Exercises Functional Analysis

April 21, 2020

1. Let (X, d) be a metric space and $E \subset X$ be compact. Show that for any $x \in X$, there exists a $e \in E$ with $d(x, E) := \inf_{y \in E} d(x, y) = d(x, e)$.

2. Let $X \subset \mathbb{R}^d$ be compact. Show that that for $1 \leq q \leq p \leq \infty$, $L^p(X) \subset L^q(X)$, and that the embedding is continuous.

3. Show that for $p \in [1, \infty]$, ℓ^p is complete.

4. Show that ℓ^∞ is not separable.

5. Let c_0 be the space of all sequences that converge to 0. Show that c_0 is a closed linear subspace of ℓ^∞. Is c_0 separable?

6. Show that $d(\{x_n\}, \{y_n\}) := \sum_{n=1}^\infty 2^{-n} \frac{|x_n - y_n|}{1+|x_n - y_n|}$ defines a metric on the linear space of real-valued sequences. Show there exists no norm $\| \, \|$ on this space such that $d(\{x_n\}, \{y_n\}) = \|\{x_n\} - \{y_n\}\|$.

7. (Gram-Schmidt) Let $\{v_i\}$ be linearly independent in an inner product space X. Show that $\{e_i\}$, defined by

$$e_i = v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, e_j \rangle}{\langle e_j, e_j \rangle} e_j.$$

   is orthogonal, and that $\text{span}\{e_j \colon 1 \leq j \leq i\} = \text{span}\{v_j \colon 1 \leq j \leq i\}$ for any $i \in \mathbb{N}$.

8. Show that there is no inner product on $C[0, 1]$ that induces the $L^1(0, 1)$-norm on that space.

9. Which of the following expressions defines an inner product on $C^2[0,1]$ (being the space of two times continuously differentiable real functions on $[0,1]$)

   - $\langle f, g\rangle = \int_0^1 f'(t)g'(t)\,dt$
   - $\langle f, g\rangle = \int_0^1 f(t)g(t) + f'(t)g'(t) + f''(t)g''(t)\,dt$

   and for those who do, do they give rise to a Hilbert space?

10. Show that $\|f\| := \max_{t\in[0,1]} |f(t)| + |f''(t)|$ defines a norm on $C^2[0,1]$, and that w.r.t. this norm, $C^2[0,1]$ is a Banach space.

11. Let V be the linear subspace of $C_{\mathbb{R}}[0,1]$ of functions f of type $f(x) = ax$ for some $a \in \mathbb{R}$. Let $\mathbb{1} : x \mapsto 1 \in C_{\mathbb{R}}[0,1]$. Compute $d(\mathbb{1}, V)$, and show that the minimal distance is attained for several $f \in V$.

12. Let V be the linear subspace of $C_{\mathbb{R}}[0,1]$ of functions f with $f(1) = 0$. Show that V is a Banach space. Let W its subspace consisting of functions g with $\int_0^1 g(x)\,dx = 0$. Show that W is a closed subspace. Let $f \in V$ be defined by $f(x) = 1 - x$. Compute the distance $d(f, W)$, and show that this distance is not attained.

13. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Hint: Compute the Fourier series of $f\colon x \mapsto x \in L^2_{\mathbb{C}}[-\pi, \pi]$, and use Parseval's theorem).

14. Let V be the linear space $C_{\mathbb{R}}[0,1]$. Define $L \in L(V,V)$ by $(Lf)(x) = xf(x)$.

    (a) Equip V with the L^∞-norm. Show that $L \in B(V,V)$, compute its is norm, and give an $f \in V$ with $\|Lf\|_{L^\infty[0,1]} = \|L\|_{B(V,V)}\|f\|_{L^\infty[0,1]}$.

    (b) Equip V with the L^2-norm. Show that $L \in B(V,V)$, compute its is norm, and show that for any $0 \neq f \in V$, $\|Lf\|_{L^2[0,1]} < \|L\|_{B(V,V)}\|f\|_{L^2[0,1]}$.

15. (Supplement to the proof of Thm. 3.54)

    (a) Show that $\mathrm{span}\{\cos^k x\colon 0 \leq k \leq n\} = \mathrm{span}\{\cos kx\colon 0 \leq k \leq n\}$ (Hint: Recall that Chebychev polynomial $T_n(t) := \cos(n \arccos(t))$ of degree n has a non-zero leading coefficient (2^{n-1} when $n \geq 1$), and use that $\cos nx = T_n(\cos x)$.)

