1. Let \((X, d)\) be a metric space and \(E \subset X\) be compact. Show that for any \(x \in X\), there exists a \(e \in E\) with \(d(x, E) := \inf_{y \in E} d(x, y) = d(x, e)\).

2. Let \(X \subset \mathbb{R}^d\) be compact. Show that that for \(1 \leq q \leq p \leq \infty\), \(L^p(X) \subset L^q(X)\), and that the embedding is continuous.

3. Show that for \(p \in [1, \infty]\), \(\ell^p\) is complete.

4. Show that \(\ell^\infty\) is not separable.

5. Let \(c_0\) be the space of all sequences that converge to 0. Show that \(c_0\) is a closed linear subspace of \(\ell^\infty\). Is \(c_0\) separable?

6. Show that \(d(\{x_n\}, \{y_n\}) := \sum_{n=1}^\infty 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}\) defines a metric on the linear space of real-valued sequences. Show there exists no norm \(\|\|\) on this space such that \(d(\{x_n\}, \{y_n\}) = \|\{x_n\} - \{y_n\}\|\).

7. (Gram-Schmidt) Let \(\{v_i\}\) be linearly independent in an inner product space \(X\). Show that \(\{e_i\}\), defined by

\[
e_i = v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, e_j \rangle}{\langle e_j, e_j \rangle} e_j.
\]

is orthogonal, and that \(\text{span}\{e_j; 1 \leq j \leq i\} = \text{span}\{v_j; 1 \leq j \leq i\}\) for any \(i \in \mathbb{N}\).

8. Show that there is no inner product on \(C[0, 1]\) that induces the \(L^1(0, 1)\)-norm on that space.
9. Which of the following expressions defines an inner product on $C^2[0,1]$ (being the space of two times continuously differentiable real functions on $[0,1]$)

- $\langle f, g \rangle = \int_0^1 f'(t)g'(t) \, dt$
- $\langle f, g \rangle = \int_0^1 f(t)g(t) + f'(t)g'(t) + f''(t)g''(t) \, dt$

and for those who do, do they give rise to a Hilbert space?

10. Show that $\|f\| := \max_{t \in [0,1]} |f(t)| + |f''(t)|$ defines a norm on $C^2[0,1]$, and that w.r.t. this norm, $C^2[0,1]$ is a Banach space.

11. Let $V$ be the linear subspace of $C_\mathbb{R}[0,1]$ of functions $f$ of type $f(x) = ax$ for some $a \in \mathbb{R}$. Let $\mathbbm{1}: x \mapsto 1 \in C_\mathbb{R}[0,1]$. Compute $d(\mathbbm{1}, V)$, and show that the minimal distance is attained for several $f \in V$.

12. Let $V$ be the linear subspace of $C_\mathbb{R}[0,1]$ of functions $f$ with $f(1) = 0$. Show that $V$ is a Banach space. Let $W$ its subspace consisting of functions $g$ with $\int_0^1 g(x) \, dx = 0$. Show that $W$ is a closed subspace. Let $f \in V$ be defined by $f(x) = 1 - x$. Compute the distance $d(f, W)$, and show that this distance is not attained.

13. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Hint: Compute the Fourier series of $f: x \mapsto x \in L^2_{\mathbb{C}}[-\pi, \pi]$, and use Parseval’s theorem).

14. Let $V$ be the linear space $C_\mathbb{R}[0,1]$. Define $L \in L(V, V)$ by $(Lf)(x) = x f(x)$.

   (a) Equip $V$ with the $L^\infty$-norm. Show that $L \in B(V, V)$, compute its is norm, and give an $f \in V$ with $\|Lf\|_{L^\infty[0,1]} = \|L\|_{B(V,V)} \|f\|_{L^\infty[0,1]}$.

   (b) Equip $V$ with the $L^2$-norm. Show that $L \in B(V, V)$, compute its is norm, and show that for any $0 \neq f \in V$, $\|Lf\|_{L^2[0,1]} < \|L\|_{B(V,V)} \|f\|_{L^2[0,1]}$.

