Additional exercises with Numerical Methods for Stationary PDEs, 2016

February 16, 2016

1. (a) With $n \in \mathbb{N}$, $h = 1/n$, $x_i = ih$ $(i = 0, \ldots, n)$, show that $\{\phi_1, \ldots, \phi_n\}$ illustrated in Figure 1 is a basis for

$$S_h := \{v \in C([0,1], \mathbb{R}) : v(0) = 0,\ v|_{[x_{i-1}, x_i]} \in P_1 (i = 1, \ldots, n)\}.$$

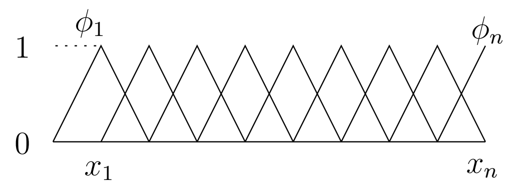


Figure 1: "Nodal" basis with zero boundary condition at the left end.

   (b) With $a(u, v) := \int_0^1 u'(x)v'(x)dx$ compute the stiffness matrix w.r.t. $\{\phi_1, \ldots, \phi_n\}$.

2. (*barycentric coordinates*) Let $\{z_1, \ldots, z_{n+1}\} \subset \mathbb{R}^n$ be $n+1$ points not on a hyperplane. For $i \in \{1, \ldots, n+1\}$, let $\lambda_i \in P_1$ be defined by $\lambda_i(z_j) = \delta_{ij}$. Show that for any $n \in \mathbb{N}$, and any $p \in P_1$, $\sum_{i=1}^{n+1} p(z_i)\lambda_i = p$.

   Note that in particular $\sum_{i=1}^{n+1} \lambda_i(x) = 1$ for any $x \in \mathbb{R}^n$. The $(n+1)$ tuple $(\lambda_1(x), \ldots, \lambda_{n+1}(x))$ are called the barycentric coordinates of x w.r.t. $\{z_1, \ldots, z_{n+1}\}$.

   The set $\{x \in \mathbb{R}^n : \lambda_i(x) \geq 0, \forall i\}$ is called an n-simplex. What is it for $n = 1, 2, 3$?

3. (Friedrich's inequality) Show that on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $p \in [1, \infty]$,
$$\|v - \bar{v}\|_{W_p^1(\Omega)} \leq C(\Omega)|v|_{W_p^1(\Omega)},$$
where $\bar{v} = \frac{1}{|\Omega|}\int_\Omega v(x)dx$.

4. Let $p \in [1, \infty]$ and $m \in \mathbb{N}_0$ such that $m + 1 > 2/p$ or $m + 1 \geq 2$ if $p = 1$. For a triangle K, and $x_1, \ldots, x_N \subset K$, invariant under permutations of the barycentric coordinates, weights $w_1, \ldots, w_N$ independent of K, let $Q_K(g) = \mathrm{vol}(K)\sum_{i=1}^N w_i g(x_i)$ be a quadrature formula that is exact on $P_m(K)$. Show that
$$|\int_K g(x)dx - Q_K(g)| \leq C\mathrm{vol}(K)^{1-1/p}\mathrm{diam}(K)^{m+1}|g|_{W_p^{m+1}(K)},$$
where C is a constant that is independent of K.

5. For some domain $\Omega \subset \mathbb{R}^n$, and with
$$Lu := -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu,$$
where $a_{ij} = a_{ji} \in L_\infty(\Omega)$, $b_i \in W_\infty^1(\Omega)$, $c \in L_\infty(\Omega)$, for $f \in L_2(\Omega)$ consider the bvp
$$\begin{cases} Lu = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$
   (a) Show that the corresponding variational formulation is given by the problem of finding $u \in H_0^1(\Omega)$ such that
   $$a(u, v) = \int_\Omega fvdx \quad (v \in H_0^1(\Omega)), \tag{1}$$
   where
   $$a(u, v) = \int_\Omega (\mathbf{A}\nabla u) \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v + cuvdx$$
   where $\mathbf{A}(x) = (a_{ij}(x))_{1 \leq i,j \leq n}$, $\mathbf{b}(x) = (b_1(x), \ldots, b_n(x)^T$.
   (b) Assuming that $\xi \cdot (\mathbf{A}(x)\xi) \geq C\|\xi\|^2$ for some absolute constant $C > 0$ independent of $\xi \in \mathbb{R}^n$ and $x \in \Omega$ (*uniform ellipticity*), $\mathbf{b} = 0$ and $c = 0$, show that (1) has a unique solution with $\|u\|_{H^1(\Omega)} \lesssim \|f\|_{L_2(\Omega)}$.
   (c) Show the same result when the assumptions $\mathbf{b} = 0$ and $c = 0$ are replaced by $-\frac{1}{2}\mathrm{div}\mathbf{b}(x) + c(x) \geq 0$ on Ω. (Hint: Show that for $u \in H_0^1(\Omega)$, $\int_\Omega (\mathbf{b} \cdot \nabla u)udx = \frac{1}{2}\int_\Omega \mathbf{b} \cdot \nabla(u^2)dx = -\frac{1}{2}\int_\Omega u^2 \mathrm{div}\mathbf{b}dx$)
   (d) Reconsidering (5a), what are the *natural boundary conditions* corresponding to the differential operator L?

