

Summary:  $X$  normed lin. space.

$F_x(x) \in X''$  by  $F_x(x)(f) = f(x)$  → second or bidual

$J_X$  — evaluation map  $J_X : X \rightarrow X'' : x \mapsto F_x(x)$  is lin. isometry.

$X$  is called **reflexive** when  $J_X$  is isomorphism

Thm  $X$  Banach. Then  $X$  refl  $\Leftrightarrow X'$  refl.

Thm Closed lin subsp of refl is refl.

For  $W \subset X$ ,  $Z \subset X'$ , **annihilators**  $W^\circ \subset X'$ ,  ${}^\circ Z \subset X$  are defined by

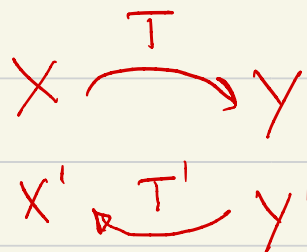
$$W^\circ = \{ f \in X' : f(x) = 0, x \in W \}$$

$${}^\circ Z = \{ x \in X : f(x) = 0, f \in Z \},$$

are closed lin. subspaces.

Thm  $W, Z$  closed subsp. of  $X$  &  $X'$ , resp. Then  ${}^\circ(W^\circ) = W$ ,  
and, if  $X$  is refl,  $({}^\circ Z)^\circ = Z$

Thm  $T \in B(X, Y)$ . Then  $T'$  def by  $(T'f)(x) = f(Tx)$  ( $f \in Y'$ ,  $x \in X$ )  
 is in  $B(Y', X')$ , and  $\|T'\| = \|T\|$   
 $T'$  is called the **dual** of  $T$ .



proof  $T'f \in X^*$ .

$$|(T'f)(x)| = |f(Tx)| \leq \|f\|_{Y'} \|Tx\|, \text{ so } T'f \in X'$$

$$\text{and } \|T'f\|_{X'} \leq \|f\|_{Y'} \|T\|. \text{ Also } T' \in L(Y', X')$$

$$\text{and } T' \in B(Y', X') \text{ with } \|T'\|_{B(Y', X')} \leq \|T\|_{B(X, Y)}.$$

$$\forall 0 \neq x \in X, \exists f \in Y' \text{ with } f(Tx) = \|Tx\| \text{ \& } \|f\|_{Y'} = 1 \text{ (HB)}$$

$$\text{So } \|Tx\| = f(Tx) = (T'f)(x) \leq \|T'f\| \|x\| \leq \|T'\| \underbrace{\|f\|}_{=1} \|x\|$$

$$\text{or } \|T\|_{B(X, Y)} \leq \|T'\|_{B(Y', X')}.$$

rem In exerc. will see that for fin. dim  $X$  &  $Y$  equipped with some bases, and  $X'$  &  $Y'$  with dual bases, matrix of  $T'$  is **transpose** of matrix of  $T$ .

⌈  
Later:  $H, K$  Hilbert,  $T \in B(H, K)$  then  $T^* \in L(K, H)$  by

$$\langle Th, k \rangle_K = \langle h, T^*k \rangle_H \quad (h \in H, k \in K) \text{ is called } \text{adjoint}$$

$$\text{of } T, \text{ and } \|T^*\|_{B(K, H)} = \|T\|_{B(H, K)}.$$

Via Riesz repr, close rel betw  $T^*$  &  $T'$ . Latter

⌋ can be viewed as gener. of  $T^*$  to normed lin spaces.

Prop  $T \in B(X, Y)$  (so  $T' \in B(Y', X')$ ). Then

$$\text{a) } \ker T' = (\text{ran } T)^\circ \quad \text{b) } \ker T = {}^\circ(\text{ran } T')$$

(for matrix  $A$ ,  $\ker A = (\text{ran } A^T)^\perp$ ,  $\text{ran } A = (\ker A^T)^\perp$ )

proof a)  $(T'f)(x) = 0 \quad \forall x$  (i.e.  $f \in \ker T'$ )  $\Leftrightarrow f(Tx) = 0 \quad \forall x$  (i.e.  $f \in (\text{ran } T)^\circ$ )

b) exerc. (HB needed)

Thm  $T \in B(X, Y)$

so  $X \simeq Y \Rightarrow X' \simeq Y'$

a)  $T$  isomorphism  $\Rightarrow T'$  isomorphism

b)  $T$  isometr. iso  $\Rightarrow T'$  isometric iso.

proof. a)  $S := T^{-1} \in B(Y, X)$ , so  $S' \in B(X', Y')$ .