15. (Supplement to the proof of Thm. 3.54)

   (a) Show that $\text{span}\{\cos^k x : 0 \leq k \leq n\} = \text{span}\{\cos kx : 0 \leq k \leq n\}$ (Hint: Recall that Chebychev polynomial $T_n(t) := \cos(n \arccos(t))$ of degree $n$ has a non-zero leading coefficient ($2^{n-1}$ when $n \geq 1$), and use that $\cos nx = T_n(\cos x)$.)
(b) Show that \( \text{span}\{\cos kx : k \geq 0\} \) is dense in \( C[0, \pi] \).

16. (Haar basis (Alfréd Haar 1909)) Let \( \phi(x) := \begin{cases} 
1 & x \in [0, \frac{1}{2}] \\
0 & x \in \mathbb{R} \setminus [0, 1] 
\end{cases} \), \( \psi(x) := \begin{cases} 
1 & x \in [0, \frac{1}{2}] \\
-1 & x \in (\frac{1}{2}, 1] \\
0 & x \in \mathbb{R} \setminus [0, 1] 
\end{cases} \), \( \psi_{\ell,k}(x) := 2^{\ell/2} \psi(2^\ell x - k) \).

(a) Show that \( \Psi := \bigcup_{\ell=0}^{\infty} \{\psi_{\ell,k} : k = 0, \ldots, 2^{\ell} - 1\} \cup \{\phi\} \) is a countable orthonormal collection in \( L^2[0, 1] \).

(b) Show that \( \text{span}(\bigcup_{\ell=0}^{N-1} \{\psi_{\ell,k} : k = 0, \ldots, 2^{\ell} - 1\} \cup \{\phi\}) = \{f \in L^2[0, 1] : f|_{(k2^{-n}, (k+1)2^{-n})} \text{ is constant } \forall k = 0, \ldots, 2^N - 1\} \).

(c) Show that \( \Psi \) is an orthonormal basis for \( L^2[0, 1] \).

17. Let \( X = \{x \in \ell^\infty : \# \text{ supp } x < \infty\} \). Define \( T : X \to X : x \mapsto (x_1, x_2/2, x_3/3, \ldots) \). Show that \( T \in B(X,X) \) and that it is bijective, but that \( T^{-1} \) is not bounded.

18. Let \( X,Y \) be Banach spaces, \( T \in B(X,Y) \) be injective. Show that \( T^{-1} : \text{Im } T \to X \) is bounded iff \( \text{Im } T \) is closed in \( Y \).

19. Let \( X_1 = (X, \|\|_1) \) and \( X_2 = (X, \|\|_2) \) be Banach spaces. Show that if for some constant \( C, \|\|_1 \leq C\|\|_2 \), then both norms are equivalent.

20. Let \( Y = C^0[0, 1] \). Let \( X \) its linear subspace of the continuous differentiable functions on \( [0, 1] \). Let \( T : X \to Y : u \mapsto u' \).

(a) Show that \( T \) is not bounded.

(b) Show that \( G(T) \) is closed (Hint: Let \( X \supset \{u_n\} \to u \), and \( \{u'_n\} \to w \). Show that \( \int_0^t w(\tau) d\tau = u(t) - u(0) \)).

Apply a theorem to conclude that \( X \) is not closed.

21. Let \( X = C^0[a,b], I : f \mapsto \int_a^b f(x) dx \in X' \), and \( \{Q_n\} \subset X' \) of the form \( Q_n f = \sum_{i=1}^{k(n)} w_i^{(n)} f(x_i^{(n)}) \) for some \( \{x_i^{(n)} : 1 \leq i \leq k(n)\} \subset [a,b] \), and \( \{w_i^{(n)} : 1 \leq i \leq k(n)\} \subset \mathbb{R} \).

(a) Show that \( \|Q_n\| = \sum_{i=1}^{k(n)} |w_i^{(n)}| \).

(b) Show that if for any polynomial \( p \), \( \lim_{n \to \infty} Q_n(p) = I(p) \), and \( \sup_n \|Q_n\| < \infty \), then for any \( f \in C[a,b], \lim_{n \to \infty} Q_n(f) = I(f) \).
(c) Show that both conditions from (21b) are also necessary conditions.

