Summary: HB (extension of lin functionals)

Corol: Let X normed lin. sp. W lin subspace, $f_W \in W'$.
Then $\exists$ extension $f_X \in X'$ with $\|f_X\|_{X'} = \|f_W\|_{W'}$.

Corol: With X & W as above, let $x \in X$ with $\delta := \inf_{w \in W} \|x-w\| > 0$.
Then $\exists f \in X'$ with $\|f\|_{X'} = 1$, $f(W) = 0$,
$f(x) = \delta$.

Corol:

a) $\forall x \in X \setminus \{0\}$ $\exists f \in X'$ with $f(x) = \|x\|$ and $\|f\|_{X'} = 1$

b) $\forall x \in X$, $\|x\| = \sup\{|f(x)| : \|f\|_{X'} \leq 1\}$

c) $\forall x \neq y \in X$, $\exists f \in X'$ with $f(x) \neq f(y)$.

compare $\|f\|_{X'} = \sup\{|f(x)| : \|x\|_X \leq 1\}$

§5.5 Second dual, reflexive spaces, and dual operators

For X normed lin space, X' and so $X'' = (X')'$ (second dual) are Banach.

Is $X'' \cong X$? (isometric isomorphic)

<u>lemma</u> $\forall x \in X$ def $F_x : X' \to \mathbb{F}$ by $F_x(f) = f(x)$.

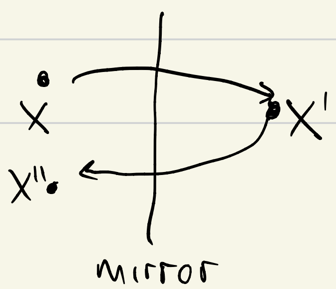
Then $F_x \in X''$ and $\|F_x\|_{X''} = \|x\|_X$.

<u>proof</u> F_x is linear. $|F_x(f)| = |f(x)| \le \|f\|_{X'} \|x\|_X$, so $F_x \in X''$ with $\|F_x\|_{X''} \le \|x\|_X$. Even

$$\|x\|_X \overset{HB}{=} \sup_{\{f \in X' : \|f\|_{X'} \le 1\}} |f(x)| = \sup_{\{f \in X' : \|f\|_{X'} \le 1\}} |F_x(f)| = \|F_x\|_{X''}$$

<u>Def</u> $J_X : X \to X''$ by $J_X(x) = F_x$. Is linear(?) and isometry.

If $\operatorname{ran} J_X = X''$, i.e. J_X is isometric isomorphism, then X is called <u>reflexive</u>.



<u>remark</u> When X isn't Banach, then not reflexive.

<u>Remark</u> If $\dim X < \infty$, then reflexive. Indeed, we know

$$\dim X \underset{\substack{\uparrow \\ \text{(using dual basis)}}}{=} \dim X' = \dim X'', \text{ and } \dim(\operatorname{ran} J_X) = \dim X$$

because isometry.

[Hilb $\subsetneq$ refl. norm space $\subsetneq$ Banach $\subseteq$ normed space]

<u>Thm</u> Any Hilbert space H is reflexive.

<u>proof</u> Recall Riesz map $T_H : H \to H'$ and sim. $T_{H'} : H' \to H''$ by

(1) $\underbrace{(T_H y)}_{f \in H'}(x) = \langle x, y \rangle_H$, $(T_{H'} f)(g) = \langle g, f \rangle_{H'} = \langle T_H^{-1} f, T_H^{-1} g \rangle_H$ (2)

T_H & $T_{H'}$ are conjugate lin. isometric isomorphisms. So any $\varphi \in H''$ is of the form $\varphi = T_{H'} T_H y$. Now

$$\varphi(g) = (T_{H'} T_H y)(g) \overset{(2)}{=} \langle y, T_H^{-1} g \rangle_H \overset{(1)}{=} g(y)$$

or $\varphi = J_H(y)$. In other words, $\boxed{J_H = T_{H'} T_H}$ and so surjective. □

<u>Thm</u> For $p \in (1, \infty)$ is ℓ^p reflexive (exerc.) (generalization above prop.)

(use that for $p \in [1, \infty)$ $(\ell^p)' \simeq \ell^q$)

Thm Let X Banach. Then X refl. $\Leftrightarrow$ X' refl.

proof. $\Rightarrow$

X refl. means $\forall \psi \in X'' \; \exists x \in X$ s.t. $\psi = J_X(x)$ i.e. $\psi(f) = f(x) \; \forall f \in X'$

X'' refl means $\forall \rho \in X''' \; \exists f \in X'$ s.t. $\rho(\psi) = \underline{\psi(f) \quad \forall \psi \in X''}$

$$\Leftrightarrow \rho(J_X(x)) = \underbrace{J_X(x)(f)}_{= f(x)} \quad \forall x \in X$$

which holds true for $f = \rho \circ J_X$

$\Leftarrow$ Suppose X is not refl. Then $\exists \; \psi \in X'' \setminus \operatorname{ran} J_X$.

Since X is Banach, $\operatorname{ran} J_X$ is Banach, so closed in X''.

So $\exists \rho \in X'''$ with $\rho(\operatorname{ran} J_X) = 0$ & $\rho(\psi) \neq 0$ (HB)

$\rho = J_{X'}(f)$ for <u>some</u> $f \in X'$ (because X' refl.)

$\forall x \in X$ it holds $f(x) \underset{\text{def } J_X}{=} (J_X(x))(f) \underset{\text{def } J_{X'}}{=} (J_{X'} f)(J_X(x))$

$$= \rho(J_X(x)) = 0$$

$\Rightarrow f = 0 \Rightarrow \rho = 0$ $\lightning$ $\square$

<u>Thm</u> Let Y closed lin subsp of refl. X. Then Y refl.