We prove  $S' = (T')^{-1}$  (i.e.  $(T')^{-1} = (T^{-1})'$ )

For  $f \in Y', y \in Y$   $(S'T'f)(y) = (T'f)(Sy) = f(TSy) = f(y)$

so  $S'T' = \text{Id}$ . Similarly  $T'S' = \text{Id}$ .

b) exerc.

isomorphic  
(known)

Thm

Another proof of  $C_0$  non-refl:  $C_0' \simeq \ell^1 \Rightarrow C_0'' \simeq (\ell^1)' \simeq \ell^\infty$

known

$\ell^\infty$  non-separable, so  $C_0''$  non-sep. But  $C_0$  is separable  
so non-refl.

lemma  $T \in B(X, Y)$ . Then  $J_Y \circ T = T'' \circ J_X$

proof  $\forall x \in X, f \in Y'$

$$J_Y(Tx)(f) = f(Tx) = (T'f)(x) = J_X(x)(T'f) = (T'' J_X(x))(f)$$

Corol If  $X \simeq Y$ , then  $X \text{ refl} \Leftrightarrow Y \text{ refl}$

proof  $X \simeq Y$  means  $\exists$  lin isomorp.  $T: X \rightarrow Y$ , so

$$T'': X'' \rightarrow Y'' \text{ lin isomorp.}$$

Let  $X \text{ refl}$ , i.e.  $J_X$  lin (isometric) isomorp.

$$\text{Then } J_Y = T'' \circ J_X \circ T^{-1} \text{ lin isomorp.}$$

□

Corol:  $\ell^1$  non-refl.

proof  $C_0$  non-refl, and so  $C_0'$  non-refl (Banach  $X$  is refl  $\Leftrightarrow X'$  is refl)  
and  $C_0' \simeq \ell^1$ .

## § 5.6 Projectors and complementary subspaces (not clear why part of Ch. 5)

Def  $X$  lin space. Subspace  $U, V$  are called complementary when  $X = U \oplus V$ . (i.e.  $\forall x \in X \exists! u_x \in U, v_x \in V$  with  $x = u_x + v_x$ )

lemma  $P: x \mapsto u_x$  is a projector (i.e.  $P$  is linear, and  $P^2 = P$ )  
(the projector on  $U$  along  $V$ )  
and  $I - P: x \mapsto v_x$

lemma  $P$  proj  $\Leftrightarrow I - P$  proj.  $\text{ran } P = \ker I - P$   
 $\ker P = \text{ran } I - P$

lemma If  $P: X \rightarrow X$  is proj, then  $\ker P$  &  $\ker I - P$  are complementary

So 1-1 correspondence between complementary subspaces and projectors.

Def  $X$  normed lin space, then  $U$  &  $V$  are called topological complementary if  $P$  is bounded.

In that case  $U$  &  $V$  closed (because kernel of bounded lin map)

lemma  $X$  Banach,  $X = U \oplus V$  and  $U, V$  closed.

Then  $U, V$  topological complementary.

proof Let  $P$  be corr. proj onto  $U$  (so  $I-P$  proj onto  $V$ )

We show  $\mathcal{G}(P)$  closed in  $X \times X$ . Then closed graph theorem shows  $P$  bounded.

Let  $(x_n, P x_n) \rightarrow (x, y)$ . Then from  $U$  closed, it follows  $y \in U$ .

From  $x - y = \lim (I - P)x_n$  and  $V$  closed, it follows  $x - y \in V$ .

So  $0 = P(x - y) = P x - P y = P x - y$  or  $(x, y) \in \mathcal{G}(P)$ .

Recall: lemma  $H$  Hilb,  $U$  closed lin subsp, then  $U$  &  $U^\perp$  topol. compl.

proof  $P: H \rightarrow U: x \mapsto$  best appr from  $U$  is orth proj.

In contrast: In Banach  $X$ , a closed subsp  $U$  not necessarily has a topological complement. Ex:  $X = \ell^\infty$ ,  $U = c_0$ .

(proof not easy)

## § 5.7 Weak & weak-\* convergence

right harpoon up

stackrel{right}{\leftarrow} \{ \text{fast} \} \{ \text{right} \dots \}

Skip Lemma 5.67 and everything related to metric  $d_w$  on  $X'$ .