(b) Show that $\operatorname{span}\{\cos kx \colon k \geq 0\}$ is dense in $C[0,\pi]$.

16. (Haar basis (Alfréd Haar 1909)) Let $\phi(x) := \begin{cases} 1 & x \in [0,1] \\ 0 & x \in \mathbb{R} \setminus [0,1] \end{cases}$, $\psi(x) := \begin{cases} 1 & x \in [0,\frac{1}{2}] \\ -1 & x \in (\frac{1}{2},1] \\ 0 & x \in \mathbb{R} \setminus [0,1] \end{cases}$, $\psi_{\ell,k}(x) := 2^{\ell/2}\psi(2^\ell x - k)$.

   (a) Show that $\Psi := \bigcup_{\ell=0}^{\infty}\{\psi_{\ell,k} \colon k = 0,\ldots,2^\ell - 1\} \cup \{\phi\}$ is a countable orthonormal collection in $L^2_{\mathbb{R}}[0,1]$.

   (b) Show that $\operatorname{span}(\bigcup_{\ell=0}^{N-1}\{\psi_{\ell,k} \colon k = 0,\ldots,2^\ell - 1\} \cup \{\phi\}) = \{f \in L^2_{\mathbb{R}}[0,1] \colon f|_{(k2^{-N},(k+1)2^{-N})} \text{ is constant } \forall k = 0,\ldots,2^N - 1\}$.

   (c) Show that Ψ is an orthonormal basis for $L^2_{\mathbb{R}}[0,1]$.

17. Let $X = \{x \in \ell^\infty \colon \#\operatorname{supp} x < \infty\}$. Define $T : X \to X \colon x \mapsto (x_1, x_2/2, x_3/3, \ldots)$. Show that $T \in B(X,X)$ and that it is bijective, but that T^{-1} is not bounded.

18. Let X, Y be Banach spaces, $T \in B(X,Y)$ be injective. Show that $T^{-1} : \operatorname{Im} T \to X$ is bounded iff $\operatorname{Im} T$ is closed in Y.

19. Let $X_1 = (X, \|\,\|_1)$ and $X_2 = (X, \|\,\|_2)$ be Banach spaces. Show that if for some constant C, $\|\,\|_1 \leq C\|\,\|_2$, then both norms are equivalent.

20. Let $Y = C_{\mathbb{R}}[0,1]$. Let X its linear subspace of the continuous differentiable functions on $[0,1]$. Let $T \colon X \to Y \colon u \mapsto u'$.

   (a) Show that T is not bounded.

   (b) Show that $\mathcal{G}(T)$ is closed (Hint: Let $X \supset \{u_n\} \to u$, and $\{u_n'\} \to w$. Show that $\int_0^t w(\tau)\,d\tau = u(t) - u(0)$.)
   Apply a theorem to conclude that X is not closed.

21. Let $X := C_{\mathbb{R}}[a,b]$, $I := f \mapsto \int_a^b f(x)\,dx \in X'$, and $\{Q_n\} \subset X'$ be a sequence of quadrature formulas of the form $Q_n f = \sum_{i=1}^{k(n)} w_i^{(n)} f(x_i^{(n)})$ for some $a \leq x_i^{(n)} < \cdots < x_{k(n)}^{(n)} \leq b$ and weights $w_1^{(n)}, \ldots, w_{k(n)}^{(n)} \in \mathbb{R}$.

   (a) Show that $\|Q_n\| = \sum_{i=1}^{k(n)} |w_i^{(n)}|$.

   (b) Show that if for any polynomial p, $\lim_{n\to\infty} Q_n(p) = I(p)$, and $\sup_n \|Q_n\| < \infty$, then for any $f \in C[a,b]$, $\lim_{n\to\infty} Q_n(f) = I(f)$.

(c) Show that both conditions from (21b) are also necessary conditions.

(For the sequence of $(n+1)$-point Newton-Cotes formulas, it holds that $\{\|Q_n\|\}$ is unbounded, whereas for the sequence of $(n+1)$-point Gauss-Legendre formulas, or the sequence of composite trapezoidal rules obtained by subdividing $[a,b]$ into n equal subintervals and by applying the trapezoidal rule to each of these subintervals, the sequence $\{\|Q_n\|\}$ is bounded, because $\sum_{i=1}^{k(n)} |w_i^{(n)}| = \sum_{i=1}^{k(n)} w_i^{(n)} = b-a$.)