6. Consider the following variational formulation of the Biharmonic problem $\begin{cases} \triangle^2 u = f & \text{on } \Omega \subset \mathbb{R}^2 \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$: Find $u \in H_0^2(\Omega)$ s.t.

$$a(u,v) := \int_\Omega \triangle u \triangle v dx = \int_\Omega f v dx =: F(v) \quad (v \in H_0^2(\Omega)). \tag{2}$$

Show that for $v \in \mathcal{D}(\Omega)$, $\int_\Omega (\frac{\partial^2 v}{\partial x_1 \partial x_2})^2 dx = \int_\Omega \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} dx$, and with that, that $|v|_{H^2}^2 \leq \|\triangle v\|_{L_2(\Omega)}^2$ $(v \in H_0^2(\Omega))$. Assuming that say $f \in L_2(\Omega)$, using Poincaré's inequality conclude that (2) has a unique solution u with $\|u\|_{H^2(\Omega)} \lesssim \|f\|_{L_2(\Omega)}$.

7. Let $\Phi : (r, \phi) \mapsto (x, y) = (r\cos\phi, r\sin\phi)$.

   (a) With $\tilde{u} = u \circ \Phi$, prove that

   $$(-\triangle u) \circ \Phi = -\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \tilde{u}}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \phi^2}.$$

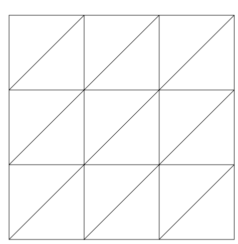
   (b) For $\alpha > \frac{1}{2}$, let $\Omega_\alpha = \mathrm{Im}\Phi|_{(0,1)\times(0,\frac{\pi}{\alpha})}$. With f given by $(f \circ \Phi)(r,\phi) = 4(\alpha+1)r^\alpha \sin\alpha\phi$, show that u given by $(u\circ\Phi)(r,\phi) = (1-r^2)r^\alpha \sin\alpha\phi$ solves

   $$\begin{cases} -\triangle u &= f & \text{on } \Omega_\alpha, \\ u &= 0 & \text{on } \partial\Omega_\alpha. \end{cases}$$

   (c) Sketch Ω_α. Show that $f \in L_2(\Omega_\alpha)$, but that $u \notin H^2(\Omega_\alpha)$ when $\alpha < 1$, i.e., when Ω_α is not convex.

8. ("superconvergence") Consider the bvp $-u'' = f$ on $(0,1)$, $u(0) = u(1) = 0$ in variational form. Let u_h denote the finite element Galerkin approximation using continuous piecewise linears, and let $I_h u$ denote the interpolant of u in this finite element space. Show that $u_h = I_h u$ by showing that $a(u_h - I_h u, w_h) = 0$ for any function w_h in the f.e. space (where $a(w,v) = \int_0^1 w'v'dx$)

9. Consider the space of continuous piecewise linears w.r.t. the triangulation of $\Omega = (0,1)^2$ illustrated below.



Use element mass and stiffness matrices to construct mass and stiffness matrices (where $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$) for both homogeneous Dirichlet boundary conditions and homogeneous Neumann boundary conditions.

10. Consider the continuous piecewise linear finite element discretization w.r.t. $n+1$ equal sized subintervals of $-u''(x) = f(x)$ on $(0,1)$, $u(0) = u(1) = 0$ in standard variational form. Show that the stiffness matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ w.r.t. the nodal basis with a lexicographical numbering of the grid points is given by

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ \vdots & & & \\ & 0 & -1 & 2 \end{bmatrix},$$

where $h = 1/(n+1)$. Show that an orthonormal basis of eigenvectors of $\mathbf{A}$ is given by $\{\mathbf{v}^{(k)} = [\sqrt{2h}\sin(jk\pi h)]_{j=1,\dots,n} : k = 1,\dots,n\}$ with corresponding eigenvalues $\lambda^{(k)} = 4h^{-1}\sin^2(hk\pi/2)$.