Def  $X$  normed lin space.

a).  $\{x_n\} \subset X$  converges weakly to  $x \in X$  when  $\lim f(x_n) = f(x) \forall f \in X'$

b)  $\{f_n\} \subset X'$  " weak-\* to  $f \in X'$  when  $\lim f_n(x) = f(x) \forall x \in X$

Notations:  $\{x_n\} \rightarrow x$ ,  $f_n \xrightarrow{*} f$ .

lemma a) weak and weak-\* limits are unique

b) weakly conv. seq are bounded; and so are weak-\*

conv seq<sup>s</sup> when  $X$  is Banach

proof: a)  $f_n \xrightarrow{*} f$ ,  $f_n \xrightarrow{*} g$ , then  $f(x) = g(x) \forall x \Rightarrow f = g$

$x_n \rightarrow x$ ,  $x_n \rightarrow y$ , then  $\forall f \in X'$   $f(x-y) = 0 \Rightarrow x=y$  (HB)

b) Let  $x_n \rightarrow x$ . Then  $\forall f \in X'$ ,  $\sup \{ |f(x_n)| : n \} < \infty$ .  $f(x_n) = J_x(x_n)(f)$

$\{ J_x(x_n) : n \in \mathbb{N} \} \subset X'' = B(X', \mathbb{F})$ . Unif. bound. principle

shows  $\{ \|J_x(x_n)\|_{X''} : n \}$  bounded Banach, and  $\|x_n\|_X = \|J_x(x_n)\|_{X''}$ .

Let  $f_n \xrightarrow{*} f$ . Then  $\forall x$ ,  $\sup \{ |f_n(x)| : n \} < \infty$ . Apply unif. bound p.

using  $X' = B(X, \mathbb{F})$  &  $X$  Ban.

lemma  $\{x_n\} \rightarrow x \Rightarrow \{x_n\} \rightharpoonup x$  (and  $\Leftarrow$  when  $\dim X < \infty$ )  
 $\{f_n\} \rightarrow f \Rightarrow \{f_n\} \rightharpoonup f \Rightarrow \{f_n\} \xrightarrow{*} f$   
 ( $\Leftarrow$  when  $X$  is refl.)

proof If  $x_n \rightarrow x$ , then  $\forall f \in X'$   $|f(x_n) - f(x)| \leq \|f\| \|x_n - x\| \rightarrow 0$

If  $\dim X < \infty$ , let  $\{\phi_1, \dots, \phi_m\}$  basis  $X$ , with dual basis  $\{\phi'_1, \dots, \phi'_m\}$  of  $X'$ .  
 Then  $x_n = \sum_i \phi'_i(x_n) \phi_i$ . If  $x_n \rightarrow x$ , then  $\phi'_i(x_n) \rightarrow \phi'_i(x)$   
 and so  $x_n \rightarrow x$

If  $f_n \rightarrow f$ , then  $f_n \rightharpoonup f$ . Latter means  $g(f_n) \rightarrow g(f)$

for all  $g \in X''$ , so in part.,  $J_x(x)(f_n) \rightarrow J_x(x)(f)$ ,

i.e.,  $f_n \xrightarrow{*} f$ . Both statements equiv when  $X$  refl.

Prop. Let  $H$  Hilbert.

$$a) x_n \rightarrow x \iff \langle x_n, y \rangle \rightarrow \langle x, y \rangle \quad \forall y$$

b) If  $\dim H = \infty$  and  $\{e_n\}$  orthon. seq, then  $e_n \rightarrow 0$

(so generally weak conv is weaker than convergence)

proof: Exerc.

Lemma Let  $S \subset X$  with  $\overline{\text{Sp} S} = X$ . Then if  $\{f_n\} \subset X'$  is bounded, and  $\{f_n(s)\}$  converges  $\forall s \in S$ . Then  $\exists f \in X'$  st  $\{f_n\} \xrightarrow{*} f$ .

proof Let  $C := \sup \{\|f_n\|_{X'} : n \in \mathbb{N}\}$ . Let  $x \in X$ . We show  $\{f_n(x)\}$  conv.  
Let  $\varepsilon > 0$  be given.  $\exists t \in \text{Sp} S$  with  $\|x - t\| < \frac{\varepsilon}{3C}$ .

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f_n(t)|}_{(1)} + \underbrace{|f_n(t) - f_m(t)|}_{(2)} + \underbrace{|f_m(t) - f_m(x)|}_{(3)}$$

$(1) \& (3) < \frac{\varepsilon}{3}$ .  $t$  is finite lin comb. of  $s \in S$ . So for  $n, m$  large enough,  $(2) < \frac{\varepsilon}{3}$ . So  $\{f_n(x)\}$  Cauchy in  $\mathbb{F}$ , so conv.

Def  $f(x) := \lim f_n(x)$ . Obv  $f \in X^*$ ,  $|f(x)| \leq \sup \{|f_n(x)| : n\} \leq C \|x\|$   
so  $f \in X'$   $\square$