{\otimes}1-3\delta)\cdot\tau^*\Gamma')\\ &= -h_X\cdot\pi_*\widetilde{\Phi}_*\tau^*\Gamma' - \pi_*\widetilde{\Phi}_*\tau^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma').\end{aligned}$$

The first term is

$$\begin{aligned}\pi_*\widetilde{\Phi}_*\tau^*\Gamma' &= i_X^*\pi_{P*}{\pi_G}^*\varphi_*\Gamma' - \pi_*\Psi_*\sigma^*\Gamma'\\ &= i_X^*\pi_{P*}{\pi_G}^*\varphi_*\Gamma' \in \mathbb{Z}\cdot h_X^{n-r-1}.\end{aligned}$$

See Lemma 1.2 and Proposition 2.1. Similarly for the second term:

$$\begin{aligned}&\pi_*\widetilde{\Phi}_*\tau^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma')\\ =&i_X^*\pi_{P*}{\pi_G}^*\varphi_*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma') - \pi_*\Psi_*\sigma^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma')\\ \equiv&-\pi_*\Psi_*\sigma^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma') \bmod \mathbb{Z}\cdot h_X^{n-r}\\ \equiv&2\Gamma \bmod \mathbb{Z}\cdot h_X^{n-r}\end{aligned}$$

Therefore, we obtain the required relation

$$2\Gamma + q_*p^*\gamma \equiv \bmod \mathbb{Z}\cdot h_X^{n-r}. \tag{2.3.1}$$

To get the formula (ii), we study the cycle class

$$\pi_*\Phi_*(\mathcal{E}\cdot\tau^*(\Gamma''\cdot g^{r-1})),$$

where Γ'' is the algebraic cycle introduced in (2) of Proposition 2.1 and $g = c_1(\varphi^*\mathscr{E}) = -h_X\hat{\otimes}1+\delta$(cf. Lemma 1.4). Notice that the cycle $\Gamma''\cdot g^{r-1}$ has dimension $r+1$. We have

$$\pi_*\tilde{\Phi}_*(\mathcal{E}\cdot\tau^*(\Gamma''\cdot g^{r-1})) = q_*p_*\gamma_\Gamma,$$

where $\gamma_\Gamma = \pi_{F*}{i_2}^*(\Gamma''\cdot g^{r-1}) \in \mathrm{CH}_{r-1}(F)$. Again use the same argument as above

$$\pi_*\tilde{\Phi}_*(\mathcal{E}\cdot\tau^*(\Gamma''\cdot g^{r-1})) = -h_X\cdot\pi_*\widetilde{\Phi}_*\tau^*(\Gamma''\cdot g^{r-1}) - \pi_*\widetilde{\Phi}_*\tau^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma''\cdot g^{r-1}).$$

Now apply the assertion (2) of the Proposition 2.1 we obtain

$$\begin{aligned}\pi_*\widetilde{\Phi}_*\tau^*(\Gamma''\cdot g^{r-1}) &= i_X^*\pi_{P*}{\pi_G}^*\varphi_*(\Gamma''\cdot g^{r-1}) - \pi_*\Psi_*\sigma^*(\Gamma''\cdot g^{r-1})\\ &\equiv -\pi_*\Psi_*\sigma^*(\Gamma''\cdot\sum_{k=0}^{r-1}\binom{r-1}{k}(-h_X\hat{\otimes}1)^k\cdot\delta^{r-k-1}) \bmod \mathbb{Z}\cdot h_X^{n-r-1}\\ &\equiv 0 \bmod \mathbb{Z}\cdot h_X^{n-r-1},\end{aligned}$$

and

$$
\begin{aligned}
&\pi_*\widetilde{\Phi}_*\tau^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma''\cdot g^{r-1})\\
&=i_X^*\pi_{P*}\pi_G{}^*\varphi_*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma''\cdot g^{r-1})-\pi_*\Psi_*\sigma^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma''\cdot g^{r-1})\\
&\equiv -\pi_*\Psi_*\sigma^*((2h_X\hat{\otimes}1-3\delta)\cdot\Gamma''\cdot\sum_{k=0}^{r-1}\binom{r-1}{k}(-h_X\hat{\otimes}1)^k\cdot\delta^{r-k-1}) \bmod \mathbb{Z}\cdot h_X^{n-r}\\
&\equiv (-1)^r 2\deg\Gamma\cdot\Gamma+3(-1)^{r+1}\Gamma \bmod \mathbb{Z}\cdot h_X^{n-r}.
\end{aligned}
$$

We thus conclude that

$$2\deg\Gamma\cdot\Gamma-3\Gamma\equiv q_*p^*\gamma_\Gamma \bmod \mathbb{Z}\cdot h_X^{n-r} \tag{2.3.2}$$

for an irreducible closed subvariety Γ.

