1. Since $\text{Sp} \overline{E}$ is a closed subspace, $H = \text{Sp} \overline{E} \oplus \text{Sp} \overline{E}^\perp$, so $\text{Sp} \overline{E} = H \iff \text{Sp} \overline{E}^\perp = \{0\}$.

$E \subset \overline{\text{Sp} \overline{E}}$ implies $\text{Sp} \overline{E}^\perp \subset E^\perp$. On the other hand, $x \in E^\perp$ implies $x \in (\text{Sp} E)^\perp$, which implies $x \in \overline{\text{Sp} E}^\perp$ by continuity of $y \mapsto \langle x, y \rangle$. So $\text{Sp} \overline{E}^\perp = E^\perp$, and the proof is completed.

2. Let $\{f_n\}$ Cauchy in $F_b(S, X)$. Then for any $s \in S$, $\{f_n(s)\}$ Cauchy in $X$, and so convergent. Define $f(s) = \lim_{n \to \infty} f_n(s)$. Let $\varepsilon > 0$ be given. There exists an $N \in \mathbb{N}$ s.t. for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon$. Thus for any $s \in S$, $\|f_n(s) - f_m(s)\| < \varepsilon$, but then $\|f_n(s) - f(s)\| \leq \varepsilon$, or $\|f_n - f\| \leq \varepsilon$.

3. (a) Expand rhs.

(b) First item shows that for any $u \in X$, $\langle Tu, w \rangle = 0$ for all $w \in X$, and thus $\langle Tu, Tu \rangle = 0$ or $Tu=0$ or $T=0$.

(c) Rotation over $90^\circ$ degrees in $\mathbb{R}^2$.

4. (a) For any $\{x_i\} \in c_0$ is $|\sum_{i=1}^{N} x_i y_i| \leq \|x\|_{\ell^\infty} \sum_{i=1}^{N} |y_i|$ with equality for $x_i = 0$ for $i > N$ or $y_i = 0$, and $x_i = |y_i|/y_i$ otherwise.

(b) For any $\{x_i\} \in c_0$, $\lim_{N \to \infty} L_N \{x_i\} = \sum_{i=1}^{\infty} x_i y_i < \infty$, thus $\sup_N |L_n\{x_i\}| < \infty$. Since $L_N \in B(c_0, \mathbb{C})$, and $c_0$ and $\mathbb{C}$ are Banach, the uniform boundedness principle shows that $\|y\|_{\ell^1} = \sup_N \|L_N\|_{c_0} < \infty$.

5. For any $x \in X$, thanks to $I - T$ being bounded, $\langle I - T \rangle \sum_{n=0}^{\infty} T^n x = \lim_{N \to \infty} (I - T) \sum_{n=0}^{N} T^n x = \lim_{N \to \infty} x - T^{N+1} x = x$, so $I - T$ is
surjective. From $\sum_{n=0}^{\infty} T^n (I - T)x = \lim_{N \to \infty} \sum_{n=0}^{N} T^n (I - T)x = \lim_{N \to \infty} x - T^{N+1}x = x$, it follows that $I - T$ is injective. Since $X$ is Banach, the open mapping theorem now shows that $I - T$ has a bounded inverse.

6. (a) $|\langle x, z \rangle| \leq \|x\|\|z\|$ with equality when $x = z$.

(b) If $X$ is Hilbert, then the Riesz representation theorem shows that any $f \in X'$ is of the form $f_z$. If $f \mapsto f_z$ is surjective, then previous item shows that $z \mapsto f_z$ is an isometric isomorphism between $X$ and $X'$. Since $X'$ is Banach, we infer that $X$ is Banach.