22. For inner product spaces X, Y, let $T \in L(X,Y)$, $S \in L(Y,X)$ such that $\langle Tx, y\rangle_Y = \langle x, Sy\rangle_X$ for all $(x,y) \in X \times Y$. Show that $\mathcal{G}(T)$ is closed. Conclude that $T \in B(X,Y)$ when X and Y are Hilbert spaces.

23. Let Y be a subspace of a linear space X. On X define the equivalence relation $x \sim z$ whenever $x - z \in Y$. The *quotient space* X/Y is the linear space of equivalence classes $[x] := \{z : x \sim z\}$. In the following, let X be a normed linear space and Y a closed linear subspace.

    (a) Show that X/Y is a normed linear space with

    $$\|[x]\| := \inf\{\|x+y\| : y \in Y\}.$$

    (b) Show that $T \in L(X, X/Y)$ defined by $x \mapsto [x]$ is bounded, and $\|T\| \leq 1$.

    (c) If X is a Banach space, show that X/Y is a Banach space. Hint: Let $\{u_i\}$ be a Cauchy sequence in X/Y. Being a Cauchy sequence, show that it has a subsequence $\{u_{n_i}\}$ with $\|u_{n_{i+1}} - u_{n_i}\| < 2^{-i}$. Show inductively existence of a sequence $\{x_{n_i}\} \subset X$ with $[x_{n_i}] = u_{n_i}$ and $\|x_{n_{i+1}} - x_{n_i}\| < 2^{-i}$. Show that $\{x_{n_i}\}$, and finally $\{u_i\}$ are convergent.

24. Let X be a Banach space, Y be a normed linear space, and let $\{L_n\}$ be a sequence in $B(X,Y)$ such that for any $x \in X$, $Lx := \lim_{n\to\infty} L_n x$ exists.

    (a) Show that $L \in B(X,Y)$.

    (b) By means of a counterexample, show that not necessarily $\lim_{n\to\infty} L_n = L$.

25. In [Rynne & Youngson, Exer. 5.8] it was shown that there exists a non-zero $f \in (\ell^\infty)'$ with $f(e_n) = 0$ for all $n \geq 1$, confirming the fact that $\ell^1 \to (\ell^\infty)' : \vec{a} \mapsto (\vec{b} \mapsto \sum_{i=1}^\infty a_i b_i)$ is not surjective (we know that the image is $(c_0)'$). It appears that an explicit construction of such an f is not possible. The following comes a bit 'closer' to an explicit construction:

    Let c be the space of $\vec{b} \in \ell_\infty$ for which $\lim_{i\to\infty} b_i$ exists.

    (a) Show that c is a separable Banach space.

    (b) *Construct* a non-zero $f \in c'$ with $f(e_n) = 0$ for all $n \geq 1$.

    (c) Conclude again that there *exists* a non-zero $f \in (\ell^\infty)'$ with $f(e_n) = 0$ for all $n \geq 1$.

26. Let Y be a subspace of a linear space X. The *codimension* of Y is defined as $\operatorname{codim} Y = \dim X/Y$. When $\operatorname{codim} Y = 1$, any element of X/Y is called a a hyperplane parallel to Y.

    (a) Show that for any $0 \neq f \in X^*(:= L(X, \mathbb{F}))$ and $c \in \mathbb{F}$, $\{x \in X : f(x) = c\}$ is a *hyperplane* parallel to $\ker f$. It determines the half-spaces $\{x \in X : f(x) \leq c\}$ and $\{x \in X : f(x) \geq c\}$.

    (b) Let X be a normed linear space, and for some $r > 0$, let $x_0 \in \{x \in X : \|x\| = r\}$. Show that there exists a hyperplane $H \ni x_0$ such that $\{x \in X : \|x\| \leq r\}$ entirely lies in one of the two hyperplanes determined by H.

27. It is known that $T : L^\infty[0,1] \to (L^1[0,1])' : v \mapsto (u \mapsto \int_0^1 v(x)u(x)\,dx)$ is a linear (isometric) isomorphism.

    (a) On $C[0,1] \subset L^\infty[0,1]$, consider $\delta : v \mapsto v(0)$. Show that $\delta \in (C[0,1])'$.

    (b) Show that $\nexists u \in L^1[0,1]$ such that $\delta(v) = \int_0^1 v(x)u(x)\,dx$ $(v \in C[0,1])$.