To extend the result to the general case, assume that a r-dimensional algebraic cycle $\Gamma\in \mathrm{CH}_r(X)$ can be written as $\sum_i n_i\Gamma_i$ for distinct irreducible components Γ_i with degrees e_i. The conclusion of (2.3.1) holds up to scalars, thus we have

$$2(\sum_j n_je_j-e_i)\cdot\Gamma_i\in q_*p^*\,\mathrm{CH}_{r-1}(F)+\mathbb{Z}\cdot h_X^{n-r} \text{ for each } i.$$

Together with (2.3.2), we obtain

$$(2\deg\Gamma-3)(\sum_i n_i\Gamma_i)\in q_*p^*\,\mathrm{CH}_{r-1}(F)+\mathbb{Z}\cdot h_X^{n-r}.$$

□

Corollary 2.4. *Let $X\subset\mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface of dimension $n\geq 3$ over a field k, and let $F(X)$ be the Fano variety of lines. Assume that there exists a degree one 1-cycle in $\mathrm{CH}_1(X)$(or degree one 0-cycle in $\mathrm{CH}_0(F(X))$). Then the cylinder homomorphism*

$$q_*p^*:\mathrm{CH}_{r-1}(F(X))\to\mathrm{CH}_r(X)$$

is surjective up to a hyperplane section class for any degree r.

Proof. The assertion for 1-cycles is due to M. Shen, see Proposition 0.1. The preceding two formulas of the algebraic cycle Γ imply the conclusion of our corollary for higher dimensional cases. □

In the following, we show that the hyperplane section classes could be generated by algebraic cycles from $F(X)$ if k is algebraically closed.

Let X be a smooth hypersurface over an algebraically closed field k. Consider the Abel-Jacobi map $p_*q^*:\mathrm{CH}^r(X)\to\mathrm{CH}^{r-1}(F)$ and the cylinder homomorphism $q_*p^*:\mathrm{CH}_{r-1}(F)\to\mathrm{CH}_r(X)$ induced by the incidence correspondence.

Choose a closed point x in X. We may set $x=[1,0,\dots,0]$. If G is the cubic equation defining X, the cubic equation has the form

$$G=X_0^2L(X_1,\dots,X_{n+1})+X_0Q(X_1,\dots,X_{n+1})+C(X_1,\dots,X_{n+1}),$$

where L(resp. Q,C) is the homogeneous polynomial of degree 1(resp. $2,3$). Denote by C_x the closed subscheme of F which parameterize the lines in X through the point x. It could be identified with the points $[\alpha_1,\dots,\alpha_{n+1}]\in\mathbb{P}^n$ such that

$$L(\alpha_1,\dots,\alpha_{n+1})=Q(\alpha_1,\dots,\alpha_{n+1})=C(\alpha_1,\dots,\alpha_{n+1})=0.$$

It is easily to see that the closed subset $q_*p^*C_x$ in X is cut out by

$$\{[x_0,\ldots,x_{n+1}]\in\mathbb{P}^{n+1}\mid L(x_1,\ldots,x_{n+1})=Q(x_2,\ldots,x_{n+1})=G(x_0,\ldots,x_{n+1})\}.$$

For a very general point x, the quadratic polynomial Q does not contain the linear factor L. So the tangent space $L=0$ at x and the quadric hypersurface $Q=0$ properly intersects, and the dimension of C_x is $n-3$. It follows that

$$q_*p^*C_x=2H_X^2\in\mathrm{CH}^2(X). \tag{2.4.1}$$

If they are not properly intersection, the closed subset $q_*p^*C_x$ is the hyperplane divisor $X\cap T_xX$. In this case, we could obtain a divisor C'_x of C_x cutting out by a general quadric hypersurface in $\mathbb{P}^n$. Then C'_x is a cycle class of dimension $n-3$ in $F(X)$ and $2H_X^2=q_*p^*C'_x$.

The equation (2.4.1) is generalized to the following:

$$q_*p^*C_{x,H^i_{X_n}}=2H^{i+2}_{X_n},\ \text{for } n-i\geq 3, \tag{2.4.2}$$

where $C_{x,H^i_{X_n}}$ is the cycle class represented by lines passing through the point x and lying in the hyperplane section $H^i_{X_n}$. Therefore, $q_*p^*C_{x,H^i_{X_n}}=2j_*H^2_{X_{n-i}}=2H^{i+2}_{X_n}$, where $j:H^i_{X_n}\hookrightarrow X_n$ is the natural inclusion map.