(For the sequence of \((n + 1)\)-point Newton-Cotes formulas, it holds that \(\{\|Q_n\|\}\) is unbounded, whereas for the sequence of \((n + 1)\)-point Gauss-Legendre formulas or the composite trapezoidal rule \(\{\|Q_n\|\}\) is bounded, because \(\sum_{i=1}^{k(n)} w_i^{(n)} = b - a.\)

22. For inner product spaces \(X, Y\), let \(T \in L(X, Y), S \in L(Y, X)\) such that \(\langle Tx, y \rangle_Y = \langle x, Sy \rangle_X\) for all \((x, y) \in X \times Y\). Show that \(\mathcal{G}(T)\) is closed. Conclude that \(T \in B(X, Y)\) when \(X\) and \(Y\) are Hilbert spaces.

23. Let \(Y\) be a subspace of a linear space \(X\). On \(X\) define the equivalence relation \(x \sim z\) whenever \(x - z \in Y\). The quotient space \(X/Y\) is the linear space of equivalence classes \([x] := \{z : x \sim z\}\).

(a) Let \(X\) be a normed linear space and \(Y\) a closed linear subspace. Show that \(X/Y\) is a normed linear space with
\[
\|[x]\| := \inf\{\|x + y\| : y \in Y\}.
\]

(b) Show that \(T \in L(X, X/Y)\) defined by \(x \mapsto [x]\) is bounded, and \(\|T\| \leq 1\).

(c) If additionally \(X\) is a Banach space, show that \(X/Y\) is a Banach space. Hint: Let \(\{u_i\}\) be a Cauchy sequence in \(X/Y\). Show that it has a subsequence \(\{u_{n_i}\}\) with \(\|u_{n_{i+1}} - u_{n_i}\| < 2^{-i}\). Show existence of a sequence \(\{x_{n_i}\} \subset X\) with \([x_{n_i}] = u_{n_i}\) and \(\|x_{n_{i+1}} - x_{n_i}\| < 2^{-i}\).

Show that \(\{x_{n_i}\}\), and finally \(\{u_i\}\) are convergent.

24. Let \(X\) and \(Y\) be Banach spaces, and let \(\{L_n\}\) be a sequence in \(B(X, Y)\) such that for any \(x \in X\), \(L_n x := \lim_{n \to \infty} L_n x\) exists.

(a) Show that \(L \in B(X, Y)\).

(b) By means of a counterexample, show that that not necessarily \(\lim_{n \to \infty} L_n = L\).

25. In [Rynne & Youngson, Exer. 5.8] it was shown that there exists a non-zero \(f \in (\ell^\infty)'\) with \(f(e_n) = 0\) for all \(n \geq 1\), confirming the fact that \(\ell^1 \to (\ell^\infty)' : \vec{a} \mapsto (\vec{b} \mapsto \sum_{i=1}^{\infty} a_i b_i)\) is not surjective (we know that
the image is $(c_0)'$). It appears that an explicit construction of such an $f$ is not possible. The following comes a bit ‘closer’ to an explicit construction:

Let $c$ be the space of $\vec{b} \in \ell_\infty$ for which $\lim_{i \to \infty} b_i$ exists.

(a) Show that $c$ is a separable Banach space.

(b) Construct a non-zero $f \in c'$ with $f(e_n) = 0$ for all $n \geq 1$.

(c) Conclude again that there exists a non-zero $f \in (\ell_\infty)'$ with $f(e_n) = 0$ for all $n \geq 1$.

26. Let $Y$ be a subspace of a linear space $X$. The codimension of $Y$ is defined as $\text{codim } Y = \dim X/Y$. When $\text{codim } Y = 1$, any element of $X/Y$ is called a a hyperplane parallel to $Y$.

(a) Show that for any $0 \neq f \in X^* (= L(X,F))$ and $c \in F$, $\{x \in X : f(x) = c\}$ is a hyperplane parallel to $\ker f$. It determines the half-spaces $\{x \in X : f(x) \leq c\}$ and $\{x \in X : f(x) \geq c\}$.

