1. HAHN-BANACH

For a linear space V over $\mathbb{F}$, we will use the notation $V^* = L(V, \mathbb{F})$, i.e., it is the linear space of linear functionals on V.

Theorem 1.1 (Real Hahn-Banach). *Let X be a real linear space, p a sublinear functional on X, W linear subspace of X, $f \in W^*$ with*

$$f_W(w) \le p(w) \quad (w \in W),$$

then f_W has an extension to an $f_X \in X^*$, with

$$f_X(x) \le p(x) \quad (x \in X).$$

The proof will be discussed later.

Remark 1.2. Let in real HB, p be a semi-norm. Then

$$-p(x) = -p(-x) \le -f_X(-x) = f_X(x) \le p(x),$$

so even $|f_X(x)| \le p(x)$.

To work towards a complex HB, we have the following definition and lemma.

Definition 1.3. For a complex linear space X, define the real linear space X_R as the same set as X but with multiplication restricted to real scalars.

Lemma 1.4. *Let X be a complex linear space. Then any $g \in X^*$ is of the form*

$$g(v) = g_R(v) - i g_R(iv),$$

for a unique $g_R \in X_R^*$; and for each $g_R \in X_R^*$ the right-hand side defines an element of X^*. Moreover, $g_R = \operatorname{Re} g$.

Proof is given in [Book, Lemma 5.15].

Theorem 1.5 (Complex Hahn-Banach). *Let X be a complex linear space, p a semi-norm on X, W a linear subspace of X, $f \in W^*$ with*

$$|f_W(w)| \le p(w) \quad (w \in W),$$

then f_W has an extension to an $f_X \in X^*$, with

$$|f_X(x)| \le p(x) \quad (x \in X).$$

Proof. For X being a real linear space, we have already seen the proof in Remark. 1.2.

Now let X be a complex linear space. For $f_{W,R} := \operatorname{Re} f_W \in W_R^*$,

$$f_{R,W}(w) \le |f_{R,W}(w)| \le |f_W(w)| \le p(w) \quad (w \in W).$$

So real HB gives extension $f_{R,X} \in X_R^*$ with $f_{R,X}(x) \le p(x)$ $(x \in X)$. Now let $x \in X$. Select $|\alpha| = 1$ with $|f_X(x)| = \alpha f_X(x)$. Then

$$|f_X(x)| = \alpha f_X(x) = f_X(\alpha x) = f_{X,R}(\alpha x) \le p(\alpha x) = |\alpha| p(x) = p(x). \quad \square$$

Remark 1.6. Notice that Remark 1.2 shows that the statement of complex HB (i.e. $|f_W(w)| \le p(w)$ for some semi-norm p implies $|f_X(x)| \le p(x)$) is also valid for real linear spaces.

The main importance of (real or complex) HB lies in the following result:

Corollary 1.7. *Let X be a normed linear space, with subspace W. Then any $f_W \in W'$ has an extension to a $f_X \in X'$ with $\|f_X\|_{X'} = \|f_W\|_{W'}$.*

Proof. Let $f \in W'$, define $p(x) := \|f\|_{W'}\|x\|_X$, which is a semi-norm. From $|f_W(w)| \leq p(w)$, HB shows existence of $f_X \in X^*$ with $|f_X(x)| \leq p(x) = \|f\|_{W'}\|x\|_X$, i.e., $f \in X'$ with $\|f_X\|_{X'} \leq \|f_W\|_{W'}$. □

Remark 1.8. In the book a 'direct' proof of this corollary is given for X being separable ([Book, proof of Theorem 5.19]). We skip this. The reason for this direct proof is that some mathematicians don't like to apply Hahn-Banach. We return to this point later.

It still remains to prove the real HB. Before we come to that first we discuss some applications:

2. Applications Hahn-Banach

Theorem 2.1. *Let X be a normed linear space. W a linear subspace. Let $x \in X$ be such that $\delta := \inf_{w \in W} \|x - w\| > 0$. Then there exists a $f \in X'$ with $\|f\|_{X'} = 1$, $f(x) = \delta$, and $f(W) = 0$.*

Proof. Set $Y := \text{span}\{x\} \oplus W$. Define $f \in Y^*$ by $f(W) = 0$, $f(x) = \delta$. Then for all $\alpha \in \mathbb{F}$, $w \in W$,

$$|f(\alpha x + w)| = |\alpha|\delta \leq |\alpha|\|x + \alpha^{-1}w\| = \|\alpha x + w\|.$$

So $f \in Y'$ with $\|f\|_{Y'} \leq 1$. For all $\varepsilon > \delta$, $\exists w \in W$ with $\|x - w\| < \varepsilon$ and so $\frac{f(x-w)}{\|x-w\|} > \frac{\delta}{\varepsilon}$. So $\|f\|_{Y'} \geq 1$, or $\|f\|_{Y'} = 1$.