On the other hand, we define a closed subscheme

$$S_{\ell,H^j_{X_n}}:=\{\ell'\in F(X_n)\mid \ell'\subset H^j_{X_n},\ell'\cap\ell\neq\emptyset \text{ for a generic } \ell\subset H^j_{X_n}\}$$

of $F(X_n)$ for an n-dimensional smooth cubic hypersurface X_n. We claim that

$$q_*p^*S_{\ell,H^j_{X_n}}=5H^{j+1}_{X_n},\ \text{for } n-j\geq 3.$$

The algebraic cycle $S_{\ell,H^j_{X_n}}$ which parametrizes the lines in $(n-j)$-dimensional complete intersection $X_{n-j}:=H^j_{X_n}$ meeting a generic line ℓ in X_{n-j}, has dimension $n-j-2$ in $F(X_{n-j})$ whenever $n-j\geq 3$. Thus $q_*p^*S_{\ell,H^j_{X_n}}$ is a divisor class $m\cdot H_{X_{n-j}}$ on the cubic hypersurface X_{n-j}. The integer m can be calculated by the result of M. Shen's 1-cycle relations on cubic hypersurfaces (see [11, Lemma 3.10])

$$q_*p^*S_{\ell,H^j_{X_n}}\cdot[\tilde\ell]=S_{\ell,H^j_{X_n}}\cdot p_*q^*[\tilde\ell]=\#\{\text{lines meeting both } \ell \text{ and } \tilde\ell\}=5.$$

Hence, we have

$$q_*p^*S_{\ell,H^j_{X_n}}=5j_*H_{X_{n-j}}=5j_*j^*H_{X_n}=5H^{j+1}_{X_n},\ \text{for } n-j\geq 3. \tag{2.4.3}$$

Corollary 2.5. *Let $X\subset\mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface of dimension $n\geq 3$ over an algebraically closed field k, and let $F(X)$ be the Fano variety of lines. Then Chow group $\mathrm{CH}_r(X)$ is generated by the Chow group $\mathrm{CH}_{r-1}(F(X))$ via the cylinder homomorphism for all integer $r<n-1$.*

Proof. For $r<n-1$, we take $i=n-r-2$ and $j=n-r-1$ in the aforementioned two formulas (2.4.2) and (2.4.3), where we obtain $3h_X^{n-r}$ and $5h_X^{n-r}$ are the images of algebraic cycles on F. Since $\gcd(3,5)=1$, it concludes that h_X^{n-r} is generated by the algebraic cycles on F via the cylinder homomorphism. Hence the assertion is a direct consequence of Corollary 2.4. □

Remark 2.6. *For cubic 4-folds, the divisor class h_X is indeed covered by the cylinder homomorphism. In the precedent paragraph, $q_*p^*S_{\ell,X}=5h_X$. On the other hand, we have $q_*p^*g^2=21h_X$, see* [12, Lemma A.4]*, where g is the canonical polarization class $c_1(\mathcal{O}(1))$ on the Fano variety of lines F. In general, let $c=c_2(\mathscr{E})$ be* *the second Chern class of the tautological vector bundle $\mathscr{E}$. Suppose $i+2j=n-2$, then $q_*p^*(g^i\cdot c^j)=k\cdot h_X$ (the integer k depends on i,j). If k is coprime to 5, the conclusion of Corollary 2.5 would be extent to the divisor classes of X.*

3. Integral Hodge conjectures and Tate conjectures

Theorem 3.1. *The integral Hodge conjecture holds for* 1*-cycles on the Fano variety of lines of a smooth complex cubic fourfold $X/\mathbb{C}$.*

Proof. Assume $\alpha\in Hdg^6(F(X),\mathbb{Z})$ is an integral Hodge class of degree 6. Since the cylinder homomorphism is a morphism of Hodge structure, thus $q_*p^*\alpha$ is an integral Hodge class of degree 4 of X. The integral Hodge conjecture of degree 4 for the smooth cubic 4-folds is affirmed by Voisin. Suppose that γ is the algebraic cycle such that $[\gamma]=q_*p^*\alpha$. By Corollary 2.5, there exists an algebraic cycle $\Gamma\in\mathrm{CH}_1(F(X))$ such that $q_*p^*\Gamma=\gamma$. Then we have

$$[\Gamma]-\alpha\in\mathrm{Ker}(q_*p^*).$$

By the Poincaré duality, the cylinder homomorphism

$$q_*p^*:H^6(F,\mathbb{Z})\to H^4(X,\mathbb{Z})$$

is dual to the Abel-Jacobi map

$$p_*q^*:H^4(X,\mathbb{Z})\to H^2(F,\mathbb{Z}).$$

Beauville and Donagi [3] proved that the Abel-Jacobi map is an isomorphism. Therefore, the integral Hodge class α is algebraic. □