    (c) Show that $\exists f \in L^\infty[0,1]'$ for which $\nexists u \in L^1[0,1]$ such that $f(v) = \int_0^1 v(x)u(x)\,dx$ $(v \in L^\infty[0,1])$.

    (d) Assume that $L^1[0,1]$ is reflexive, i.e. that the linear isometry $J_{L^1[0,1]} : L^1[0,1] \to L^1[0,1]''$ is surjective. Show that in this case for any $f \in L^\infty[0,1]'$ and $v \in L^\infty[0,1]$, for $w = J^{-1}_{L^1[0,1]}(T')^{-1}g$ it holds that $f(v) = \int_0^1 v(x)w(x)\,dx$, and conclude a contradiction.

(e) Show that $L^\infty[0,1]$ is not reflexive.

28. Let W be a finite dimensional linear subspace of a normed linear space X.

    (a) Show that W is closed.

    (b) Show that for any $x \in X$, there exists a $w \in W$ with $\|x - w\| = \inf_{v \in W} \|x - v\|$. (Hint: Note that it is sufficient to consider v with $\|v\| \leq 2\|x\|$.)

29. Consider the odd function f, on $[0, \pi]$ defined by $f(x) = x(\pi - x)$. Compute $\|f\|^2_{L^2[-\pi,\pi]}$, and with that, $\sum_{n=0}^\infty \frac{1}{(2n+1)^6}$, and finally $\sum_{n=0}^\infty \frac{1}{n^6}$.

30. Let Y be a closed linear subspace of a normed space X. Show that for $x \in X \setminus Y$ there exists a $f \in X'$ with $f(x) = 1$ and $f(Y) = 0$.
    (This property is used on many places in [Rynne & Youngson], for example in the proof of Thm. 5.43.)

31. Let X and Y be finite dimensional linear spaces equipped with bases $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Psi = \{\psi_1, \ldots, \psi_n\}$, respectively, and let $T \in L(X,Y) = B(X,Y)$.

    (a) When $x = \sum_{j=1}^m x_j \phi_j$, show that $Tx = \sum_{i=1}^n y_i \psi_i$ where $\vec{y} = \mathbf{T}\vec{x}$ and $\mathbf{T} = [\psi_i'(T\phi_j)]_{1\leq i\leq n, 1\leq j\leq m} \in \mathbb{F}^{n\times m}$, and $\Psi' = \{\psi_1', \ldots, \psi_n'\}$ is the basis for Y' that is dual to Ψ. The matrix $\mathbf{T}$ is the *representation* of $T \in L(X,Y)$ equipped with Φ or Ψ, respectively.

    (b) Show that $\{J_X\phi_1, \ldots, J_X\phi_m\}$ is a basis for X'' that is dual to $\Phi' = \{\phi_1', \ldots, \phi_m'\}$, being the basis for X' that is dual to Φ.

    (c) Show that the representation of $T' \in L(Y', X')$, equipped with Ψ' and Φ', is given by the transpose $\mathbf{T}^\top$.

32. (Part of the *closed range theorem*) Let X and Y be normed linear spaces, and $T \in B(X,Y)$.

    (a) Show that $\operatorname{Im} T$ is closed if and only if $\operatorname{Im} T = {}^\circ \ker T'$.

    (b) Now let X be a Banach space. Show that if $\inf_{0\neq x\in X} \frac{\|Tx\|_Y}{\|x\|_X} > 0$ and $\ker T' = \{0\}$, then T is bijective, and even boundedly invertible.

33. Let X be a normed linear space, and U a one-dimensional linear subspace. Show that there exists a projection P on X with $\operatorname{Im} P = U$ and $\|P\| = 1$.

34. (a) Let H be a Hilbert space, and let $T_H : H \to H'$ be the Riesz operator, i.e. $(T_H y)(x) = \langle x, y \rangle_H$. Show that for $W \subset H$, $W^0 = T_H W^\perp$, and ${}^\circ(T_H W) = W^\perp$.

    (b) For Hilbert spaces H and K, and $T \in B(H, K)$, by using (34a) and [Rynne & Youngson, Lemma 5.52] show that $\ker T^* = (\operatorname{Im} T)^\perp$, and $\ker T = (\operatorname{Im} T^*)^\perp$.