Extend f with preservation of its norm to an element of X'. □

Remarks 2.2. The proof given on p135 of the book of this result is ridiculous.

Notice that the conditions of the above theorem are fulfilled when W is closed and $x \notin W$.

Corollary 2.3. *a) $\forall x \in X \setminus \{0\}$, $\exists f \in X'$ with $f(x) = \|x\|$ and $\|f\| = 1$.*
b) $\forall x \in X$, $\|x\| = \sup\{|f(x)| \colon \|f\| \leq 1\}$
(note symmetry with $\|f\| = \sup\{|f(x)| \colon \|x\| \leq 1\}$)
c) $\forall x \neq y \in X$, $\exists f \in X'$ with $f(x) \neq f(y)$.

Proof. a) Previous theorem for $W := \{0\}$.
b) $\geq$ ok. $\leq$ from a).
c) Read x in a) as $x - y$. □

Theorem 2.4. *Let X be a normed linear space. If X' separable, then X separable.*

Proof. Let $B := \{f \in X' \colon \|f\| = 1\}$. From X' separable it follows that there exists a sequence $\{f_n\} \subset X'$ whose closure is B. So for any $g \in B$, $\exists f_i$ with $\|g - f_i\| < \varepsilon/2$, but then $|1 - \|f_i\|| < \varepsilon/2$, and so $\|g - \frac{f_i}{\|f_i\|}\| \leq \|g - f_i\| + \|f_i - \frac{f_i}{\|f_i\|}| < \varepsilon$. In other words $\{h_n := \frac{f_n}{\|f_n\|}\} \subset B$ is dense in B.

For all n, select $w_n \in X$ with $\|w_n\| = 1$ and $|h_n(w_n)| \geq \frac{1}{2}$, and define $W = \overline{\text{span}\{w_n \colon n \in \mathbb{N}\}}$. Suppose $W \neq X$. Then, HB, $\exists f \in B$ with $f(W) = 0$. So for all n, $\frac{1}{2} \leq |h_n(w_n)| = |h_n(w_n) - f(w_n)| \leq \|h_n - f\|$ which gives a contradiction with the density of $\{h_n\}$ in B. So $W = X$, but then also $X = \overline{\{\sum_{i=1}^N \alpha_i w_i \colon N \in \mathbb{N},\ \alpha_i \in \mathbb{Q} + i\mathbb{Q}\}}$. □

Remarks 2.5. a) The converse of Theorem 2.4 is not true. Indeed, we have seen that for $p \in [1, \infty)$, $(\ell^p)'$ is isometric isomorphic to ℓ^q. In particular, $(\ell^1)'$ is isometric isomorphic to ℓ^∞. So if the converse of Theorem 2.4 would be true, then from $\ll^1$ separable, it would follow that $(\ell^1)'$ is separable, which would imply that ℓ^∞ is separable, which is known not to be true.

b) Also we see again that $(\ell^\infty)'$ is not isomorphic to ℓ^1. Indeed, if it would be true, then from ℓ^1 being separable and thus $(\ell^\infty)'$ separable, Theorem 2.4 would show that ℓ^∞ is separable, which is known not to be true.

3. Proof of the real Hahn-Banach theorem

This proof is given in [Book, Section 5.4] until halfway p139. The remainder of that section we will skip. As you will read, the proof is based on Zorn's lemma, which is actually not so much a lemma as well as an *axiom* that is equivalent to the *axiom of choice* from set theory. On

`https://en.wikipedia.org/wiki/Axiom_of_choice`

you can read *Although originally controversial, the axiom of choice is now used without reservation by most mathematicians*. This doesn't sound very comforting. Mathematicians that called themselves intuitists don't accept this axiom as a valid part of mathematics. Intuitionism was introduced by Brouwer

`(https://plato.stanford.edu/entries/intuitionism/)`

professor of Mathematics at the UvA, and probably the most well-known Dutch mathematician. Maybe you have heard about Brouwer's fixed point theorem (not to be confused with the easier Banach's fixed point theorem)

Apart from Zorn' lemma, another ingredient of the proof of Hahn-Banach is [Book, Lemma 5.18] which shows that an extension as meant in the real Hahn-Banach theorem can always be constructed on $W \oplus \operatorname{span}\{z\}$ for any $z \notin W$.