(b) Let $X$ be a normed linear space, and for some $r > 0$, let $x_0 \in \{x \in X : \|x\| = r\}$. Show that there exists a hyperplane $H \ni x_0$ such that $\{x \in X : \|x\| \leq r\}$ entirely lies in one of the two hyperplanes determined by $H$.

27. It is known that $T : L_\infty[0,1] \to (L^1[0,1])' : v \mapsto (u \mapsto \int_0^1 v(x)u(x) \, dx)$ is a linear (isometric) isomorphism.

(a) On $C[0,1] \subset L_\infty[0,1]$, consider $\delta : v \mapsto v(0)$. Show that $\delta \in (C[0,1])'$.

(b) Show that $\exists u \in L^1[0,1]$ such that $\delta(v) = \int_0^1 v(x)u(x) \, dx$ ($v \in C[0,1]$).

(c) Show that $\exists f \in L_\infty[0,1]'$ for which $\exists u \in L^1[0,1]$ such that $f(v) = \int_0^1 v(x)u(x) \, dx$ ($v \in L_\infty[0,1]$).

(d) Assume that $L^1[0,1]$ is reflexive, i.e. that the linear isomorphism $J_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]'$ is surjective. Show that in this case for any $f \in L_\infty[0,1]'$ and $v \in L_\infty[0,1]$, for $w = J_{L^1[0,1]}^{-1}(T')^{-1}g$ it holds that $f(v) = \int_0^1 v(x)w(x) \, dx$, and conclude a contradiction.

(e) Show that $L_\infty[0,1]$ is not reflexive.
28. Let $W$ be a finite dimensional linear subspace of a normed linear space $X$.

(a) Show that $W$ is closed.

(b) Show that for any $x \in X$, there exists a $w \in W$ with $\|x - w\| = \inf_{v \in W} \|x - v\|$. (Hint: Note that it is sufficient to consider $v$ with $\|v\| \leq 2\|x\|$.)

29. Consider the odd function $f$, on $[0, \pi]$ defined by $f(x) = x(\pi - x)$.

Compute $\|f\|_{L^2[-\pi, \pi]}$, and with that, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}$, and finally $\sum_{n=0}^{\infty} \frac{1}{n^6}$.

30. Let $Y$ be a closed linear subspace of a normed space $X$. Show that for $x \in X \setminus Y$ there exists a $f \in X'$ with $f(x) = 1$ and $f(Y) = 0$.

(This property is used on many places in [Rynne & Youngson], for example in the proof of Thm. 5.43.)

31. Let $X$ and $Y$ be finite dimensional linear spaces equipped with bases $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Psi = \{\psi_1, \ldots, \psi_n\}$, respectively, and let $T \in L(X,Y) = B(X,Y)$.

(a) When $x = \sum_{j=1}^{m} x_j \phi_j$, show that $Tx = \sum_{i=1}^{n} y_i \psi_i$ where $\bar{y} = T\bar{x}$ and $T = [\psi_i'(T\phi_j)_{1 \leq i \leq n, 1 \leq j \leq m}] \in \mathbb{F}^{n \times m}$, and $\Psi' = \{\psi'_1, \ldots, \psi'_n\}$ is the basis for $Y'$ that is dual to $\Psi$. The matrix $T$ is the representation of $T \in L(X,Y)$ equipped with $\Phi$ or $\Psi$, respectively.

(b) Show that $\{J_X\phi_1, \ldots, J_X\phi_m\}$ is a basis for $X''$ that is dual to $\Phi' = \{\phi'_1, \ldots, \phi'_m\}$, being the basis for $X'$ that is dual to $\Phi$.

(c) Show that the representation of $T' \in L(Y', X')$, equipped with $\Psi'$ and $\Phi'$, is given by the transpose $T^\top$.

32. (Part of the closed range theorem) Let $X$ and $Y$ be normed linear spaces, and $T \in B(X,Y)$.

(a) Show that $\text{Im} T$ is closed if and only if $\text{Im} T = \circ \ker T'$.