35. Let X and Y be finite dimensional Hilbert spaces equipped with bases $\Phi = \{\phi_1, \ldots, \phi_m\}$ and $\Psi = \{\psi_1, \ldots, \psi_n\}$, respectively, and let $T \in L(X, Y) = B(X, Y)$.

    (a) Setting $\mathcal{F}_X \colon \mathbb{F}^m \to X \colon \vec{x} \mapsto \sum_{j=1}^m x_j \phi_j$, and analogously $\mathcal{F}_Y \colon \mathbb{F}^n \to Y$, show that the matrix $\mathbf{T}$ that represents T is given by $\mathcal{F}_Y^{-1} T \mathcal{F}_X$.

    (b) Equipping $\mathbb{F}^m$ with $\langle \vec{x}, \vec{y} \rangle_{\mathbb{F}^m} := \sum_{i=1}^m x_i \bar{y}_i$, show that Φ is an orthonormal basis if and only if $\mathcal{F}_X^{-1} = \mathcal{F}_X^*$. (Hint: Expand $\langle \mathcal{F}_X \vec{x}, \mathcal{F}_X \vec{y} \rangle_X$.)

    (c) Show that if Ψ is orthonormal, then $\langle \mathbf{T}\vec{x}, \vec{y} \rangle_{\mathbb{F}^n} = \langle T \mathcal{F}_X \vec{x}, \mathcal{F}_Y \vec{y} \rangle_Y$ $(\vec{x} \in \mathbb{F}^m, \vec{y} \in \mathbb{F}^n)$.

    (d) Let Φ and Ψ be orthonormal.

        i. Show that the representation of T^* is $\mathbf{T}^*$.

        ii. Show that $T^* = T^{-1}$ if and only if $\mathbf{T}^* = \mathbf{T}^{-1}$.

    (e) Let $X = Y$, $\Phi = \Psi$ be orthonormal. Show that $T^* = T$ if and only if $\mathbf{T}^* = \mathbf{T}$, and $T^*T = TT^*$ if and only if $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^*$.

36. For a Banach space X, let $A, B \in B(X)$.

    (a) Show that if AB is boundedly invertible, and A and B commute, then both A and B are boundedly invertible (this is needed for showing the fourth $\Longleftrightarrow$-implication in the proof of [Rynne & Youngson, Thm 6.39])

    (b) Give an example showing that the condition that A and B commute cannot be dropped.

37. Give an example showing that [Rynne & Youngson, Lemma 6.29] is generally not valid in a real inner product space.

38. Let H be a complex Hilbert space and $T \in B(H)$. Show that if $\langle Tx, x\rangle$ is real for all $x \in H$, then T is self-adjoint.

39. Let H be a Hilbert space. Let M, N be closed linear subspaces of a Hilbert space H such that

$$\alpha := \inf_{0\neq m\in M} \sup_{0\neq n\in N} \frac{|\langle m, n\rangle|}{\|m\|\|n\|} > 0, \tag{1}$$

$$\forall 0 \neq n \in N,\ \exists m \in M \text{ with } \langle m, n\rangle \neq 0. \tag{2}$$

   (a) With Q_N denoting the orthogonal projector onto N, let $R := (Q_N)|_M \in B(M, N)$. Show that $\|Rm\| \geq \alpha\|m\|$ $(m \in M)$, $\ker R^* = \{0\}$, and conclude that R is boundedly invertible.

   (b) Defining $P := R^{-1}Q_N \in B(H, H)$, show that P is a projector with $\operatorname{Im} P = M$.

   (c) Using that $\langle Q_N u, n\rangle = \langle u, n\rangle$ $(u \in H,\ n \in N)$, and $Q_N R^{-1} = I$, show that $\operatorname{Im}(I - P) \subset N^\perp$. Show that $PN^\perp = \{0\}$, and conclude that $\operatorname{Im}(I - P) = N^\perp$.

   (d) Show that $\operatorname{Im} P^* = N$, $\operatorname{Im}(I - P^*) = M^\perp$. The operators P and P^* are called *biorthogonal* projectors. Note that $P = P^*$ if and only if $M = N$.

   (e) Conversely, let $P \in B(H)$ be a projector. Show that $M := \operatorname{Im} P$, $N = \operatorname{Im}(I - P)^\perp$ are closed linear subspaces that satisfy (1) and (2).

40. Let H be a Hilbert space. An operator $T \in B(H)$ is called nilpotent when for some $n \in \mathbb{N}$, $T^n = 0$.

   (a) Show that the spectrum of a *nilpotent* operator is $\{0\}$.