The surjectivity of the cylinder homomorphism connects the integral Hodge conjectures for a smooth cubic hypersurface and its Fano variety of lines. Let $X/\mathbb{C}$ be a complex smooth cubic hypersurfaces of dimension n, the associated Fano variety of lines $F_1(X)$ is a smooth projective variety of dimension $2n-4$. By Lefschetz hyperplane theorem, the non-trivial part of the cohomology of a smooth hypersurface is at the middle degree. We may assume the dimension n is even. If the kernel of the cylinder homomorphism on cohomology

$$\Psi:H^{3n-6}(F_1(X),\mathbb{Z})\to H^n(X,\mathbb{Z}).$$

is generated by algebraic classes, the same argument in the cubic fourfolds case shows that: the integral Hodge conjecture for $\frac{n}{2}$-cycles of the smooth cubic X induces the integral Hodge conjecture for $\frac{3n-6}{2}$-cycles on the variety $F_1(X)$.

According to the paper [13, §1] by Ichiro Shimada, for an even number n, the kernel of Ψ with rational coefficient is generated by algebraic classes. Indeed, let G be the Grassmannian of lines in $\mathbb{P}^{n+1}$, $\mathrm{Ker}\,\Psi_{\mathbb{Q}}$ is contained in the image

$$\mathrm{im}:(H^{3n-6}(G,\mathbb{Q})\to H^{3n-6}(F_1(X),\mathbb{Q}))$$

via the natural restriction map. Therefore, our result only predict the above connection for rational Hodge conjectures. For instance, if X is a smooth cubic 8-fold, the rational Hodge conjecture for dimension 3-cycles on the Fano variety $F_1(X)$ is true due to [14].

Tate Conjecture Inspired by Theorem (3.1), the integral Tate conjecture for 2-cycles on X is supposed to be equivalent to the integral Tate conjecture for 1-cycles on $F(X)$ if the cylinder homomorphism on Chow groups is surjective.

Lemma 3.2. *Let X be a smooth cubic fourfold in $\mathbb{P}^5_k$ over a field k finitely generated over its prime subfield, and let $F := F(X)$ be the Fano variety of lines on X. Then the cylinder homomorphism on cohomology*

$$q_*p^* : H^6_{\text{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(1))^U \to H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^U$$

is an isomorphism, where U runs over the open subgroups of the absolute Galois group $G := \mathrm{Gal}(\bar{k}/k)$.

Proof. By the Poincaré duality, it suffices to prove that the Abel-Jacobi map

$$q_*p^* : H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2)) \to H^2_{\text{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(1))$$

is a U-equivariant isomorphism. Suppose the prime subfield of k is a finite field $\mathbb{F}_q$. The proof could be carried out by smooth and proper bases change. We denote by $W(\mathbb{F}_q)$ the ring of the Witt vectors of the field $\mathbb{F}_q$, and let $\mathfrak{X}$ be a smooth proper lifting of X over $W(\mathbb{F}_q)$. One could embed $W(\mathbb{F}_q)$ into the complex numbers $\mathbb{C}$. Therefore, we have the following fiber products diagrams

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] & X \arrow[l] \arrow[d] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(W(\mathbb{F}_q)) & \mathrm{Spec}\,\mathbb{F}_q \arrow[l]
\end{tikzcd}.$$

By the proper and smooth base change theorem, for any prime number ℓ invertible in k, there is the canonical isomomorphism of étale cohomology groups

$$H^{2i}_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) \simeq H^{2i}_{\text{et}}(X_{\mathbb{C}}, \mathbb{Z}_\ell(i)). \tag{3.2.1}$$

Moreover, due to the comparison theorem, we have

$$H^{2i}_{\text{et}}(X_{\mathbb{C}}, \mathbb{Z}_\ell(i)) \simeq H^{2i}_{sing}(X(\mathbb{C})^{an}, \mathbb{Z}(i)) \otimes \mathbb{Z}_\ell.$$

Therefore, the Abel-Jacobi homomorphism

$$p_*q^* : H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2)) \to H^2_{\text{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(1)),$$

induced by the incidence correspondence is a U-equivariant isomorphism.