(b) Now let $X$ be a Banach space. Show that if $\inf_{0 \neq x \in X} \frac{\|Tx\|_Y}{\|x\|_X} > 0$ and $\ker T' = \emptyset$, then $T$ is invertible, and thus boundedly invertible when also $Y$ is a Banach space.

33. Let $X$ be a normed linear space, and $U$ a one-dimensional linear subspace. Show that there exists a projection $P$ on $X$ with $\text{Im} P = U$ and $\|P\| = 1$.  

6
34. (a) Let $H$ be a Hilbert space, and let $T_H : H \to H'$ be the Riesz operator, i.e. $(T_H y)(x) = \langle x, y \rangle_H$. Show that for $W \subset H$, $W^0 = T_H W^\perp$, and $\circ (T_H W) = W^\perp$.

(b) For Hilbert spaces $H$ and $K$, and $T \in B(H, K)$, by using (34a) and [Rynne & Youngson, Lemma 5.52] show that $\ker T^* = (\text{Im } T)^\perp$, and $\ker T = (\text{Im } T^*)^\perp$.

35. Let $X$ and $Y$ be finite dimensional Hilbert spaces equipped with bases $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Psi = \{\psi_1, \ldots, \psi_n\}$, respectively, and let $T \in L(X, Y) = B(X, Y)$.

(a) Setting $F_X : \mathbb{F}^m \to X : \vec{x} \mapsto \sum_{j=1}^m x_j \phi_j$, and analogously $F_X : \mathbb{F}^n \to Y$, show that the matrix $T$ that represents $T$ is given by $F_Y^{-1} T F_X$.

(b) Equipping $\mathbb{F}^m$ with $\langle \vec{x}, \vec{y} \rangle_{\mathbb{F}^m} := \sum_{i=1}^m x_i \bar{y}_i$, show that $\Phi$ is an orthonormal basis if and only if $F_X^{-1} = F_X^*$. (Hint: Expand $\langle F_X \vec{x}, F_X \vec{y} \rangle_X$.)

(c) Show that if $\Psi$ is orthonormal, then $\langle T \vec{x}, \vec{y} \rangle_{\mathbb{F}^m} = \langle T F_X \vec{x}, F_Y \vec{y} \rangle_Y$ ($\vec{x} \in \mathbb{F}^m, \vec{y} \in \mathbb{F}^n$).

(d) Let $\Phi$ and $\Psi$ be orthonormal.

i. Show that the representation of $T^*$ is $T^*$.

ii. Show that $T^* = T^{-1}$ if and only if $T^* = T^{-1}$.

(e) Let $X = Y$, $\Phi = \Psi$ be orthonormal. Show that $T^* = T$ if and only if $T^* = T$, and $T^* T = TT^*$ if and only if $T^* T = TT^*$.

36. For a Banach space $X$, let $A, B \in B(X)$.

(a) Show that if $AB$ is boundedly invertible, and $A$ and $B$ commute, then both $A$ and $B$ are boundedly invertible (this is needed for showing the fourth $\iff$-implication in the proof of [Rynne & Youngson, Thm 6.39])

(b) Give an example showing that the condition that $A$ and $B$ commute cannot be dropped.

37. Give an example showing that [Rynne & Youngson, Lemma 6.29] is generally not valid in a real inner product space.

38. Let $H$ be a complex Hilbert space and $T \in B(H)$. Show that if $\langle Tx, x \rangle$ is real for all $x \in H$, then $T$ is self-adjoint.
39. Let $H$ be a Hilbert space. Let $M, N$ be closed linear subspaces of a Hilbert space $H$ such that

\[ \alpha := \inf_{0 \neq m \in M, 0 \neq n \in N} \sup \frac{\langle m, n \rangle}{\|m\| \|n\|} > 0, \]

\[ \forall 0 \neq n \in N, \exists m \in M \text{ with } \langle m, n \rangle \neq 0. \] (1)

(a) With $Q_N$ denoting the orthogonal projector onto $N$, let $R := (Q_N)|_M \in B(M, N)$. Show that $\|Rm\| \geq \alpha \|m\|$ ($m \in M$), $\ker R^* = \{0\}$, and conclude that $R$ is boundedly invertible.