   (b) Construct a non-zero nilpotent operator $T \in B(\ell^2)$.

41. For a complex Hilbert space H, let $T \in B(H)$. On $\rho(T) = \mathbb{C} \setminus \sigma(T)$, define the *resolvent* $R_T(\lambda) := (\lambda I - T)^{-1} \in B(H)$.

   (a) Show that for all $n \in \mathbb{N}$, $r_\sigma(T) \leq \|T^n\|^{\frac{1}{n}}$, and so $r_\sigma(T) \leq \liminf_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

(b) Show that

$$R_T(\lambda) - R_T(\nu) = (\nu - \lambda) R_T(\lambda) R_T(\nu),$$

and for $\lambda \neq \nu$,

$$\frac{R_T(\lambda) - R_T(\nu)}{\lambda - \nu} + R_T(\nu)^2 = (R_T(\nu) - R_T(\lambda)) R_T(\nu).$$

(c) Given $\eta \in B(H)'$, set $f = \eta \circ R_T \colon \rho(T) \to \mathbb{C}$. Show that for $\lambda \neq \nu$,

$$\Big| \frac{f(\lambda) - f(\nu)}{\lambda - \nu} + \eta(R_T(\nu)^2) \Big| \leq \|\eta\| \|R_T(\nu)\| \|R_T(\nu) - R_T(\lambda)\|,$$

and conclude that f is analytic.

(d) Show that for $|\lambda| > \|T\|$, $R_T(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1} T)^n$, and consequently,

$$f(\lambda) = \lambda^{-1} \sum_{n=0}^{\infty} \eta((\lambda^{-1} T)^n). \tag{3}$$

(e) Since f is analytic, complex function theory learns us that f has a unique Laurent expansion about the origin that is valid in every $\lambda \in \rho(T)$, and thus that (3) is valid for all $|\lambda| > r_\sigma(T)$.

Conclude that for $|\lambda| > r_\sigma(T)$, $\{(\lambda^{-1} T)^n\} \rightharpoonup 0$, and thus that $\{(\lambda^{-1} T)^n\} \subset B(H)$ is bounded.

(f) Show that for $|\lambda| > r_\sigma(T)$, $\limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq |\lambda|$, so that $\limsup_{n \to \infty} \|T^n\|^{\frac{1}{n}} \leq r_\sigma(T)$, and eventually

$$\lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = r_\sigma(T).$$

42. Let X and Y be normed linear spaces and $T \in B(X, Y)$.

(a) Show that if in X, $\{x_n\} \rightharpoonup x$, then in Y, $\{T x_n\} \rightharpoonup T x$.

(b) Show that if, additionally, T is compact, then $\{T x_n\} \to T x$. (Hint: Assuming that the statement is wrong, conclude that there exists an $\eta > 0$ and a subsequence $\{T x_{n(m)}\}$ with $\|T x_{n(m)} - T x\| > \eta$. Now derive a contradiction.)

43. Let X be an infinite dimensional normed linear space. Show that any set $A \subset X$ that has non-empty interior is not compact.

44. (Existence of a minimizer) Let V be a normed space over $\mathbb{R}$, $K \subseteq V$. An $f\colon K \to \mathbb{R}$ is called *weakly lower semi-continuous* (w.l.s.c) if $\{v_n\} \subseteq K$ and $v_n \rightharpoonup v \in K$ imply $f(v) \leq \liminf_{n\to\infty} f(v_n)$.
A set $K \subseteq V$ is called *weakly closed* if $\{v_n\} \subseteq K$ and $v_n \rightharpoonup v$ imply $v \in K$.

(a) Show that the norm on V is w.l.s.c. (Hint: given $v \in V$, select a suitable $\ell \in V'$).

(b) If V is reflexive, $K \subseteq V$ is bounded and weakly closed, and $f\colon K \to \mathbb{R}$ is w.l.s.c, show that there exists a $u \in K$ with $f(u) = \inf_{v\in K} f(v)$. (Hint: Consider $\{v_n\} \subset K$ with $\lim_{n\to\infty} f(v_n) = \inf_{v\in K} f(v)$.)

(c) Show that in the previous item the condition of K being bounded can be omitted when f additionally satisfies $f(v) \to \infty$ for $\|v\| \to \infty$. (Hint: Pick any $v_0 \in K$ and consider $K_0 := \{v \in K : f(v) \leq f(v_0)\}$.)