Answers midterm Functional Analysis March 26, 2019

1. Since $\overline{\operatorname{Sp} E}$ is a closed subspace, $H = \overline{\operatorname{Sp} E} \oplus \overline{\operatorname{Sp} E}^{\perp}$, so $\overline{\operatorname{Sp} E} = H \iff \overline{\operatorname{Sp} E}^{\perp} = \{0\}$.

   $E \subset \overline{\operatorname{Sp} E}$ implies $\overline{\operatorname{Sp} E}^{\perp} \subset E^{\perp}$. On the other hand, $x \in E^{\perp}$ implies $x \in (\operatorname{Sp} E)^{\perp}$, which implies $x \in \overline{\operatorname{Sp} E}^{\perp}$ by continuity of $y \mapsto \langle x, y\rangle$. So $\overline{\operatorname{Sp} E}^{\perp} = E^{\perp}$, and the proof is completed.

2. Let $\{f_n\}$ Cauchy in $F_b(S, X)$. Then for any $s \in S$, $\{f_n(s)\}$ Cauchy in X, and so convergent. Define $f(s) = \lim_{n\to\infty} f_n(s)$. Let $\varepsilon > 0$ be given. There exists an $N \in \mathbb{N}$ s.t. for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon$. Thus for any $s \in S$, $\|f_n(s) - f_m(s)\| < \varepsilon$, but then $\|f_n(s) - f(s)\| \leq \varepsilon$, or $\|f_n - f\| \leq \varepsilon$.

3. (a) Expand rhs.

   (b) First item shows that for any $u \in X$, $\langle Tu, w\rangle = 0$ for all $w \in X$, and thus $\langle Tu, Tu\rangle = 0$ or Tu=0 or $T = 0$.

   (c) Rotation over $90°$ degrees in $\mathbb{R}^2$.

4. (a) For any $\{x_i\} \in c_0$ is $|\sum_{i=1}^N x_i y_i| \leq \|x\|_{\ell^\infty} \sum_{i=1}^N |y_i|$ with equality for $x_i = 0$ for $i > N$ or $y_i = 0$, and $x_i = |y_i|/y_i$ otherwise.

   (b) For any $\{x_i\} \in c_0$, $\lim_{N\to\infty} L_N\{x_i\} = \sum_{i=1}^\infty x_i y_i < \infty$, thus $\sup_N |L_n\{x_i\}| < \infty$. Since $L_N \in B(c_0, \mathbb{C})$, and c_0 and $\mathbb{C}$ are Banach, the uniform boundedness principle shows that $\|y\|_{\ell^1} = \sup_N \|L_N\|_{c_0'} < \infty$.

5. For any $x \in X$, thanks to $I - T$ being bounded, $(I - T)\sum_{n=0}^\infty T^n x = \lim_{N\to\infty}(I - T)\sum_{n=0}^N T^n x = \lim_{N\to\infty} x - T^{N+1}x = x$, so $I - T$ is

surjective. From $\sum_{n=0}^{\infty} T^n(I-T)x = \lim_{N\to\infty}\sum_{n=0}^{N} T^n(I-T)x = \lim_{N\to\infty} x - T^{N+1}x = x$, it follows that $I-T$ is injective. Since X is Banach, the open mapping theorem now shows that $I-T$ has a bounded inverse.

6. (a) $|\langle x, z\rangle| \leq \|x\|\|z\|$ with equality when $x = z$.

   (b) If X is Hilbert, then the Riesz representation theorem shows that any $f \in X'$ is of the form f_z. If $f \mapsto f_z$ is surjective, then previous item shows that $z \mapsto f_z$ is an isometric isomorphism between X and X'. Since X' is Banach, we infer that X is Banach.