(b) Defining $P := R^{-1}Q_N \in B(H, H)$, show that $P$ is a projector with $\text{Im} P = M$.

(c) Using that $\langle Q_N u, n \rangle = \langle u, n \rangle$ ($u \in H, n \in N$), and $Q_NR^{-1} = I$, show that $\text{Im}(I - P) \subset N^\perp$. Show that $PN^\perp = \{0\}$, and conclude that $\text{Im}(I - P) = N^\perp$.

(d) Show that $\text{Im} P^* = N$, $\text{Im}(I - P^*) = M^\perp$. The operators $P$ and $P^*$ are called biorthogonal projectors. Note that $P = P^*$ if and only if $M = N$.

(e) Conversely, let $P \in B(H)$ be a projector. Show that $M := \text{Im} P$, $N = \text{Im}(I - P)^\perp$ satisfy (1) and (2).

40. Let $H$ be a Hilbert space. An operator $T \in B(H)$ is called nilpotent when for some $n \in \mathbb{N}$, $T^n = 0$.

(a) Show that the spectrum of a nilpotent operator is $\{0\}$.

(b) Construct a non-zero nilpotent operator $T \in B(\ell^2)$.

41. For a complex Hilbert space $H$, let $T \in B(H)$. On $\rho(T) = \mathbb{C} \setminus \sigma(T)$, define the resolvent $R_T(\lambda) := (\lambda I - T)^{-1} \in B(H)$.

(a) Show that for all $n \in \mathbb{N}$, $r_\sigma(T) \leq \|T^n\|^\frac{1}{2}$, and so $r_\sigma(T) \leq \liminf_{n \to \infty} \|T^n\|^\frac{1}{2}$.

(b) Show that

\[ R_T(\lambda) - R_T(\nu) = (\nu - \lambda)R_T(\lambda)R_T(\nu), \]

and for $\lambda \neq \nu$,

\[ \frac{R_T(\lambda) - R_T(\nu)}{\lambda - \nu} + R_T(\nu)^2 = (R_T(\nu) - R_T(\lambda))R_T(\nu). \]
(c) Given \( \eta \in B(H)' \), set \( f = \eta \circ R_T : \rho(T) \to \mathbb{C} \). Show that for \( \lambda \neq \nu \),

\[
\left| \frac{f(\lambda) - f(\nu)}{\lambda - \nu} + \eta(R_T(\nu)^2) \right| \leq \|\eta\| \|R_T(\nu)\| \|R_T(\nu) - R_T(\lambda)\|,
\]

and conclude that \( f \) is analytic.

(d) Show that for \( |\lambda| > \|T\| \), \( R_T(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1}T)^n \), and consequently,

\[
f(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} \eta((\lambda^{-1}T)^n).
\]

(3)

(e) Since \( f \) is analytic, complex function theory learns us that \( f \) has a unique Laurent expansion about the origin that is valid in every \( \lambda \in \rho(T) \), and thus that (3) is valid for all \( |\lambda| > r_\sigma(T) \).

Conclude that for \( |\lambda| > r_\sigma(T) \), \( \{((\lambda^{-1}T)^n\} \to 0 \), and thus that \( \{((\lambda^{-1}T)^n\} \subset B(H) \) is bounded.

(f) Show that for \( |\lambda| > r_\sigma(T) \), \( \limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq |\lambda| \), so that \( \limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq r_\sigma(T) \), and eventually

\[
\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r_\sigma(T).
\]

42. Let \( X \) and \( Y \) be normed linear spaces and \( T \in B(X,Y) \).

(a) Show that if in \( X \), \( \{x_n\} \to x \), then in \( Y \), \( \{Tx_n\} \to Tx \).

(b) Show that if, additionally, \( T \) is compact, then \( \{Tx_n\} \to Tx \).

(Hint: Assuming that the statement is wrong, conclude that there exists an \( \eta > 0 \) and a subsequence \( \{Tx_{n(m)}\} \) with \( \|Tx_{n(m)} - Tx\| > \eta \). Now derive a contradiction.)

43. Let \( X \) be an infinite dimensional normed linear space. Show that any set \( A \subset X \) that has non-empty interior is not compact.