11. Show that on shape regular triangles T,

$$\|v\|_{L_2(\partial T)} \lesssim h_T^{-1/2}\|v\|_{L_2(T)} + h_T^{1/2}|v|_{H^1(T)} \quad (v \in H^1(T))$$

12. For a triangle T, let $\{\phi_1, \phi_2, \phi_3\} \subset P_1(T)$ be the standard nodal basis corresponding to evaluation in the vertices. Let $\{\psi_1, \psi_2, \psi_3\} \subset P_1(T)$ let be the collection that is dual to $\{\phi_1, \phi_2, \phi_3\}$, i.e., $\langle \psi_i, \phi_j\rangle_{L_2(T)} = \delta_{ij}$. Express all ψ_i as a linear combinations of ϕ_1, ϕ_2, ϕ_3.

13. Show that any monotone, bounded function on $(0,1)$ can be approximated in L_∞-norm at order $\mathcal{O}(N^{-1})$ with piecewise constant, free knot approximation with N knots.

14. Let τ_0 be a conforming initial partition. In this exercise, we consider exclusively partitions τ that are created from τ_0 by *newest vertex bisection*.
    (a) Show all τ are uniformly shape regular (only dependent on τ_0). Hint: Show that all descendants of a $T \in \tau_0$ fall into 4 similarity classes.
    (b) Assume that τ_0 satisfies the matching condition. Show that any uniform refinement τ of τ_0 (meaning that all its triangles have the same generation) is conforming.
    (c) Let τ_0 satisfy the matching condition, τ be conforming, and $T, T' \in \tau$. Show that if T' contains the refinement edge of T, then either
        - $\mathrm{gen}(T') = \mathrm{gen}(T)$, and T and T' share their refinement edge, or
        - $\mathrm{gen}(T') = \mathrm{gen}(T) - 1$, and T shares its refinement edge with one of both children of T'.
    (d) Let τ_0 satisfies the matching condition, τ be conforming, and $T \in \tau$. Show that $\texttt{refine}(\tau, T)$ terminates.

15. (Inverse inequalities) Show that for uniformly shape regular triangles T, and fixed $k \in \mathbb{N}$,

$$\|\Delta z\|_{L_2(T)} \lesssim h_T^{-1}\|z\|_{H^1(T)} \quad (z \in P_k(T)),$$
$$\|\nabla z\|_{L_2(e)^2} \lesssim h_T^{-\frac{1}{2}}\|z\|_{H^1(T)} \quad (z \in P_k(T)).$$

16. For τ_0 being a conforming initial partition, let $\mathbb{T}$ denote the set of all conforming partitions that can be created from $\mathcal{T}_0$ by newest vertex bisection. Given $\mathcal{T}', \mathcal{T}'' \in \mathbb{T}$, let $\mathcal{T}$ be the partition defined as the set of leaves of the tree defined as the union of the binary trees underlying $\mathcal{T}'$ and $\mathcal{T}''$. Show that $\mathcal{T} \in \mathbb{T}$, and that it is the smallest common refinement of $\mathcal{T}'$ and $\mathcal{T}''$. Furthermore show that

$$\#\mathcal{T} + \#\mathcal{T}_0 \leq \#\mathcal{T}' + \#\mathcal{T}''.$$

(Hint: Note that it is sufficient to show the last inequality for $\#\mathcal{T}_0 = 1$).

17. Consider the model problem $\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ with $\Omega = (0,1)^2$ and $f = 1$. Compute the continuous piecewise linear finite element solution u_0 w.r.t. the triangulation $\mathcal{T}_0$ of Ω into 4 triangles by drawing the 2 diagonals.

    With the center of Ω being the newest vertex of all 4 triangles of $\mathcal{T}_0$, construct $\mathcal{T}_1$ by applying two uniform refinements, resulting in a triangulation that has 16 triangles. Compute the continuous piecewise linear finite element solution u_1 w.r.t. the triangulation $\mathcal{T}_1$. Compare u_0 and u_1.

    The bulk chasing parameter θ can be chosen such that u_0 and u_1 will be in the sequence of finite element solutions produced by the AFEM. Is there a contradiction with the convergence result we have seen for the AFEM? Explain your answer.

18. Given $k \in \mathbb{R}$, find the variational formulation of $-u''+ku'+u = f$ on $(0,1)$ for (a) homogeneous Dirichlet boundary conditions, and (b) Neumann boundary conditions. Show that in the first case the bilinear form is coercive (Hint: write $vv' = \frac{1}{2}(v^2)'$), and give a k for which in the second case the bilinear form is not coercive.