Proof Any elem from Y' is of the form $f|_Y$ for some $f \in X'$.
(indeed such $f|_Y \in Y'$, and given an element from Y', you can extend it to an elem from X' (HB))

So to show that $\forall g \in Y'' \; \exists y \in Y$ s.t

$$g(f|_Y) = f(y) \quad (\forall f \in X')$$

Let $g \in Y''$. First we extend $g \in Y''$ to some $\bar{g} \in X''$:

Def $\bar{g}(f) = g(f|_Y)$. Then $|\bar{g}(f)| \leq \|g\|_{Y''} \|f|_Y\|_{Y'} \leq \|g\|_{Y''} \|f\|_{X'}$

so $\bar{g} \in X''$. X refl, so $\exists x \in X$ with $\bar{g}(f) = f(x) \;\; \forall f \in X'$

Remains to show $x \in Y$. Let $x \notin Y$. Y closed, so $\exists f \in X'$ with $f(Y) = 0$ & $f(x) = 1$ (HB).

$$1 = f(x) = \bar{g}(f) = g(\underbrace{f|_Y}_{=0}) = 0 \quad \text{↯} \quad \square$$

Def Let $\emptyset \neq W \subseteq X$, $\emptyset \neq Z \subseteq X'$. Annihilators of W or Z are

$$W^0 = \{ f \in X' : f(x) = 0 \quad \forall x \in W \}$$
$$^0 Z = \{ x \in X : f(x) = 0 \quad \forall f \in Z \}$$

(For X Hilbert is f(x) of the form $\langle x, y \rangle$, so annihilators generalize orth. complements)

Lemma Let $\emptyset \neq W_1 \subseteq W_2 \subseteq X$, $\emptyset = Z_1 \subseteq Z_2 \subseteq X'$. Then

a) $W_2^0 \subseteq W_1^0$, $^0 Z_2 \subseteq {^0 Z_1}$

b) $W_1 \subseteq {^0(W_1^0)}$ and $Z_1 \subseteq (^0 Z_1)^0$

c) W_1^0 & $^0 Z_1$ are closed lin subspaces.

exercise.

(compare: For a closed subspace Y of Hilb H, $Y^{\perp\perp} = Y$)

<u>Thm</u> Let W and Z be non-empty <u>closed lin subspaces</u> of X or X'.
Then ${}^{\circ}(W^{\circ}) = W$, and <u>if X is reflexive</u>, $({}^{\circ}Z)^{\circ} = Z$

<u>proof</u> • Let $p \in {}^{\circ}(W^{\circ}) \setminus W$. W closed, so $\exists f \in X'$ with $f(W)=0$ $f(p) \neq 0$
(HB)

So $f \in W^{\circ}$ and $p \notin {}^{\circ}(W^{\circ})$ $\lightning$.

• Let $g \in ({}^{\circ}Z)^{\circ} \setminus Z$. Z closed, so $\exists \psi \in X''$ with $\psi(Z)=0$, $\psi(g) \neq 0$
(HB)
$\Downarrow$
(doesn't mean $\psi \in {}^{\circ}Z$)

X refl, so $\psi = J_X(q)$ for some $q \in X$.

$\forall f \in Z$, $0 = \psi(f) = f(q)$ so $q \in {}^{\circ}Z$, and

$0 \neq \psi(g) = g(q)$ so $g \notin ({}^{\circ}Z)^{\circ}$ $\lightning$. $\square$

Next, for $X = c_0$, we construct closed subsp $Z \subsetneq X'$ with $({}^{\circ}Z)^{\circ} = X'$.

So X refl. necessary for 2nd statement.

Recall $T: \ell^1 \to c_0' : \{a_n\} \mapsto \left(\{b_n\} \mapsto \sum_{n=1}^{\infty} a_n b_n\right)$ is linear, isometric isomorphism

(for $p \in [1,\infty)$ idem for $\ell^q \to (\ell^p)'$)

Let $V := \{\{a_n\} \in \ell^1 : \sum_n (-1)^n a_n = 0\}$ is proper, closed subs ℓ^1,

so $Z := T(V)$ is proper, closed subspace c_0'.

Let $\{p_n\} \in {}^{\circ}Z = \{\{q_n\} \in c_0 : f(\{q_n\}) = 0 \ \forall f \in T(V)\}$

$= \{ \quad " \quad : \sum_1^{\infty} a_n p_n = 0 \quad \forall \{a_n\} \in V\}$

Taking, for any k, $\{a_n\} = e_k + e_{k+1} \Rightarrow p_{k+1} = -p_k \ \forall k$.

From $\lim p_n = 0 \Rightarrow \{p_n\} = 0$. So ${}^{\circ}Z = \{0\}$,

but then $({}^{\circ}Z)^{\circ} = c_0' \neq Z$.

<u>Corol</u>: c_0 non-refl, and since closed in ℓ^{∞}, also ℓ^{∞} non-refl.

Summary: X normed lin. space.

$F_X(x) \in X''$ by $F_X(x)(f) = f(x)$

$J_X : X \to X'' : x \longmapsto F_X(x)$ is lin. isometry.

X is called reflexive when J_X is isomorphism

Thm X Banach. Then X refl $\Leftrightarrow$ X' refl.

Thm Closed lin subsp of refl is refl.

For $W \subset X$, $Z \subset X'$, annihilators $W^\circ \subset X'$, ${}^\circ Z \subset X$ are defined by ....

Thm W, Z closed subsp of X & X'. Then ${}^\circ(W^\circ) = W$, and, if X is refl, $({}^\circ Z)^\circ = Z$