When the prime subfield of k is a number field, the argument is more direct by base change from $\bar{\mathbb{Q}}$ to $\mathbb{C}$. □

The following integral analog Tate conjecture is proved by F. Charles and A. Pirutka.

Theorem 3.3. [5, Theorem 1.1] *Let k be a field finitely generated over its prime subfield with characteristic different from 2 and 3, and let $\bar{k}$ be the algebraic closure of k. If $X \subset \mathbb{P}^5_k$ is a smooth cubic fourfold. The integral Tate conjecture is true for the codimension 2-cycles on X. Precisely, for all prime number $\ell \neq \mathrm{char}\, k$, the cycle class map*

$$\mathrm{CH}^r(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \varinjlim_U H^{2r}_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(r))^U$$

is surjective, where the direct limit is over the open subgroups U of the absolute Galois group $G := \mathrm{Gal}(\bar{k}/k)$.

Notice that the Chow groups in the map (0.2.4) is defined over an algebraically closed field, where the surjectivity of the cylinder homomorphism always holds without extra conditions, we obtain the following integral analog of the Tate conjecture for 1-cycles on the Fano variety of lines.

Corollary 3.4. *Let X be a smooth cubic fourfold in $\mathbb{P}^5_k$ over a field k finitely generated over its prime subfield with characteristic different from 2 and 3. Let $F := F(X)$ be the Fano variety of lines on X. Then the integral analog of the Tate conjecture for 1-cycles of F is true. In the other words, for all prime number $\ell \neq \operatorname{char} k$, the cycle class map*

$$cl : \mathrm{CH}^3(F_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \varinjlim_U H^6_{\mathrm{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(3))^U \tag{3.4.1}$$

is surjective, where U runs over the open subgroups of the absolute Galois group $G := \mathrm{Gal}(\bar{k}/k)$.

Proof. Let $[\alpha] \in \varinjlim_U H^6_{\mathrm{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(3))^U$ be any cohomology class. Suppose that $\alpha \in H^6_{\mathrm{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(3))^U$ for some open subgroup U is a representative of $[\alpha]$. It follows from Theorem 3.3 that the class $q_*p^*\alpha \in H^4_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^U$ lifts to an algebraic cycle $\Gamma \in \mathrm{CH}^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell$. By the surjectivity of the cylinder homomorphism, it gives an algebraic 1-cycle $\gamma \in \mathrm{CH}_1(F(X_{\bar{k}})) \otimes \mathbb{Z}_\ell$ such that $q_*p^*\gamma = \Gamma$. By Lemma 3.2, the map $q_*p^* : H^6_{\mathrm{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(1))^U \to H^4_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^U$ is an isomorphism for all open subgroups U. Thus we have $cl(\gamma) = \alpha$. □

With the notations in the aforementioned theorem, the question whether the cycle class map

$$\mathrm{CH}^2(X_k) \otimes \mathbb{Z}_\ell \to H^4_{\mathrm{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^G \tag{3.4.2}$$

is surjective is still open, where prime number $\ell \neq \operatorname{char} k$.

Nevertheless, we speculate that: *The cycle class map (3.4.2) is surjective if and only if the map (0.2.3) is. In other words, the integral Tate conjecture for 2-cycles on a smooth cubic fourfold X/k holds if and only if the integral Tate conjecture for 1-cycles on $F(X)$ is true.*

In particular, when k is a finite field, the following result about the rational k-lines on X is due to Debarre, Laface and Roulleau

Theorem 3.5. [6] *Let X be a smooth cubic hypersurfaces of dimension n defined over a finite field $\mathbb{F}_q$. Then X contains a line defined over $\mathbb{F}_q$ in each of the following case:*

(1) $n = 3$ and $q \leq 11$;
(2) $n = 4$, and $q = 2$ or $q \geq 5$;
(3) $n \geq 5$.

If we further assume that for a smooth cubic 4-fold $X/\mathbb{F}_q$ with $q \geq 5$ and a $\mathbb{F}_q$-line $\mathfrak{l}$ on X, one could find smooth complete intersections of hyperplanes of X containing $\mathfrak{l}$, then the same arguments in the proof of Corollary 2.5 allows us to conclude that the cylinder homomorphism of $X/\mathbb{F}_q$ is surjective. Moreover, our speculation above could be confirmed.

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